BAIRE CLASS OF THE LYAPUNOV INDICES

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1. Following Millionshchikov [1, 2, 3], we consider the family of the morphisms

$$(X(m), \chi(m)): (E, p, B) \to (E, p, B), \quad m \in \mathbf{N}, \tag{1}$$

of a certain vector bundle (E, p, B) with a fiber $\mathbf{R}^n$, whose base B is a complete metric space. We assume that the mapping X(m) is nonsingular on fibers, i.e., for all $m \in \mathbf{N}$, and $b \in B$ the linear mapping

$$X(m, b): p^{-1}(b) \to p^{-1}(\chi(m) b),$$

defined as the restriction of the mapping X(m) to the fiber $p^{-1}(b)$, has an inverse $[X(m, b)]^{-1}$: $p^{-1}(\chi(m) b) \to p^{-1}(b)$.

A certain Riemannian metric is fixed on the vector bundle (E, p, B) and we suppose that there exists a function $a(\cdot): B \to \mathbf{R}^+$, such that $\max(\| X(m, b)\|, \| [X(m, b)]^{-1}\|) \leqslant \exp(ma(b))$ for all $m \in \mathbf{N}$ and $b \in B$.

The Lyapunov indices of the family (1) were defined in [1]:

$$\lambda_k(b) = \min_{\mathbf{R}^{n-k+1} \in G(\mathbf{R}^n)} \max_{\xi \in \mathbf{R}^{n-k+1}} (1/m) \ln | X(m, b) \xi |, \quad k = 1, \ldots, n.$$

Then their properties were studied as functions on B. In particular, Theorem 1, which shows that the functions $\lambda_k(\cdot), k = 1, \ldots, n$ belong to the second Baire class (see [4, 5]), was proved. Since, by definition, each Baire class is contained in the next one, there naturally arises the problem of unimprovability of this theorem, i.e., about the strict belongingness of the Lyapunov indices to the second Baire class (see [5]). For abstractly defined morphism (1), this problem cannot have a unique solution. Thus, e.g., if the base consists of a single point, then each function on it is of zero class. We can give examples of the morphisms (1) with discontinuous Lyapunov indices. However, the present problem will be solved in this case if we indicate objects for which the functions $\lambda_k(\cdot)$, $k = 1, \ldots, n$ belong to the second, but do not belong to any preceeding class.

We show how the required result follows from [6, 7] (in each case, for the function $\lambda_1(\cdot)$); then we give a direct proof of the theorem on the realization of the second Baire class by the functions $\lambda_1(\cdot), \ldots, \lambda_n(\cdot)$, in conformity with linear systems of ordinary differential equations.

2. Let there be given a uniformly continuous mapping $A(\cdot): \mathbf{R} \to \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ that satisfies the condition $\sup_{x \in \mathbf{R}} \| A(x) \| < +\infty$.

By a well-known method (see [8, p. 534 in the Russian original]) we define the metric space $\mathbf{R}_A$ of all possible translates A(t + x) of the mapping A(t) by introducing the metric

$$\rho(A_1, A_2) = \sup_{x \in \mathbf{R}} \min [| A_1(x) - A_2(x) |, 1/|x|].$$

Let us complete $\mathbf{R}_A$. The obtained complete metric space is denoted by $\overline{\mathbf{R}}_A$.

The equation $(f^t A)(x) = A(t + x)$ defines a dynamical system f^t in $\overline{\mathbf{R}}_A$.

Now let A(•) be an almost periodic mapping. For each $\tilde{A} \in \overline{\mathbf{R}}_A$ we consider the linear system of ordinary differential equations $\xi' = \tilde{A}(t) \xi$, $\xi \in \mathbf{R}^n$.

Setting $B = \overline{\mathbf{R}}_A$, $E = B \times \mathbf{R}^n$, and $p = pr_1$ (pr_1 is the projection of the product $B \times \mathbf{R}^n$ onto the first factor), we define the trivial vector bundle (E, p, B). Not depending on $b \in B$, the Euclidean structure of the fiber $\mathbf{R}^n$ defines a Riemannian metric on (E, p, B). The equations

Mathematics and Mechanics Institute, Academy of Sciences of the Kazakh SSR. Translated from Matematicheskie Zametki, Vol. 31, No. 6, pp. 925-931, June, 1982. Original article submitted June 25, 1981.

