AN ANALYTIC SOLUTION FOR THE J_2 PERTURBED EQUATORIAL ORBIT

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Abstract. An analytic solution for the J_2 perturbed equatorial orbit is obtained in terms of elliptic functions and integrals. The necessary equations for computing the position and velocity vectors, and the time are given in terms of known functions. The perturbed periapsis and apoapsis distances are determined from the roots of a characteristic cubic.

1. Introduction

The motion of a satellite constrained to lie in the equatorial plane and subject to forces due to an inverse-square gravitational attraction and a perturbation due to the Earth's oblateness as the J_2 term can be analytically determined in terms of elliptic functions and elliptic integrals. Whittaker (1944) discusses the integrable cases of central forces in which the magnitude depends only on the distance r. He shows which power of the distance is soluble by circular or elliptic functions. The present problem with a perturbation power of -4 is shown to be integrable in terms of elliptic functions. Sterne (1957) investigates the solutions of non-circular equatorial orbits using a canonical approach. Ramnath (1973) was also aware that this solution was integrable but choose to investigate the motion in terms of asymptotic solutions. A recent paper by Cohen and Lyddane (1981) implies that the J_2 perturbed equatorial problem does not possess an analytic solution. They examine this problem using Lie series showing that the solution diverges under certain conditions.

General solutions for the motion of an artificial Earth satellite subject to perturbations can be applied to this problem. Brouwer's (1959) theory using Delaunay variables can be considered one of the fundamental solutions using canonical variables. The limitations on small eccentricities and inclinations were eliminated by Lyddane (1963). Vinti's (1961) satellite theory in terms of oblate spherical coordinates describes a gravitational potential which accounts exactly for zonal harmonics through second order and a portion of the fourth zonal. Limitations on small inclinations were removed by Vinti (1962) and satellite prediction formulae were expressed in elliptic functions by O'Mathuna (1970). These and other solutions are available for predicting the motion of a satellite constrained to the equatorial plane and subject to the J_2 perturbation only.

In this paper we will develop the analytic solution for the J_2 perturbed equatorial orbit problem. There is a twofold utility for such a solution: (1) it leads to a clearer

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understanding of the motion by examining possibly the simplest model; and (2) it can be used as an intermediary orbit for the motion subject to perturbations due to zonal harmonics and small inclinations. One possible method for obtaining this latter result is the procedure developed by Jezewski (1983). The solution for the position and velocity vectors will be shown to be determined by an incomplete elliptic integral of the first kind. Explicit equations and tests will be given such that the solution is easily obtained. In addition, the time will be shown to be determined by an incomplete elliptic integral of the third kind and is also readily calculated. An additional bonus of this solution is that the perturbed periapsis and apoapsis distance can be obtained from the roots of a characteristic cubic; a precalculation of elliptic functions and integrals.

2. Problem Formulation

The differential equations of motion for a particle constrained to lie in the equatorial plane and subject to forces of an inverse-square gravitational attraction and a perturbation due to the Earth's oblateness as the J_2 term can be expressed in polar coordinates as

$$\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} + \frac{J_0}{r^4} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 0, \tag{1}$$

where r, θ are the radius and polar angle, respectively, μ is the gravitational constant, and the dots refer to differentiation with respect to the independent variable, time. The constant J_0 is

$$J_0 = \tfrac{3}{2}\mu J_2 r_e^2,$$

where r_e is the equatorial radius of the Earth. From Equation (1) we note an integral of the motion (the angular momentum), which we shall designate as

$$r^2\dot{\theta} = h_0, \tag{2}$$

where h_0 is a constant. It is convenient to transform both the dependent and the independent variables of the problem; the independent variable is transformed from time, t, to the angular variable, θ, using Equation (2) and the dependent variable by

$$r = \frac{1}{u}.$$

First and second derivatives of the radius are expressed as

$$\dot{r} = -h_0 u', \qquad \ddot{r} = -h_0^2 u^2 u'',$$

where the prime refers to differentiation with respect to θ. Using these transformations in Equation (1), the motion can be described by the second-order, nonlinear differential equation

$$u'' + u - \frac{\mu}{h_0^2} - \frac{J_0 u^2}{h_0^2} = 0. \tag{3}$$

A first integral of this differential equation can be obtained by multiplying through by the integrating factor u'. We obtain

