ORBITAL RATES OF EARTH SATELLITES AT RESONANCES TO TEST THE ACCURACY OF EARTH GRAVITY FIELD MODELS

JAROSLAV KLOKOČNÍK

Astronomical Institute of the Czechoslovak Academy of Sciences, 251 65 Ondřejov Observatory, Czechoslovakia

(Received 6 December, 1982; Accepted 2 February, 1983)

Abstract. Differences among the Earth gravity field models, which were (in Klokočník and Pospíšilová, 1981) expressed as dispersions of the relevant lumped geopotential coefficients, are here transformed to the differences in variations of orbital quantities.

Theoretical formulae, the Lagrange (planetary) equations, describing the orbital rates near resonances due to the geopotential, are derived in a simple and unified form. They are then applied to estimate the orbital uncertainty as a function of Earth models differences. The first set of the Earth models (set I) consists of 11 models from the decade 1970–1980, of greatly varying quality; the set II contains several recent models; we present a test (for the 13th- to 15th-order) based on standard deviations of the lumped values of GEM 10B, which were estimated by means of independent resonant data (in Klokočník, 1982).

Maxima of the differences in the variations of the elements for the set I reach 8×10^{-4} deg day^{-1}, 10–12 m day^{-1} or 200 m day^{-1} in I, a, or $L_0 = \omega + M_0 + \Omega$, respectively, for close and polar orbits ($\sim$ 15 revs day^{-1}); the values are not higher than 10^{-4} deg day^{-1}, 1–2 m day^{-1} or 20 m day^{-1} in I, a, L_0 for higher orbits ($\sim$ 6–7 revs day^{-1}). For the set II, calibrated by resonant data, the maximum inaccuracy ($\pm 3\sigma$) is about 3×10^{-4} deg day^{-1}, $\leqslant 6$ m day^{-1} or $\leqslant 100$ m day^{-1} for I, a, and L_0 at 15 revs day^{-1}, and is not larger than $\sim 1 \times 10^{-4}$ deg day^{-1}, 2 m day^{-1} or 25 m day^{-1} for 13 revs day^{-1}.

1. Introduction

The Lagrange Planetary Equations (LPE), describing variations of some of the orbital elements of an Earth's artificial satellite owing to the Earth's gravitational potential, have been transformed into a form suitable for investigating the orbital resonances of satellites by Allan (1971) and Gooding (1971, 1975). Here, we present similar equations for additional elements; the case of nearly circular orbit is described in a simple and unified form, using the lumped geopotential coefficients.

The reasons for our derivations rise from practical demands. The requirements for the prediction of orbits of close Earth's satellites can be at a submeter level of accuracy to fully utilize certain satellite-determined data for geoscience applications. The usually available orbital accuracy is, however, about several meters, using recent Earth gravitational field models (EM), and only exceptionally $\sim \pm 1$ m (in radial direction) is achievable (Lerch *et al.*, 1982a, b). A considerable problem is posed by the uncertainty in the parameters characterizing the Earth's gravitational field. The investigation of the accuracy of the EMs has been carried out in various ways, and here, a further approach is presented. The differences among the EMs, originally available as differences among the relevant lumped coefficients, will be transformed to the differences in the orbital elements, expressed in terms of the LPE. Instead of the differences in 'abstract', dimensionless harmonics, we will have 'live', dimensioned orbital rates (separately for the individual orders of harmonics). This test may be prepared with various sets of the EMs, for arbitrary orbital inclination and need not be connected to any Earth satellite and so limited by its inclination. Alternative methods will be mentioned in Section 4, where numerical examples of our tests will be discussed.

2. Functions and Symbols Used

We suppose the Earth's gravitational potential written as a function of the harmonic coefficients $\bar{C}_{lm}$, $\bar{S}_{lm}$ (l degree, m order), fully normalized (labelled by the bar), of the inclination functions $\bar{F}_{lmp}(I)$, also fully normalized (e.g., Allan, 1967), of the eccentricity functions $G_{lpq}(e) = X_{l-2p+q}^{-l-1,l-2p}(e)$, Kaula (1966), and of

$$\psi_{lmpq} \equiv (l-2p)\omega + (l-2p+q)M + m(\Omega - S - \lambda_{lm}), \tag{1}$$

where S is the sidereal Greenwich time,

$$2 \leqslant l < \infty, \qquad 0 \leqslant m \leqslant l, \qquad 0 \leqslant p \leqslant l, \qquad -\infty < q < \infty,$$

$m\lambda_{lm} = \arctan(\bar{S}_{lm}/\bar{C}_{lm})$.

We make use of the usual orbital elements $a, e, I, \omega, \Omega, M$. LPE, describing their variations due to the geopotential, will be used in the form derived in Allan (1967).

The 'resonant' indices for $\bar{F}_{lmp}(I)$ and $\bar{C}_{lm}$, $\bar{S}_{lm}$ for given orbital resonance β/α (β nodal revolution of a satellite per α sideral days; α, β mutually prime integers) will be choosen from all (l, m, p) by this way (Klokočnik, 1976):

$$\begin{aligned} \beta/\alpha:\ \alpha\gamma &= l - 2p + q, \qquad \beta\gamma = m, \qquad \gamma = 1, 2, 3, \ldots \quad \text{(resonance levels)};\\ l_i &= \beta\gamma + \delta + 2i,\\ m_i &= \beta\gamma,\\ 2p_i &= (\beta - \alpha)\gamma + \delta + q + 2i,\\ i &= 0, 1, 2, \ldots;\\ \delta &= 0 \ldots \quad \text{for} \quad (\beta-\alpha)\gamma + q \quad \text{even},\\ \delta &= 1 \ldots \quad \text{for} \quad (\beta-\alpha)\gamma + q \quad \text{odd},\\ \varepsilon &= 1 - \delta. \end{aligned} \tag{2}$$

The original quantity of (1) will then be transformed to

$$\psi_{lmpq\,(\text{res})} = \gamma\phi_{\beta/\alpha} - m\lambda_{lm} - q\omega,$$

where $\phi_{\beta/\alpha}$ is the so-called resonant angle

$$\phi_{\beta/\alpha} \equiv \alpha(\omega + M) + \beta(\Omega - S). \tag{3}$$

Further symbols:

a_0 is a length (conveniently taken as the best estimation of the Earth' equatorial radius), the scaling factor associated with $\bar{C}_{lm}$, $\bar{S}_{lm}$;

$M = \int n \, dt + M_0$,

n is the osculating mean motion,
$L = \tilde{\omega} + \Omega$ is the mean longitude,
$\lambda = \tilde{\omega} + \Omega \cos I$ is called along-track component, and
$\tilde{\omega} = \omega + M$;
$j = \sqrt{-1}$.

The term 'lumped (geopotential) coefficient' has been introduced by Gooding (1971) and Allan (1971, 1973); the harmonic coefficients of various degree (either even, or odd) of the same order $m = \beta\gamma$ are 'lumped' to linear combinations; these quantities are directly observable from the resonant phenomena in satellite orbits. The lumped values $\bar{R}^{q,k}_{\beta\gamma}(I)$ are defined in this way:

$$\bar{R}^{q,k}_{\beta\gamma}(I) = \sum_{i=0}^{\infty} Q^{q,k}_{\beta\gamma+\delta+2i,\beta\gamma}(I) \times \bar{R}_{\beta\gamma+\delta+2i,\beta\gamma}, \tag{4}$$

where the influence coefficients $Q(I)$ are functions of $\bar{F}_{lmp}(I)$, see Section 3; the symbol R substitutes C or S.