0001-4346/82/3156-0467$07.50

$$X(m)\,e = (f^m \tilde{A},\ \Xi(m, 0; \tilde{A})\,\xi),$$
$$\chi(m)\,\tilde{A} = f^m \tilde{A},$$

where $e = (\tilde{A}, \xi) \in E$, $\tilde{A} \in \overline{\mathbf{R}}_A$, $\xi \in \mathbf{R}^n$, and $\Xi(\theta, \tau; \tilde{A})$ is the Cauchy operator of the system $\xi' = A(t)\,\xi$, define the mappings X(m): $E \to E$ and $\chi(m): B \to B$, $m \in \mathbf{N}$. It is clear that the mappings $(X(m), \chi(m))$, $m \in \mathbf{N}$ satisfy all the necessary conditions.

Let A be a point of discontinuity of the function $\lambda_1(\cdot)$. The existence of such a point has been proved in [7]. By virtue of the remark to the theorem of [7], all the points $\tilde{A}$ of $\tilde{\mathbf{R}}_A$ are also points of discontinuity of this function. It follows from [5, p. 243, Theorem VI in the Russian translation] that the restriction of each function of the first class, defined on a complete metric space, to an arbitrary closed set has at least one point of continuity in this set. Hence, by virtue of [1, Theorem 1], the function $\lambda_1(\cdot): \overline{\mathbf{R}}_A \to \mathbf{R}$ is strictly of second Baire class. Analogous arguments are possible also with respect to the functions $\lambda_2(\cdot), \ldots, \lambda_n(\cdot)$. However, let us observe that in the above proof we have essentially relied on the remark to the theorem of [7], which, in its turn, followed from this theorem and the remark to Theorem 2 of [6]. But in [7] only the index $\lambda_1(\bullet)$ has been considered. Consequently, more investigation is needed to obtain the conclusions about other indices. We use a proof of this interesting fact that is independent of [6, 7].

3. Let us consider the construction, suggested in [2, Sec. 2] in the particular case $\mathscr{B} = \mathbf{R}$, $f^t x = x + t$, $B = S \times \mathscr{B}$, and S is the metric space of the continuous (piecewise continuous) mappings A(•): $\mathbf{R} \to \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n)$, such that $A(\cdot): \sup_{x \in \mathscr{B}} \|A(x)\| < +\infty$. The metric in S is given by the equality

$$\rho(A_1, A_2) = \sup_{x \in \mathscr{B}} \|A_1(x) - A_2(x)\|.$$

For all $A \in S$, and $x \in \mathscr{B}$ we consider the linear system of ordinary differential equations

$$\xi' = A(f^t x)\,\xi,\ \xi \in \mathbf{R}^n.$$

Let $\Xi(\theta, \tau; x, A)$ be the Cauchy operator of this system. Let there be given a trivial vector bundle (E, p, B). We set $E = B \times \mathbf{R}^n$, and $p = pr_1$. The mappings $X(m): E \to E$, $\chi(m): B \to B$ are defined by the equations

$$X(m)(A, x, \xi) = (A, m + x, \Xi(m, 0; x, A)\,\xi),$$
$$\chi(m)(A, x) = (A, x + m),$$

where $A \in S$, $x \in \mathscr{B}$, and $\xi \in \mathbf{R}^n$.

It is clear that the Lyapunov indices of the system $\xi' = A(x + t)\,\xi$ depend only on the first component of the basic element b = (A, x). By the same token, we can set $\lambda_k(b) = \lambda_k(A)$, $k = 1, \ldots, n$.

The following theorem holds for the families of the morphisms (1), defined in this manner.

<u>THEOREM 1</u>. For each $k \in \{1, \ldots, n\}$ and $n \geqslant 2$ the function $\lambda_k(\cdot): B \to \mathbf{R}$ is strictly of the second Baire class.

<u>Proof</u>. I. Let us consider the case n = 2. Let us define a mapping $F(\cdot): [0, 1] \to S$: in the following manner: For each $\omega \in [0, 1]$ we set $F(\omega) = A_\omega(\cdot)$, where

$$A_\omega(t) = \begin{pmatrix} a(t) & (1 - a(t))\,\pi\omega \\ (a(t) - 1)\,\pi\omega & -a(t) \end{pmatrix},$$
$$a(t) = \begin{cases} 1 & \text{for } t \in [\zeta(2k), \zeta(2k + 1)),\ k \in \mathbf{N}^+ \\ 0 & \text{for all other values of t;} \end{cases}$$
$$\zeta(k) = \sum_{i=1}^{k} 3^{E[(1/2)(\sqrt{1+4i}-1)]},$$

E(x) being the integral part of the number x.

The inequality $\|F(\omega_1) - F(\omega_2)\| < \pi\,|\omega_1 - \omega_2|$ implies the continuity of the mapping F(•). Therefore, F([0, 1]) is a closed set in S.