$$u'u'' - \frac{J_0 u^2 u'}{h_0^2} + uu' - \frac{\mu u'}{h_0^2} = 0.$$

Integrating this equation, we have

$$\gamma^2 (u')^2 = u^3 + a_2 u^2 + a_1 u + a_0, \tag{4}$$

where

$$\gamma^2 = \frac{3h_0^2}{2J_0}, \qquad a_2 = -\gamma^2, \qquad a_1 = \frac{3\mu}{J_0}, \qquad a_0 = \gamma^2 s_0$$

and s_0 is an integral of the motion and can be computed from the initial conditions. Taking the square root and separating the variables, we express the solution by the quadrature

$$\int d\theta = \gamma \int \frac{du}{(u^3 + a_2 u^2 + a_1 u + a_0)^{1/2}}. \tag{5}$$

The right-hand side of this equation is an incomplete elliptic integral of the first kind. However, before proceeding with the formal mathematical solution of Equation (5), let us analyze it from an orbital mechanics viewpoint.

3. Analysis

Consider orbits which are 'elliptic'. These trajectories will exhibit points of closest and furthest approach to the attracting mass, which we shall designate the 'periapsis' and 'apoapsis' radii, r_p and r_a, respectively. At these points, $\dot{r} = 0$, and from the transformation equations, we have

$$u' = -\frac{\dot{r}}{h_0} = 0 \qquad \text{(apsis condition)}.$$

Therefore, for these orbits, the cubic expressed in the right-hand side of Equation (4) will exhibit two real zeros per orbit. Since imaginary roots must appear in conjugate pairs, we conclude that the roots of the right-hand side of Equation (4) are real, and at least two are positive. Designating the roots and their order as

$$a > b > c$$

we can express Equation (5) as

$$\int \mathrm{d}\theta = \gamma \int \frac{\mathrm{d}u}{\{(u-a)(u-b)(u-c)\}^{1/2}}. \tag{6}$$

Having determined that the roots are real and that the variable u is bounded between two successive values, the two possible ranges for u are

$$a > u > b > c, \qquad a > b > u > c.$$

In the first case, if u were greater than b and c but less than a, then the argument under the radical in Equation (6) would have to be negative and two roots would be imaginary. But this contradicts the previous conclusion that all the roots are real; therefore, u cannot lie between a and b.

For u greater than c but less than a or b, the argument under the radical in Equation (6) remains positive. The solution is given by (233.00).* Additionally, the roots b and c have a special significance. They are the inverses of the periapsis and the apoapsis radii, respectively, of the perturbed orbit, or

$$b = \frac{1}{r_p}, \qquad c = \frac{1}{r_a}.$$

Hence we can find two additional constants of the perturbed trajectory by simply evaluating the roots of a cubic.

4. Position and Velocity

The solution given by (233.00) can be expressed as

$$\theta = \theta_c + \gamma g \tau, \tag{7}$$

where θ_c is an integration constant determined by the initial conditions

$$\theta_c = \theta_0 - \gamma g \tau_0. \tag{8}$$

θ_0 is the initial value θ,

$$g = \frac{2}{\sqrt{a-c}}$$

and τ is the incomplete elliptic integral of the first kind designated as

$$\tau = F(\phi, k).$$

The constant argument k is the modulus of the Jacobian elliptic functions and integrals and, for this solution, has the value

$$k = \sqrt{\frac{b-c}{a-c}}.$$

* Numbers in parenthesis will refer to mathematical equations in Byrd and Friedman (1971).