The lumped coefficients can be assorted (Klokočnik and Pospíšilová, 1981) as follows:

–*one day resonances* ($\alpha = 1$):
$\gamma = 1, q = 0 \ldots$ 'basic terms' $\bar{R}^{0,1}_{\beta}$;
$\gamma = 1, q = \pm 1 \ldots$ '*e*-terms' $\bar{R}^{1,0}_{\beta}$, $\bar{R}^{-1,2}_{\beta}$ (of 'side-band' resonance);
$\gamma = 2, q = 0 \ldots$ 'double terms' $\bar{R}^{0,2}_{2\beta}$ (overtones);
$\vdots$
$\gamma \geqslant 2, |q| > 0 \ldots$ higher terms (usually indeterminate from satellite orbits).

– *two-day resonances* ($\alpha = 2$):
$\gamma = 1, q = 0 \ldots$ 'basic terms' $\bar{R}^{0,2}_{\beta}$;
$\vdots$
$\gamma = 2, q = 0 \ldots$ 'double terms' $\bar{R}^{0,4}_{2\beta}$, etc.

The index $k = \alpha\gamma - q$ has been suggested by Gooding (1975).

Further on we will deal with $\bar{R}^{0,\alpha}_{\beta}(I)$ and $\bar{R}^{\pm 1,\alpha \mp 1}_{\beta}(I)$ in theoretical formulae for I and a, and just with the basic terms for the 'longitudinal' quantities and numerical examples; these terms are the most important for nearly circular orbits.

3. Derivation

There are two main steps in the derivations:

(i) the selection of the resonant terms for given combination of α, β, γ, q.

(ii) the inclusion of the lumped coefficients. The pattern of the derivations is the same for all the elements (the details for the inclination are in Klokočník, 1976).

3.1. Inclination I

Following Allan (1967):

$$\begin{aligned} \mathrm{d}I/\mathrm{d}t = n(1-e^2)^{-1/2} \sum_{lmpq} J_{lm}(a_0/a)^l \times \\ \times \mathscr{R}\{F_{lmp}(I) G_{lpq}(e) j[(l-2p)\cot I - m \operatorname{cosec} I] \times \\ \times \exp(j\psi_{lmpq})\}, \end{aligned} \tag{5}$$

$\mathscr{R}\{.\}$ denotes the real part of the respective complex expression.

One can select the resonant terms by means of (2), introduce (3) instead of (1) and to use (4). Various subsidiary formulae are needed, e.g.:

$$\begin{aligned} \bar{J}_{lm}\mathscr{R}\{j^{2i}\exp(j\psi_{lmp0})\} &= (-1^i(\bar{S}_{lm}\sin\phi_{\beta/\alpha} + \bar{C}_{lm}\cos\phi_{\beta/\alpha}), \\ -\bar{J}_{lm}\mathscr{R}\{j^{2i+1}\exp(j\psi_{lmp0})\} &= (-1)^{i+1}(\bar{C}_{lm}\sin\phi_{\beta/\alpha} - \bar{S}_{lm}\cos\phi_{\beta/\alpha}). \end{aligned} \tag{6}$$

By simplifying for $\gamma = 1$, $q = 0, \pm 1$, the final 'resonant' LPE for I reads (compare with Allan, 1973; Gooding, 1975; Klokočník, 1976; or King–Hele and Walker, 1981):

$$\begin{aligned} (\mathrm{d}I/\mathrm{d}t)_{\beta/\alpha} = f_I\{[\bar{\bar{C}}_\beta^{0,\alpha}\sin\phi_{\beta/\alpha} - \bar{\bar{S}}_\beta^{0,\alpha}\cos\phi_{\beta/\alpha}] + \\ + e[\bar{\bar{C}}_\beta^{\pm 1,\alpha\mp 1}\sin(\phi_{\beta/\alpha}\mp\omega) - \bar{\bar{S}}_\beta^{\pm 1,\alpha\mp 1}\cos(\phi_{\beta/\alpha}\mp\omega)]\} + \\ + O(\gamma, q), \end{aligned} \tag{7}$$

where in $[.]$ is the summation over the e-terms, $O(\gamma, q)$ are neglected double- and higher-terms,

$$f_I = n(\beta - \alpha\cos I)\operatorname{cosec} I \times [\![\bar{F}_{\beta+\delta,\beta,1/2(\beta-\alpha)+1/2\delta}(I)(a_0/a)^{\beta+\delta}]\!] \tag{8}$$

taking $q = 0$, or

$$f_I = n\mathscr{F}(\beta - \alpha\cos I)\operatorname{cosec} I \tag{9}$$

with $\mathscr{F} = [\![.]\!]$; and the lumped coefficients are

$$\begin{aligned} \begin{Bmatrix} \varepsilon\bar{\bar{C}} - \delta\bar{\bar{S}} \\ \delta\bar{\bar{C}} + \varepsilon\bar{\bar{S}} \end{Bmatrix}_\beta^{0,\alpha} = 1/\bar{F}_{\beta+\delta,\beta,1/2(\beta-\alpha)+1/2\delta}(I)\sum_{i=0}^{\infty}(-1)^i(a_0/a)^{2i} \times \\ \times \bar{F}_{\beta+2i+\delta,\beta,1/2(\beta-\alpha)+1/2\delta+i}(I)\begin{Bmatrix} \bar{C} \\ \bar{S} \end{Bmatrix}_{\beta+2i+\delta,\beta} \end{aligned} \tag{10}$$

$$\begin{Bmatrix} \varepsilon\bar{\bar{C}} - \delta\tilde{\bar{S}} \\ \delta\bar{\bar{C}} + \varepsilon\bar{\bar{S}} \end{Bmatrix}_{\beta}^{\pm 1, \alpha \mp 1} = [\beta - (\alpha \mp 1)\cos I] \times$$
$$\times [2(\beta - \alpha\cos I)\bar{F}_{\beta+\varepsilon, \beta, 1/2(\beta-\alpha)+1/2\varepsilon}(I)]^{-1} \times$$
$$\times \sum_{i=0}^{\infty} (-1)^i (\beta - \alpha - \varepsilon + x\alpha(+2i)(a_0/a)^{2i+\delta-\varepsilon} \times$$
$$\times \bar{F}_{\beta+2i+\delta, \beta, 1/2(\beta-\alpha)+1/2\delta \pm 1/2 + i}(I) \times$$
$$\times \begin{Bmatrix} \bar{C} \\ \bar{S} \end{Bmatrix}_{\beta+2i+\delta, \beta}, \tag{11}$$

where $x = 3$ for $q = +1$ and $x = -1$ for $q = -1$.