II. Let there be given subsets σ_1 and σ_2 of the segment [0, 1]. Writing each number $\omega \in [0, 1]$ in the ternary system $\omega = \alpha_1 3^{-1} + \alpha_2 3^{-2} + \ldots$, where $\alpha_i \in \{0, 1, 2\}$, i = 1, 2, . . ., we put all finite fractions $\alpha_1 3^{-1} + \ldots + \alpha_r 3^{-r}$ with the sum of the numbers $\alpha_1, \ldots, \alpha_r$

even in the set σ_1 and all the infinite fractions $\alpha_1 3^{-1} + \alpha_2 3^{-2} + \ldots$, in which $\alpha_i \neq 1$ only for a finite number of the indices i, in the set σ_2. It is obvious that each of the sets σ_1 and σ_2 is dense in [0, 1].

Let us establish that $\lambda_1(A_\omega) = 1/2$ and $\lambda_2(A_\omega) = -1/2$ for $\omega \in \sigma_1$ and $\lambda_1(A_\omega) = \lambda_2(A_\omega) = 0$ for $\omega \in \sigma_2$.

Let us set

$$q_m = (3/2)(3^m(m-1)+1),$$
$$\Delta_i^m = [q_m + i3^{m+1},\ q_m + (i+1)3^{m+1}),$$
$$X_\omega^{m,i} = \Xi(q_m + 2(i+1)3^{m+1},\ q_m + 2i3^{m+1};\ 0,\ A_\omega).$$

Since

$$A_\omega(t) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{for } t \in \Delta_{2i}^m, \\ \begin{pmatrix} 0 & \pi\omega \\ -\pi\omega & 0 \end{pmatrix} & \text{for } t \in \Delta_{2i+1}^m, \end{cases}$$

where $i = 0, \ldots, m+1$, $m \in \mathbf{N}^+$, we get

$$X_\omega^{m,i} = \begin{pmatrix} \cos(\pi\omega\, 3^{m+1}) & \sin(\pi\omega\, 3^{m+1}) \\ -\sin(\pi\omega\, 3^{m+1}) & \cos(\pi\omega\, 3^{m+1}) \end{pmatrix} \cdot \begin{pmatrix} \exp 3^{m+1} & 0 \\ 0 & \exp 3^{-m-1} \end{pmatrix}. \tag{2}$$

III. We fix any number ω_1 in the set σ_1. Since it has the form $\omega_1 = \alpha_1 3^{-1} + \ldots + \alpha_r 3^{-r}$, where $\alpha_i \in \{0, 1, 2\}$, $i = 1, \ldots, r$, and the sum $\alpha_1 + \ldots + \alpha_r$ is an even number, it follows that $\omega_1 3^{m+1}$ is an integer and is an even number for $m \geqslant r-1$. Therefore, from Eq. (2) we get

$$X_{\omega_1}^{m,i} = \begin{pmatrix} \exp 3^{m+1} & 0 \\ 0 & \exp 3^{-m-1} \end{pmatrix} \tag{3}$$

for $m \geqslant r-1$.

For each $t \geqslant q_r$ we determine an $m_t \in \mathbf{N}^+$, such that $\zeta(2m_t) \leqslant t < \zeta(2m_t+2)$ and $k_t, i_t \in \mathbf{N}^+$, $i_t < k_t$, such that $\zeta(2m_t) = q_{m_t} + 2i_t 3^{k_t+1}$. If

$$x_j(t) = \Xi(t,\ q_r;\ 0,\ A_{\omega_1})\,\xi_j,$$

where $\xi_1 = (1, 0)$ and $\xi_2 = (0, 1)$, then by (3) we have

$$(1/t)\ln\|x_j(t)\| = \frac{\zeta(2m_t)}{t}\frac{1}{\zeta(2m_t)}\ln\|x_j(\zeta(2m_t))\| + \frac{1}{t}\ln\frac{\|x_j(t)\|}{\|x_j(\zeta(2m_t))\|} =$$
$$= \frac{\zeta(2m_t)}{t}\frac{1}{\zeta(2m_t)}\left(\sum_{k=r}^{k_t}\sum_{i=0}^{k}\ln\|X_{\omega_1}^{k,i}\xi_j\| + \sum_{i=0}^{i_t}\ln\|X_{\omega_1}^{k,i}\xi_j\|\right) + \ln\frac{\|x_j(i)\|}{\|x_j(\zeta(2m_t))\|} =$$
$$= \frac{\zeta(2m_t)}{t}\frac{1}{\zeta(2m_t)}(-1)^{j+1}\frac{1}{2}(\zeta(2m_t) - q_{m_r}) + \frac{1}{t}\ln\frac{\|x_j(t)\|}{\|x_j(\zeta(2m_t))\|}.$$

Now, taking into account the fact that

$$\lim_{t\to+\infty}\frac{\zeta(2m_t)}{t} = 1, \quad \lim_{t\to+\infty}\frac{1}{t}\ln\frac{\|x_j(t)\|}{\|x_j(\zeta(2m_t))\|} = 0,$$

we get

$$\lim_{t\to+\infty}(1/t)\ln\|x_j(t)\| = (-1)^{j+1}1/2.$$

Hence, it follows from the equation $\mathrm{Sp}\,A_{\omega_1}(t) = 0$ that the basis $x_1(t)$, $x_2(t)$ is normal. Therefore, 1/2, −1/2 are the Lyapunov indices of the system $\xi' = A_\omega(t)\,\xi$ for $\omega \in \sigma_1$.