The variable argument ϕ is known as the amplitude of τ and, for this solution, is defined by

$$\sin^2 \phi = \frac{u - c}{b - c}, \qquad 0 \leqslant \phi \leqslant \pi/2.$$

Elliptic functions and integrals are periodic functions with a period of $4K$, where in standard notation K is the complete elliptic integral of the first kind defined by

$$K(k) = F(\pi/2, k).$$

Since these functions are periodic, we need to establish the correct quadrant where the solution is initially located. This is accomplished by examining the Jacobian sn and cn functions. From (233.00) the sn function is defined by

$$\operatorname{sn} \tau = \sqrt{\frac{u - c}{b - c}} \tag{9}$$

and the cn function from the derivative of the sn function, (731.01), by

$$\operatorname{cn} \tau = \frac{gu'}{2(b - c) \operatorname{sn} \tau \operatorname{dn} \tau}, \tag{10}$$

where the Jacobian dn function is defined by

$$\operatorname{dn} \tau = \sqrt{1 - k^2 \operatorname{sn}^2 \tau}.$$

We are now ready to determine the value of τ_0 in Equation (8) by first evaluating Equations (9) and (10) using $u(\theta_0)$ and $u'(\theta_0)$, and, second by using the following test to determine the correct quadrant and subsequent value of τ_0:

$$\begin{array}{llll}
\operatorname{sn} \tau_0 > 0 & \text{and} & \operatorname{cn} \tau_0 > 0, & \tau_0 = F(\phi_0, k), \\
\operatorname{sn} \tau_0 > 0 & \text{and} & \operatorname{cn} \tau_0 < 0, & \tau_0 = 2K - F(\phi_0, k), \\
\operatorname{sn} \tau_0 < 0 & \text{and} & \operatorname{cn} \tau_0 < 0, & \tau_0 = 2K + F(\phi_0, k), \\
\operatorname{sn} \tau_0 < 0 & \text{and} & \operatorname{cn} \tau_0 > 0, & \tau_0 = 4K - F(\phi_0, k),
\end{array}$$

where $\phi_0 = \phi(\theta_0)$. This concludes the computations necessary to compute the integration constant θ_c in Equation (8).

To compute the values of $u(\theta)$ and $u'(\theta)$ for any value of θ, we first solve Equation (7) for τ.

$$\tau = \frac{\theta - \theta_c}{\gamma g}.$$

Designating M as the integer number of $4K$ periods in τ or

$$M = \frac{\tau}{4K}$$

we define the principal value of τ by ρ as

$$\rho = \tau - 4MK.$$

We can now compute sn ρ by (908.01) or some other similar computation, and cn ρ and dn ρ as

$$\text{cn}\,\rho = \pm\sqrt{1 - \text{sn}^2\,\rho}, \qquad \text{dn}\,\rho = \sqrt{1 - k^2\,\text{sn}^2\,\rho},$$

where, if $K \leqslant \rho < 3K$, cn ρ must be negative. We then compute $u(\theta)$ and $u'(\theta)$ from Equations (9) and (10) as

$$u(\theta) = c + (b - c)\,\text{sn}^2\,\rho,$$

$$u'(\theta) = \frac{2(b - c)\,\text{sn}\,\rho\,\text{cn}\,\rho\,\text{dn}\,\rho}{g} \tag{11}$$

Finally, the position and velocity vectors **R** and **V**, respectively, can be expressed as in Battin (1964)

$$\mathbf{R} = \frac{\xi}{u(\theta)}, \qquad \mathbf{V} = h_0\{u(\theta)\xi' - u'(\theta)\xi\},$$

where the unit vectors ξ and ξ' are given by

$$\xi = \mathbf{A}\sin\theta + \mathbf{B}\cos\theta, \qquad \xi' = \mathbf{A}\cos\theta - \mathbf{B}\sin\theta.$$

A and **B** are constant unit orthogonal vectors defined by

$$\mathbf{A} = \xi_0\sin\theta_0 + \xi_0'\cos\theta_0, \qquad \mathbf{B} = \xi_0\cos\theta_0 - \xi_0'\sin\theta_0$$

and

$$\xi_0 = \left.\frac{\mathbf{R}}{r}\right|_{\theta_0}, \qquad \xi_0' = \left.\frac{(r\dot{\mathbf{R}} - \dot{r}\mathbf{R})}{h_0}\right|_{\theta_0}.$$

5. Time

The only equation where time, t, occurs in the formulation is in the magnitude of the angular momentum vector expressed in Equation (2).

$$h_0 = r^2\dot{\theta},$$

Recalling that $r = 1/u$, we can express this relationship by

$$h_0\,\mathrm{d}t = \frac{\mathrm{d}\theta}{u^2}.$$

From Equations (7) and (9), the differential change in θ and the function u can be expressed in terms of the variable τ by

$$\mathrm{d}\theta = \gamma g\,\mathrm{d}\tau, \qquad u = c + (b - c)\,\text{sn}^2\,\tau,$$