The normalization in (10) by the leading inclination function (with the lowest l, p for given β) is to keep the compatibility with the formulae in King–Hele and Walker, (1981). As for two-bar values, they are chosen such that the argument $[\bar{\bar{C}}\sin(.) - \bar{\bar{S}}\cos(.)]$ in (7) holds for any $(\beta - \alpha)\gamma + q$. Another possibility is to define $\bar{C}_{\beta\gamma}^{q,k} = f(\bar{C}_{l,\beta})$ and $\bar{S}_{\beta\gamma}^{q,k} = f(\bar{S}_{l,\beta})$ for each $(\beta - \alpha)\gamma + q$ and hence to have the arguments as follows:

$$[(\varepsilon\bar{C}_{\beta\gamma}^{q,k} + \delta\bar{S}_{\beta\gamma}^{q,k})\sin(.) + (\delta\bar{C}_{\beta\gamma}^{q,k} - \varepsilon\bar{S}_{\beta\gamma}^{q,k})\cos(.)].$$

3.2. Semimajor axis a

According to Allan (1967):

$$\mathrm{d}a/\mathrm{d}t = 2na \sum_{lmpq} J_{lm}(a_0/a)^l \times$$
$$\times \mathscr{R}\{j(l - 2p + q)F_{lmp}(I)G_{lpq}(e)\exp(j\psi_{lmpq})\}. \tag{12}$$

Repeating the same procedure as for I, we arrive at

$$(\mathrm{d}a/\mathrm{d}t)_{\beta/\alpha} = f_a\{[\bar{\bar{C}}_{\beta}^{0,\alpha}\sin\phi_{\beta/\alpha} - \bar{\bar{S}}_{\beta}^{0,\alpha}\cos\phi_{\beta/\alpha}] +$$
$$+ e[\bar{\bar{C}}_{\beta}^{\pm 1,\alpha\mp 1}\sin(\phi_{\beta/\alpha} \mp \omega) - \bar{\bar{S}}_{\beta}^{\pm 1,\alpha\mp 1}\cos(\phi_{\beta/\alpha} \mp \omega)]\} +$$
$$+ O(\gamma, q), \tag{13}$$

with $f_a = -2\alpha na\mathscr{F}$, taking $q = 0$.

The form of (13) is the same as of (7) and the lumped coefficients are also the same. There is a simple relation between (13) and (7); dividing them mutually, we can obtain

$$\mathrm{d}a/\mathrm{d}t = [2\alpha a(\alpha\cos I - \beta)^{-1}\sin I](\mathrm{d}I/\mathrm{d}t), \qquad (\gamma = 1, q = 0), \tag{14}$$

which yields a useful check or a tool for predicting one quality from the other (Batrakov, 1965; Wagner and Klosko, 1975).

3.3. Longitude L_0

The analyses of $L = \omega + M + \Omega$ at shallow resonances have been successfully used for direct determination of $\bar{C}_{lm}$, $\bar{S}_{lm}$ of the 12th–14th order (e.g., Reigber and Rummel, 1979). Our approach involves the use of the lumped coefficients. Two terms of M will not be included: $\mathrm{d}_2M/\mathrm{d}t$ (Kaula, 1966, p. 49, the effect from the integration of a) and $\mathrm{d}M_{\text{indirect}}/\mathrm{d}t$ (Wagner, 1975, p. 4090); they are functions of $(1/\dot{\psi}_{lmpq})$ and hardly can be estimated generally for various satellite orbits and epochs. This is why we cannot speak about mean longitude L, but only about a 'longitudinal' term, say L_0, and $L_0 = \omega + M_0 + \Omega$. Such an omission could lead to a serious defect in orbit predictions, but here we are interested only in a transformation of the differences among the EMs from one of their form to another.

Following Allan (1967), we have at our disposal:

$$\begin{aligned}\mathrm{d}\omega/\mathrm{d}t = n \sum_{lmpq} J_{lm}(a_0/a)^l \mathscr{R}\{[(1/e)(1-e^2)^{1/2} F_{lmp}(I) G'_{lpq}(e) - \\ -(1-e^2)^{-1/2} F'_{lmp}(I) G_{lpq}(e) \cot I] \exp(j\psi_{lmpq})\};\end{aligned} \tag{15}$$

$$\begin{aligned}\mathrm{d}M_0/\mathrm{d}t = n \sum_{lmpq} J_{lm}(a_0/a)^l \mathscr{R}\{F_{lmp}(I)[2(l+1)G_{lpq}(e) - \\ -(1/e)(1-e^2)G'_{lpq}(e)] \exp(j\psi_{lmpq})\};\end{aligned} \tag{16}$$

$$\begin{aligned}\mathrm{d}\Omega/\mathrm{d}t = n(1-e^2)^{-1/2} \operatorname{cosec} I \times \\ \times \sum_{lmpq} J_{lm}(a_0/a)^l \mathscr{R}\{F'_{lmp}(I) G_{lpq}(e) \exp(j\psi_{lmpq})\},\end{aligned} \tag{17}$$

where $F'(I) = \partial F(I)/\partial I$, $G'(e) = \partial G(e)/\partial e$.

The simplification for $e \to 0$ in $\mathrm{d}L_0/\mathrm{d}t$ is substantial*. Repeating the procedure from Sections 3.1 and 3.2 leads to

$$(\mathrm{d}L_0/\mathrm{d}t)_{\beta/\alpha} = f_L(\tilde{\tilde{C}}{}^{0,\alpha}_{\beta} \sin \phi_{\beta/\alpha} - \tilde{\tilde{S}}{}^{0,\alpha}_{\beta} \cos \phi_{\beta/\alpha}) + O(e), \tag{18}$$

where

$$f_L = (-1)^\delta n\mathscr{F}. \tag{19}$$

A new argument at the inclination functions has appeared in $\mathrm{d}L_0/\mathrm{d}t$ during the derivation; it was necessary to define

$$\tilde{F}_{lmp}(I) = 2(\beta + 2i + \delta + 1)\bar{F}_{lmp}(I) + \bar{F}'_{lmp}(I) \tan(I/2) \tag{20}$$

to keep the form (10) of the definition of the lumped coefficients, i.e.:

$$\begin{aligned}\begin{Bmatrix} \delta\tilde{\tilde{C}} - \varepsilon\tilde{\tilde{S}} \\ \varepsilon\tilde{\tilde{C}} + \delta\tilde{\tilde{S}} \end{Bmatrix}^{0,\alpha}_{\beta} = 1/\bar{F}_{\beta+\delta,\beta,1/2(\beta-\alpha)+1/2\delta}(I) \sum_{i=0}^{\infty} (-1)^i (a_0/a)^{2i} \times \\ \times \tilde{F}_{\beta+2i+\delta,\beta,1/2(\beta-\alpha)+1/2\delta+i}(I) \begin{Bmatrix} \bar{C} \\ \bar{S} \end{Bmatrix}_{\beta+2i+\delta,\beta}.\end{aligned} \tag{21}$$

* $[\mathrm{d}\omega/\mathrm{d}t] + [\mathrm{d}M_0/\mathrm{d}t] + [\mathrm{d}\Omega/\mathrm{d}t] = [(1/e)(1-e^2)^{1/2} G'(e)F(I) - (1-e^2)^{-1/2} F'(I)G(e) \cot I +$
$+ 2(l+1)G(e)F(I) - (1/e)(1-e^2)G'(e)F(I) + F'(I)G(e) \operatorname{cosec} I]$,
$e \to 0, e^2/2 \twoheadrightarrow 0$:
$[.] = G(e)\{2(l+1)F(I) + F'(I)\tan(I/2)\}$.

Since the explicit equation (21) is not known from literature, we should emphasize its differences from (10) and (11): the position of δ and ε differs; (21) cannot be replaced by (10).