IV. Now we fix a number ω_2 in the set σ_2. It can be represented in the form $\omega_2 = \alpha_1 3^{-1} + \ldots + \alpha_r 3^{-r} + 3^{-r-1} + 3^{-r-2} + \ldots$, where $\alpha_i \in \{0, 1, 2\}$, $i = 1, \ldots, r$. Therefore, for $m \geqslant r-1$ the number $\omega_2 3^{m+1}$ can be written as $K_m + 1/2$, where $K_m \in \mathbf{N}^+$. Consequently, it follows from (2) that

$$X_{\omega_2}^{m,i+1} X_{\omega_2}^{m,i} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

if $q_m + 2l3^{m+1} \leqslant q_{m+1}$, $l = i$, $i + 1$, and $m \geqslant r - 1$. Now, as in the paragraph III, we establish that the indices of the system $\xi' = A_\omega(t)\xi$ are equal to zero for $\omega \in \sigma_2$.

V. It follows from the continuity of the mapping F(•) that the sets $F(\sigma_1)$ and $F(\sigma_2)$ are dense in F([0, 1]). By virtue of what we have proved, the functions $\lambda_k(\cdot) : (F([0, 1]), 0) \to \mathbf{R}$, $k = 1, 2$ are discontinuous at each point of the set $(F([0, 1]), 0) \subset B$. In this case, as already observed above, the functions $\lambda_k(\cdot) : B \to \mathbf{R}$, $k = 1, 2$ are strictly of the second Baire class.

VI. For n > 2, for each function $\lambda_k(\cdot)$, supplementing the system of second order by a system of (n − 2)-th order, we can construct a closed set that consists entirely of points of discontinuity of the restriction of the function $\lambda_k(\cdot)$ to this set. Here we define the function $\lambda_k(\cdot)$ by the system of second order, constructed in the paragraph I, and the n − 2 functions $\lambda_i(\cdot)$ by any convenient equations $\xi_i' = a_i(t)\xi_i$, that preserve the ordering of the functions $\lambda_1(\cdot), \ldots, \lambda_n(\cdot)$. The theorem is proved.

Remark. The following proposition follows immediately from Theorem 1 on taking into account the fact that $\mathrm{Sp}\, A_\omega(t) \equiv 0$ for all $\omega \in [0, 1]$,

Proposition. For each $k \in \{1, \ldots, n\}$, where $n = 2m$, $m \in \mathbf{N}^+$, the restriction of the function $\lambda_k(\cdot)$: $B \to \mathbf{R}$ to the set of the linear Hamiltonian systems (see [9, p. 69]) is strictly a function of the second Baire class.

In conclusion, the author thanks V. M. Millionshchikov for assistance with the note.

LITERATURE CITED

1. V. M. Millionshchikov, "The Baire Classes of functions and the Lyapunov indices. I," Differents. Uravn., 16, No. 8, 1408-1416 (1980).
2. V. M. Millionshchikov, "The Baire classes of functions and the Lyapunov indices. II," Differents. Uravn., 16, No. 9, 1587-1598 (1980).
3. V. M. Millionshchikov, "The Baire classes of functions and the Lyapunov indices. III," Differents. Uravn., 16, No. 10, 1766-1785 (1980).
4. R. Baire, Theory of Discontinuous Functions [Russian translation], GITTL, Moscow–Leningrad (1932).
5. F. Hausdorff, Set Theory, 2nd Ed., Chelsea.
6. V. M. Millionshchikov, "On the connection between the stability of characteristic indices and almost reducibility of systems with almost periodic coefficients," Differents. Uravn., 3, No. 12, 2127-2134 (1967).
7. V. L. Novikov, "On the instability of the characteristic indices of systems of linear differential equations with almost periodic coefficients," Differents. Uravn., 8, No. 5, 795-800 (1972).
8. V. V. Nemytskii and V. V. Stepanov, Qualitative Theory of Differential Equations, Princeton Univ. Press (1960).
9. B. F. Bylov, R. É. Vinograd, D. M. Grobman, and V. V. Nemytskii, Theory of the Lyapunov Indices [in Russian], Nauka, Moscow (1966).