Using this relationship, the time equation can be expressed as an incomplete elliptic integral of the third kind

$$h_0 \int \mathrm{d}t = \frac{\gamma g}{c^2} \int \frac{\mathrm{d}\tau}{(1 - \alpha^2 \operatorname{sn}^2 \tau)^2}, \tag{12}$$

where α^2 is known as the parameter of the elliptic integral of the third kind and has a value for this problem of

$$\alpha^2 = \frac{c - b}{c}.$$

(Note that α^2 is negative here.) There are two possible solutions to Equation (12) based on whether $-\alpha^2 > k$ or $-\alpha^2 < k$.

5.1. $k < -\alpha^2 < \infty$

This range of α^2 corresponds to the root $c > 0$. The solution to Equation (12) is given by (432.06) and may be expressed as

$$c_1 t = t_c + \frac{1}{c_0}\left(\alpha^2 E(\phi, k) + c_3 \tau - \frac{\alpha^4 \operatorname{sn} \rho \operatorname{cn} \rho \operatorname{dn} \rho}{1 - \alpha^2 \operatorname{sn}^2 \rho} - c_2(\tau \Lambda_0 - \Omega_5)\right) \tag{13}$$

where t_c is an integration constant defined by the initial conditions, and the constants c_0, c_1, c_2, c_3, and Λ_0 are

$$c_0 = 2(1 - \alpha^2)(\alpha^2 - k^2),$$

$$c_1 = \frac{h_0 c^2}{\gamma g},$$

$$c_2 = \alpha^2 \frac{(k^2(3 - 2\alpha^2) - \alpha^2(2 - \alpha^2))\pi}{K\sqrt{2\alpha^2 c_0}},$$

$$c_3 = \frac{\alpha^2(1 + k^2) - 2k^2}{1 - \alpha^2},$$

$$\Lambda_0 = \frac{2}{\pi}((E(k) - K)F(\beta, k') + KE(\beta, k')).$$

The function $E(\phi, k)$ is the incomplete elliptic integral of the second kind and requires, in addition the following test for its evaluation.

If

$$\begin{aligned}
0 < \rho \leqslant K, &\quad E(\phi, k) = 4ME(k) + E(\phi, k),\\
K < \rho \leqslant 2K, &\quad E(\phi, k) = 2(2M + 1)E(k) - E(\phi, k),\\
2K < \rho \leqslant 3K, &\quad E(\phi, k) = 2(2M + 1)E(k) + E(\phi, k),\\
3K < \rho \leqslant 4K, &\quad E(\phi, k) = 2(2M + 2)E(k) - E(\phi, k).
\end{aligned}$$

The arguments β and k' in the incomplete elliptic integrals, $E(\beta, k')$ and $F(\beta, k')$ are given as

$$\beta = \sin^{-1}\frac{1}{\sqrt{1-\alpha^2}}, \qquad k' = \sqrt{1-k^2}.$$

The function Ω_5 can be expressed as

$$\Omega_5 = \tau + \frac{2K}{\pi}\tan^{-1}\left[\frac{\sum_1^\infty(-1)^{m+1}q^{m^2}\sin(2mv)\sinh(2m(p-w))}{\frac{1}{2}+\sum_1^\infty(-1)^m q^{m^2}\cos(2mv)\cosh(2m(p-w))}\right],$$

where

$$p = \frac{\pi K'}{2K}, \qquad K' = K(k')$$

$$q = e^{-2p},$$

$$v = \frac{\pi\tau}{2K},$$

$$w = \frac{\pi F(\beta, k')}{2K}.$$

In Equation (13), the dependent variable time, t, is expressed as a function of the independent variable θ. There is no easy way to invert this analogue of Kepler's equation. However, since ϕ', ρ', and τ' are known or can be readily computed, for a given t the variable θ can be determined using a Newton–Raphson method.

5.2. $0 < -\alpha^2 < k$

This range of α^2 corresponds to the root $c < 0$. The solution is given by (431.06), which is similar in form to (432.06). The details of this solution can be obtained from Byrd and Friedman (1971).

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