3.4. Along-track component λ_0

$\lambda = \omega + M + \Omega \cos \mathrm{I}$ has been recommended and used for the analysis of resonant phenomena by Wagner *et al.* (e.g., Wagner and Klosko, 1975; Klosko and Wagner, 1982; Dunn, 1981). The formulae for $(\mathrm{d}\lambda_0/\mathrm{d}t)_{\mathrm{res}}$, omitting the same terms as for L_0, may be derived similarly as (18), starting from (15)–(17). A change arises owing to presence of the term 'cos I': instead of $\bar{F}(I)$, we need to introduce

$$*\bar{F}_{lmp}(I) = 2(\beta + 2i + \delta + 1)\bar{F}_{lmp}(I) \tag{22}$$

or by means of (20)

$$*\bar{F}_{lmp}(I) = \bar{F}_{lmp}(I) - \bar{F}'_{lmp}(I)\tan(I/2). \tag{23}$$

The LPE is

$$(d\lambda_0/\mathrm{d}t)_{\beta/\alpha} = f_\lambda(*C_\beta^{0,\alpha} \sin \phi_{\beta/\alpha} - *S_\beta^{0,\alpha} \cos \phi_{\beta/\alpha}) + O(e), \tag{24}$$

where $f_\lambda = f_L$ and

$$\begin{Bmatrix} \delta{*C} - \varepsilon{*S} \\ \varepsilon{*C} + \delta{*S} \end{Bmatrix}_\beta^{0,\alpha} = 1/\bar{F}_{\beta+\delta,\beta,1/2(\beta-\alpha)+1/2\delta}(I) \times$$
$$\times \sum_{i=0}^{\infty} (-1)^i (a_0/a)^{2i} \times$$
$$\times *\bar{F}_{\beta+2i+\delta,\beta,1/2(\beta-\alpha)+1/2\delta+i}(I) \begin{Bmatrix} \bar{C} \\ \bar{S} \end{Bmatrix}_{\beta+2i+\delta,\beta} \tag{25}$$

Both $\mathrm{d}\lambda_0/\mathrm{d}t$ and $\mathrm{d}L_0/\mathrm{d}t$ contain the same combination of δ, ε, both differ from (10) and (11). From (10), (21), or (25) we see that the harmonic coefficients of even degree cannot be evaluated from nearly circular orbits; but this was known already to Batrakov (1965).

3.5. Comments

It should be emphasized that the 'longitudinal' coefficients (25) are formally very similar to (10), but they differ by the factor $*\bar{F}(I)/\bar{F}(I)$, (22), which depends on i (or l), so that $\bar{\bar{R}}_\beta^{0,1}$-values cannot be used in place of $*R_\beta^{0,1}$.

Now it would be easy to repeat the derivation for ω, Ω, M_0 or for the angle $\tilde{\omega} = \omega + M$. The result for $(\mathrm{d}\tilde{\omega}/\mathrm{d}t)_{\mathrm{res}}$ would be similar to (24), $f_{\tilde{\omega}} = f_\lambda$, and $\hat{R}_\beta^{0,\alpha}$ (instead of $*R_\beta^{0,\alpha}$) would be compatible to (25), when $*\bar{F}(I)$ would be changed to

$$\hat{F}_{lmp}(I) = 2(\beta + 2i + \delta + 1)\bar{F}_{lmp}(I) - \bar{F}'_{lmp}(I)\cot I. \tag{26}$$

For a check, we can use the 'perturbation factors' from Kaula (1966), p. 40 or general Allan's equations (Allan, 1967; p. 1843) to verify our Q-coefficients for all Equations (10), (11), (20), (21), (22), (25) or (26). We have also confronted (7)–(10), (13), and (18)–(21) with the formulae in Klokočník and Kostelecký (1979), which were derived there for geostationary orbits ($\beta/\alpha = 1/1$; $\gamma = 1$, $q = 0$); they coincide well.

The rates of change of L_0 and λ_0 are in deg day^{-1} in (18) and (24), if n is in deg day^{-1}, and can be converted to **m** day^{-1} (Section 4):

$$\mathrm{d}E/\mathrm{d}t_{[\mathrm{m/day}]} = (a_{[m]}/(180/\pi)_{[\mathrm{deg}]})(\mathrm{d}E/\mathrm{d}t)_{[\mathrm{deg/day}]}. \tag{27}$$

4. Application

4.1. Differences among Earth Models

To achieve the aim, which was described in Section 1, it is sufficient to take into account the lumped coefficients with $q = 0$ and $q = \pm 1$ ($\gamma = 1, 2, \ldots$); the lumped values for $q = \pm 2, \pm 4, \ldots$ (or $\pm 3, \pm 5, \ldots$) contain the same harmonic coefficients (followed by different inclination functions) as those for $q = 0$ (or ± 1). We have numerical examples for $q = 0$, $\alpha = \gamma = 1$.

To assess the differences in the variations of the orbital parameters, corresponding to the differences among the EMs, one can directly apply (7), (13), (18), or (24) for each model separately, and then to subtract resulting $(\mathrm{d}E/\mathrm{d}t)_{\beta/\alpha}$. More conveniently, the differences among the lumped coefficients of a set of EMs, say $\Delta \bar{R}_\beta^{q,k}(I)$, can be converted to the differences in the orbital rates, namely

$$\Delta(\mathrm{d}E/\mathrm{d}t)_{\beta/\alpha} = f_E(\Delta \bar{C}_\beta^{0,\alpha} \sin \phi_{\beta/\alpha} - \Delta \bar{S}_\beta^{0,\alpha} \cos \phi_{\beta/\alpha}), \tag{28}$$

where f_E and $\bar{R}_\beta^{0,\alpha}(I)$ are computed by the formulae from Sections 3.1–3.4.

Numerical values of $\bar{F}_{l_{\min},\beta,p_{\min}}(I)\bar{\bar{R}}_\beta^{0,1}(I)$ – and not $\bar{\bar{R}}_\beta^{0,1}(I)$ themselves – have been available from the set of figures (1–20) in Klokočník and Pospíšilová (1981), those for the longitudinal terms from the authors' paper (1984); all values for $30 \leqslant I \leqslant 140°$ (in step 1°) and order $6 \leqslant \beta \leqslant 15$. The actual computing equation was accordingly:

$$\Delta(\mathrm{d}E/\mathrm{d}t)_{\beta/1} = (f_E/\bar{F}_{\min}) \times \\ \times [\Delta(\bar{F}_{\min}\bar{C}_\beta^{0,1}) \sin \phi_{\beta/1} - \Delta(\bar{F}_{\min}\bar{S}_\beta^{0,1}) \cos \phi_{\beta/1}], \tag{28'}$$

where $\bar{F}_{\min} = \bar{F}_{l_{\min},\beta,p_{\min}}(I)$. The values of (28′) were computed for $0 < \phi_{\beta/1} \leqslant 360°$, in step of 5°. We do not work with real satellites, so that we need not know their orbital elements and time to define (3); thus we decided to choose maximum values of $\Delta(\mathrm{d}E/\mathrm{d}t)_{\beta/1}$, observed at a $\phi_{\beta/1}$, say $\bar{\phi}_{\beta/1}$, and these maxima will be labelled $|\max \Delta(\mathrm{d}E/\mathrm{d}t)_{\beta/1}|$. These amplitudes are plotted in figures – Figures 1, 2, 3 for I, a, L_0, respectively.

The results $\Delta(\mathrm{d}E/\mathrm{d}t)$ will evidently depend on our choice of the EMs. We want

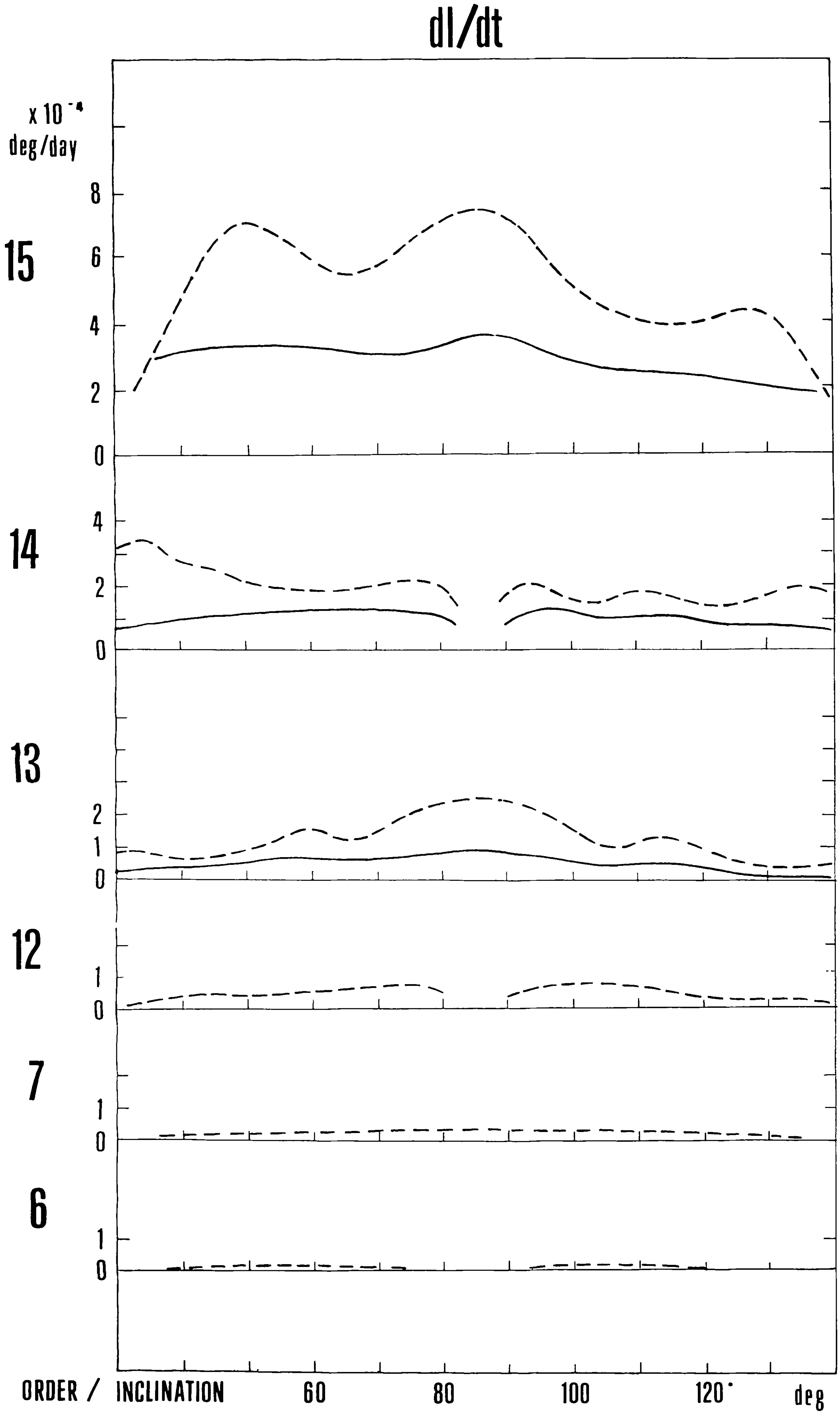

Fig. 1. The amplitudes of $|\max \Delta(\mathrm{d}I/\mathrm{d}t)|$ – dashed curves – and of $3\sigma_{\mathrm{d}I/\mathrm{d}t}$ (full curves) with set I and II of the EMs.

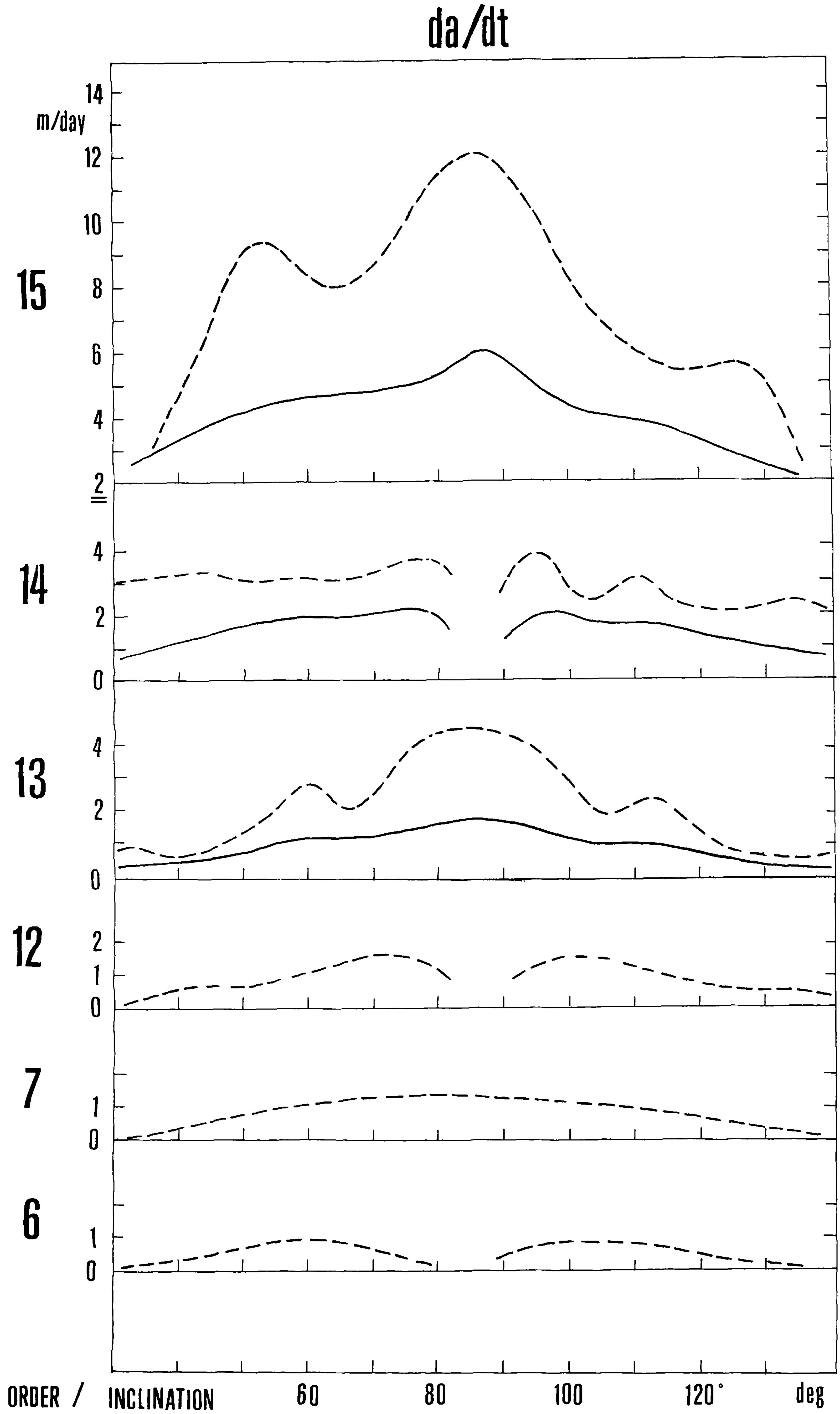

Fig. 2. The amplitudes of $|\max \Delta(\mathrm{d}a/\mathrm{d}t)|$ – dashed curves – and of $3\sigma_{\mathrm{d}a/\mathrm{d}t}$ (full curves) with set I and II of the EMs.

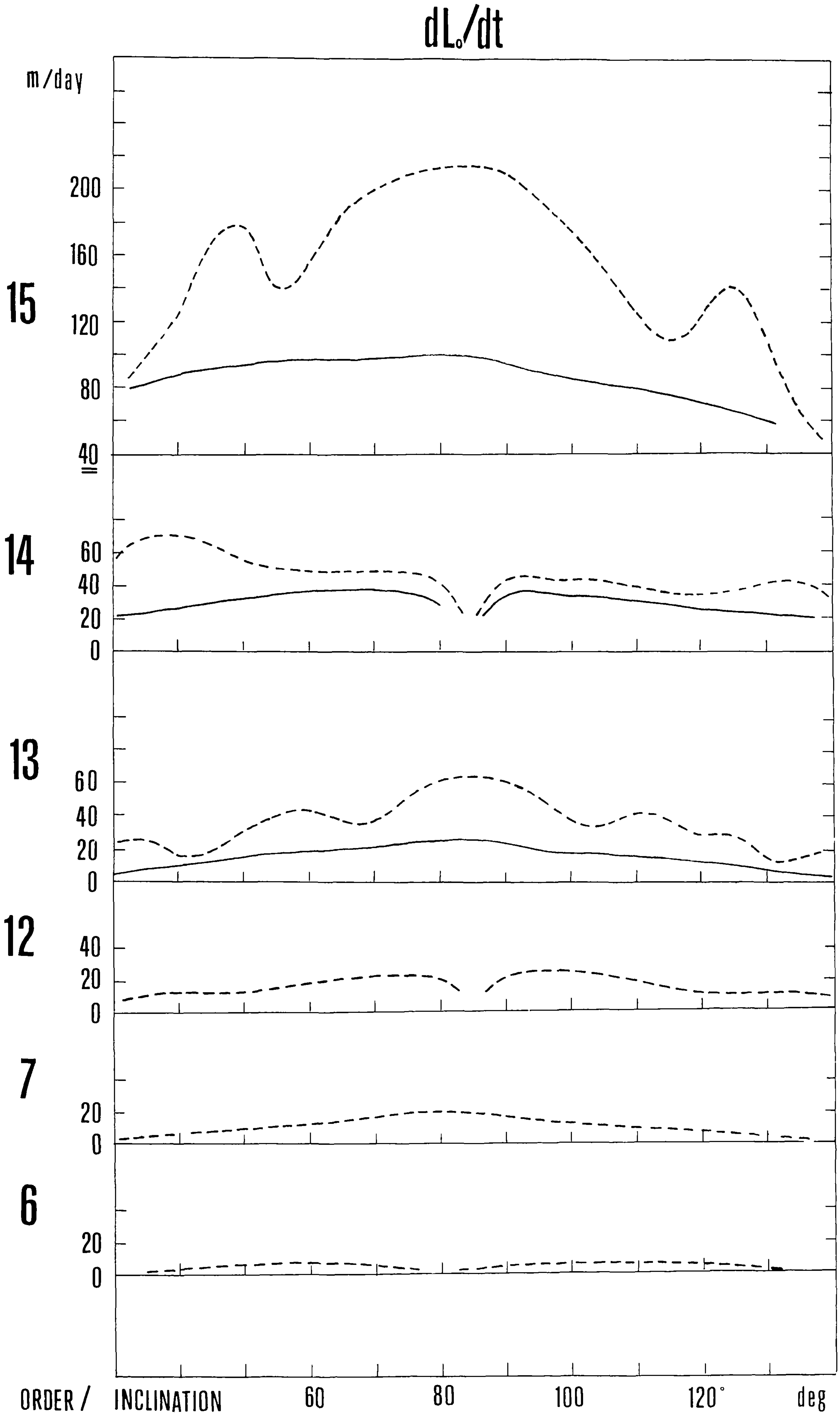

Fig. 3. The amplitudes of $|\max \Delta(dL_0/dt)|$ – dashed curves – and of $3\sigma_{dL_0/dt}$ (full curves) with set I and II of the EMs.

to take into account many of the EMs, which are available, to avoid a biased conclusion based on only a few of them. At the same time, we adopt the selection of the EMs by means of the truncation error tests from Klokočník and Pospíšilová (1981, Section 3.2) to be sure that all compared models have a 'sufficient number' of harmonic coefficients for given order. Some of older EMs are effectively excluded. Nevertheless, EMs will never form a homogeneous set from the point of view of the accuracy. Moreover, many of EMs are mutually dependent, because parts of their data are common.

Our set I consists of: SAO SE 2, 3, 5 (or 6); GSFC GEM 6, 7, 8, 10B; 'Koch 74' (or 'Chovitz and Koch 78') for $\beta \leqslant 8$; Harmograv, 'Rapp 77' and GRIM 2 to keep the compatibility with the tests in Klokočník and Pospíšilová (1981) (see there also for the references of the EMs).

Our set II contains: 'Rapp 81' (Rapp 1981), GRIM 3 (Reigber *et al.*, 1982), PGS–S3 and S4 (Lerch *et al.*, 1982a), GEM–L2 (Lerch *et al.*, 1982b) and again GEM 10B.

The results $\Delta(\mathrm{d}E/\mathrm{d}t)$ will also be influenced by our definition of the differences among the lumped values. We make use of 'mean characteristic differences' in the set of figures ($30 \leqslant I \leqslant 140°$, step 5°); where the typical differences were ambiguous, a smoothing took place.

Figures 1–3 show how the amplitudes decrease with decreasing β, i.e. with increasing semimajor axis, but the effect for $\beta = 13$, e.g., is sometimes greater than that for $\beta = 14$.

The extreme differences among the EMs (Klokočník and Pospíšilová, 1981; Table 3) in set I have relevant 'extreme' max $\Delta(\mathrm{d}E/\mathrm{d}t)$. Let us give several examples: For L_0 and $\beta = 15$, the typical maximum differences are roughly 30% of the extreme differences between SE 5 or 'Rapp 77' and all other tested models ($80 \leqslant I \leqslant 100°$). The departures of GRIM 2 or SE 2 from a mean value, for some parts of $30 \leqslant I \leqslant 140°$, lead up to twice the value of $|\max \Delta(\mathrm{d}L_0/\mathrm{d}t)_{14/1}|$; similarly for Harmograv for $\beta = 13$. The extreme $|\max \Delta(\mathrm{d}L_0/\mathrm{d}t)_{7/1}|$, originating from the difference between 'Koch 74' and the other models, is twice or even three times that of the typical $|\max \Delta(\mathrm{d}L_0/\mathrm{d}t)_{7/1}|$ at $I \sim 60$, $\sim 110°$ or $65 \leqslant I \leqslant 95°$. These facts indicate that orbit predictions with those EMs would show observable differences from the predictions with the other EMs, for some I and β. By applying 'Koch 74', for example, for a satellite in polar (circular) orbit near 7 revs day^{-1}, the amplitude of $\Delta(\mathrm{d}L_0/\mathrm{d}t)_{7/1}$ increases from 17 to 48 m day^{-1}.

The following conclusions from the tests with set I are possible:

(i) The uncertainty in $\mathrm{d}I/\mathrm{d}t$ may reach 8×10^{-4} deg day^{-1} for low and polar orbits ($I \sim 90°$, 15th order resonance); the value of 5×10^{-4} (~ 50 m in geocentric position) is exceeded for majority of $30 \leqslant I \leqslant 140°$. For higher orbits, the geopotential uncertainty effects are very small.

(ii) 'Radial' uncertainty $|\max \Delta(\mathrm{d}a/\mathrm{d}t)|$ is about 10–12 m day^{-1} for the low and polar orbits and is not higher than about 4 m day^{-1} for $\beta \leqslant 14$.

(iii) The values of $|\max \Delta(\mathrm{d}L_0/\mathrm{d}t)|$ are the largest; they approach 200 m day^{-1}

for $\beta = 15$, 20–60 m day^{-1} for $\beta = 13$ or 14 and do not exceed 20 m day^{-1} for the higher orbits.

For some of the EMs, the differences may be still greater, as discussed above.

These results are very pessimistic. The majority of the EMs of set I have, however, only historical value now and no one would use them today for orbit predictions. With set II, we will try to estimate the present state of art in the Earth gravity field description. The difference between sets I and II will reflect, to a certain extent, an improvement during the last decade.

The scatter in $\bar{R}_{\beta}^{0,\alpha}(I)$ among the different EMs cannot, however, measure the actual accuracy of the EMs: it may be used for a relative comparison of the models and can reveal some of their imperfections. An absolute accuracy test is possible via the independent data, e.g., by means of the lumped coefficients, observed from resonant perturbations.

4.2. Test of the Accuracy

The 15th-order lumped coefficients odd degree have been evaluated from resonant variations of 23 satellites (King–Hele and Walker, 1981). The important fact is that these results are independent on the lumped coefficients, which can be reconstituted from the harmonic coefficients of the 15th-order of GEM 10B/C (or from the older GSFC' models). Other EMs ('Rapp 81', GRIM 3, etc.) utilize some of resonant results.

The test of the accuracy of some of the EMs has been performed for the 13th–15th order with the aid of the lumped coefficients (Klokočník, 1982), the most strict being for the 15th-order. The limit of $\sigma_{l,15} = \pm 5 \times 10^{-9}$ for the 15th-order harmonic coefficients (fully normalized, odd degree) in the GEM 10B has been deduced. This limit gives $\sigma_{\mathrm{FR}_{15}}(I) \approx (2\text{–}3) \times 10^{-9}$ in the lumped values of $\bar{F}_{15,15,7}(I)\ \bar{\bar{R}}_{15}^{0,1}(I)$ for $30 \leqslant I \leqslant 140°$ (see Figures 3a, b and Table 2 in Klokočník, 1982). The values of $\sigma_{\mathrm{FR}_{\beta}}(I)$ can be transformed to $\sigma_{\mathrm{d}E/\mathrm{d}t(\beta/\alpha)}$, using slightly modified Equation (28′).

For our purpose, we decided to use the limit $\pm 3\sigma_{\mathrm{FR}_{\beta}}(I)$ (risk $< 1\%$) of GEM 10B ($30 \leqslant I \leqslant 140°$, step 5°). We have plotted the values of $\bar{F}_{\min}\bar{\bar{R}}_{\beta}^{0,1}$ of all EMs of set II ($13 \leqslant \beta \leqslant 15$). It is observed that the belt of $\pm 3\sigma_{\mathrm{FR}_{\beta}}(I)$ of GEM 10B covers completely the differences in the lumped values of set II (the differences are often inside $\pm 1\sigma_{\mathrm{FR}_{\beta}}$ of GEM 10B, but this interval would not be sufficient if various resonant solutions for $13 \leqslant \beta \leqslant 15$ are taken into account, see Klokočník, 1982). Thus, we have the values of $\pm 3\sigma_{\mathrm{FR}_{\beta}}$ as an upper limit for the inaccuracy of all the EMs in the set II (valid at least for $\beta = 15$, perhaps also for $\beta = 14$ and 13). The values of $\sigma_{\mathrm{d}E/\mathrm{d}t} = f(3\sigma_{\mathrm{FR}_{\beta}})$ are plotted in Figures 1–3 (full curves) for the comparison with $|\max \Delta(\mathrm{d}E/\mathrm{d}t)_{\beta/1}|$ (dashed).

From the tests, described in this section, the following conclusions are suggested:
(i) The inaccuracy in $\mathrm{d}I/\mathrm{d}t$, i.e. $\pm 3\sigma_{\mathrm{d}I/\mathrm{d}t}$, is not higher than 3.5×10^{-4} deg day^{-1}

for close and polar orbits (15/1 ... 470–650 km above the Earth's surface) and is $\sim 5 \times 10^{-5}$–1×10^{-4} deg day^{-1} for 14/1 (800–950 km) or 13/1 (1200–1300 km).

(ii) The inaccuracy of $\mathrm{d}a/\mathrm{d}t$, $\pm 3\sigma_{\mathrm{d}a/\mathrm{d}t}$, varies from 2 to 6 m day^{-1} for 15/1, and is below 2 m day^{-1} for the higher orbits.

(iii) The values of $\pm 3\sigma_{\mathrm{d}\hat{L}_0/\mathrm{d}t}$ may reach 100 m day^{-1} for 15/1, and are expected to be lower than 40 m day^{-1} for $\beta = 14$ and 25 m day^{-1} for $\beta = 13$.

4.3. Comparisons

Klosko and Wagner (1982) have suggested an alternative method. They use the linear theory of Kaula (1966) to estimate the errors in any of the orbital elements by normalizing the lumped coefficients' differences to an equivalent error in the dominant ordinary harmonic coefficient. They work with real satellite orbits (the terms with $q \neq 0$ are included). They use only several of the EMs, mainly the GEMs. Their results are valid for specific inclinations and time intervals (argument of perigee). Our results are prepared for the individual orders of the harmonic coefficients separately. We do not use any real orbits and this results in no limitation for I and $\phi_{\beta/\alpha}$. Our numerical values are not available for $q \neq 0$ yet. The disadvantage is that only the maxima of the differences $|\max \Delta(\mathrm{d}E/\mathrm{d}t)|$ or of the inaccuracies $\sigma_{\mathrm{d}E/\mathrm{d}t}$ are estimated; therefore, our results must be considered as pessimistic 'upper limits'.

Lerch *et al.* (1982a) have analyzed the orbit of SEASAT–A ($I \doteq 108°$); the radial ephemeris error, using GEM 9, 10B, and PGS–S4 is estimated to be 3–5, 1, and 0.7 m. We have estimated the maximum possible contribution of the uncertainty in the 14th-order odd degree harmonic coefficients to the orbital uncertainty, namely $|\max \Delta(\mathrm{d}a/\mathrm{d}t)_{14/1}| \doteq 2$ m day^{-1} or $\pm 1\sigma_{\mathrm{d}a/\mathrm{d}t} \doteq 0.6$ m day^{-1} (Figure 2). Only a comparison of the order of the effects is possible due to various differences between our methods; taking this into account, the agreement is good.

Our results are not in disagreement also with Wakker *et al.* (1982), where the effects of different EMs on several satellite orbits have been studied.

5. Conclusions

Differences among the Earth models (EM), which were expressed as dispersions of the relevant lumped coefficients (in our previous paper), were here transformed to the differences in the orbital rates $\mathrm{d}I/\mathrm{d}t$, $\mathrm{d}a/\mathrm{d}t$, and $\mathrm{d}L_0/\mathrm{d}t$. The formulae are given in Section 3. Examples have been evaluated for nearly circular orbits ($q = 0$), for individual orders of harmonic coefficients ($6 \leqslant \beta \leqslant 15$) and arbitrary inclination in $30 \leqslant I \leqslant 140°$. In Figures 1–3, the maximum differences (or inaccuracies) in the variations of I, a, and L_0 are plotted for two sets of the EMs.

The first set consists of the EMs from 1970–1980 of greatly varying quality (SAO SE 2, 3, 5, 6; GSFC GEM 6, 7, 8, 10B; 'Koch 74', 'Chovitz and Koch 78'; Harmograv, 'Rapp 77' and GRIM 2). The maxima of the differences in the variations reach 8×10^{-4} deg day^{-1}, 10–12 m day^{-1} or 200 m day^{-1} in I, a, or L_0 for close and polar orbits ($\sim$ 15 revs day^{-1}); the values are not higher than 10^{-4} deg day^{-1}, 1–2 m day^{-1} or 20 m day^{-1} in these rates for higher orbits ($\sim$ 6–7 revs day^{-1}); see the dashed curves in Figures 1–3. Majority of the EMs of this set have only historical value now and no one would use them for orbit predictions.

The second set consists of more recent EMs: 'Rapp 81', GRIM 3, PGS–S3, and S4, GEM–L2 and again GEM 10B. The testing of their accuracy is based on the calibration by means of more or less independent resonant lumped coefficients. The limit of $\sigma_{l,15} = \pm 5 \times 10^{-9}$ for $\bar{C}_{l,15}$, $\bar{S}_{l,15}$ (odd degree) in GEM 10B has been deduced in previous works. This limit corresponds to $\sigma_{\mathrm{FR}_\beta}(I)$ in the lumped values $\bar{F}_{\min}\bar{\bar{R}}_\beta^{0,1}(I)$ and further, these values are transformed to the inaccuracy in $\mathrm{d}I/\mathrm{d}t$, $\mathrm{d}a/\mathrm{d}t$, and $\mathrm{d}L_\Omega/\mathrm{d}t$. The belts of $\pm 3\sigma_{\mathrm{FR}_\beta}(I)$ of GEM 10B cover completely the differences in $\bar{F}_{\min}\bar{R}_\beta^{0,1}(I)$ for all EMs in set II ($13 \leqslant \beta \leqslant 15$, $30 \leqslant I \leqslant 140°$). The values of $\pm 3\sigma_{\mathrm{FR}_\beta}(I)$ (GEM 10B) have been used for the accuracy estimates. The inaccuracy $\pm 3\sigma_{\mathrm{d}I/\mathrm{d}t}$ is not higher than 3.5×10^{-4} deg day^{-1} at 15 revs day^{-1} and is $\sim 5 \times 10^{-5}$–1×10^{-4} deg day^{-1} for 13–14 revs day^{-1}. The inaccuracy $\pm 3\sigma_{\mathrm{d}a/\mathrm{d}t}$ varies from 2 to 6 m day^{-1} for 15/1, and is below 2 m day^{-1} for the higher orbits. Finally, the longitudinal terms have the accuracy $\pm$ 100 m day^{-1} or better for 15/1, and better than 40 m day^{-1} for 14/1 and 25 m day^{-1} for 13/1.

Our results are not in disagreement with the analyses of satellite orbits in Klosko and Wagner (1982), Lerch *et al.* (1982a), and Wakker *et al.* (1982).

Acknowledgements

I thank Drs S. M. Klosko, G. Balmino, and L. Sehnal for their critical and inspiring remarks.

References

Allan, R. R.: 1967, *Planet. Space Sci.* **15**, 1829.

Allan, R. R.: 1971, Roy. Aircr. Establ. Techn. Rep. 71245.

Allan, R. R.: 1973, *Planet. Space Sci.* **21**, 205; see also: in B. D. Tapley and V. G. Szebehely (eds.), *Recent Advances in Dynamical Astronomy*, D. Reidel Publ. Co., Dordrecht, Holland, p. 333.

Batrakov, Yu. V.: 1965, in *Proc. Internat Astronaut. Congr.* (*Paris 1963*), IV. 7, Gauthier-Villars, Paris, and PWN-Warsawa, p. 99.

Dunn, P.: 1981, *Bull. Geod.* **55**, 143.

Gooding, R. H.: 1971, Roy. Aircr. Establ. Techn. Rep. 71068; see also: 1971, *Nature Phys. Sci.* **231**, 168.

Gooding, R. H.: 1975, 'Studies of Resonance in the Orbits of Ariel Satellites', (unpublished RAE Techn. Rep.), private communication.

Kaula, W. M.: 1966, *Theory of Satellite Geodesy*, Blaisdell Publ. Co, Waltham.

King-Hele, D. G. and Walker, D. M. C.: 1981, Roy. Aircr. Establ. Techn. Rep. 81006; see also: 1982, *Proc. Roy. Soc. London* **A379**, 247.

Klokočník, J.: 1976, *Bull. Astron. Inst. Czech.* **27**, 287.

Klokočník, J.: 1982, *Bull. Astron. Inst. Czech.* **33**, 89.

Klokočník, J. and Kostelecký, J.: 1979, 'Earth Gravitational Field Effect on a Nearly-Geostationary Satellite Orbit', *Edice VÚGTK* **5**, 2, Praha.

Klokočnik, J. and Pospíšilová, L.: 1981, *Planet. Space Sci.* **29**, 653.
Klokočnik, J. and Pospíšilová, L.: 1984, 'Intercomparisons of Earth Models by Means of Lumped Coefficients: Terms for Longitude and Eccentricity, to be published. (*Planet. Space Sci.*)
Klosko, S. M. and Wagner, C. A.: 1982, *Planet. Space Sci.* **30**, 5; also pres. XVII GA of IUGG, Canberra, (1979).
Lerch, F. J., Marsh, J. G., Klosko, S. M., and Williamson, R. G.: 1982a, *J. Geophys. Res.* **87**, 3281.
Lerch, F. J., Klosko, S. M., and Patel, G. B.: 1982b, *Geophys. Res. Letters*, to be published.
Marsh, J. G. and Williamson, R. G.: 1980, *J. Astronaut. Sci.* **28**, 345.
Rapp, R. H.: 1981, Ohio State Univ. Reports Department of Geod. Sci. No. 322.
Reigber, Ch. and Rummel, R.: 1979, in G. Veis and E. Livieratos (eds.), *Proc. Internat. Symp. The Use of Artif. Satell. for Geodesy and Geodynamics*, II, Athens, p. 217.
Reigber, Ch., Balmino, G., Moynot, B., and Müller, H.: 1982, 'The GRIM 3 Earth Model Parameters', DGFI and GRGS Internal Rep.
Wagner, C. A.: 1975, *J. Geophys. Res.* **80**, 4089.
Wagner, C. A. and Klosko, S. M.: 1975, Goddard Space Flight Center X-921-75-187, see also: 1977, *Celest. Mech.* **16**, 143.
Wakker, K. F. and Ambrosius, B. A. C.: 1982, Delft Univ. Techn. Rep. LR-342.
Wakker, K. F., Ambrosius, B. A. C., and der Ploeg, T.: 1982, Delft Univ. Techn. Rep. LR-352.