Arch. Math., Vol. 41, 363—370 (1983) 0003-889X/83/4104-0363 $ 3.10/0

Cogrowth of groups and simple random walks

By

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Introduction. Let G be a finitely generated discrete group having the presentation $G = F_t/N$ where F_t is the free group on $x_1, \ldots, x_t$ ($t \geqq 2$) and N is a normal subgroup of F_t.

Cohen [3] and Grigorchuk [5] independently introduced the notion of *cogrowth*:

Let E_n be the set of words $w \in F_t$ of length $|w| = n$. If N is non-trivial then the *cogrowth coefficients* $\gamma_n = |E_n \cap N|$ ($|A|$ is the cardinality of the set A) satisfy either

a) $\lim\limits_{n \to \infty} \gamma_n^{1/n} = \gamma$ or

b) $\lim\limits_{n \to \infty} \gamma_{2n}^{1/2n} = \gamma$ and $\gamma_{2n-1} = 0 \quad \forall n \in \mathbb{N}$, and $\gamma \in (\sqrt{2t-1},\ 2t-1]$.

Cohen [3] calls the number $\eta = \log(\gamma)/\log(2t-1)$ the cogrowth of G with respect to the given presentation.

In the results which are going to be proved in this paper the following is contained:

— If G is infinite then $\gamma_n/|E_n|$ (and in fact even γ_n/γ^n) tends to zero as $n \to \infty$.

— If G is finite then either

$$\lim_{n \to \infty} \gamma_n/|E_n| = 1/|G| \quad \text{or} \quad \lim_{n \to \infty} \gamma_{2n}/|E_{2n}| = 2/|G|$$

in accordance with the two cases a), b) mentioned above.

Statement of results. Further notations have to be introduced: Let π be the natural projection of F_t onto $G = F_t/N$. If $w \in F_t$ then $\tilde{w} = \pi(w)$. e denotes the empty word, the unit element of F_t. μ_n is the uniform distribution on E_n, i.e. $\mu_n(w) = 1/|E_n|$ if $w \in E_n$ and $\mu_n(w) = 0$ otherwise. Observe that $|E_0| = 1$ and $|E_n| = 2t(2t-1)^{n-1}$ for $n \geqq 1$. σ_n denotes the projection of the probability distribution μ_n onto G: $\sigma_n(\tilde{w}) = \sum\limits_{\pi(w) = \tilde{w}} \mu_n(w)$.

In a wider sense than above, the cogrowth coefficients are considered as *functions* on G: If $\tilde{w} \in G$ then $\gamma_n(\tilde{w}) = |E_n \cap wN|$ where $\pi(w) = \tilde{w}$. Thus $\gamma_n(.)$ counts the words of length n in the different cosets of N in F_t, and $\gamma_n(\tilde{w}) = |E_n|\, \sigma_n(\tilde{w})$.

The following Lemma generalizes Proposition 1 (resp. Theorem 1) of [3]:

Lemma 1. *If γ is the number defined in the introduction, $w \in F_t$ and $\tilde{w} = \pi(w)$ then either*

a) $\lim\limits_{n\to\infty} \gamma_n(\tilde{w})^{1/n} = \gamma$ *or*

b) $\lim\limits_{n\to\infty} \gamma_{2n+|w|}(\tilde{w})^{1/2n} = \gamma$ *and* $\gamma_{2n+|w|-1}(\tilde{w}) = 0 \quad \forall n \in \mathbb{N}$

in accordance with the two cases from the introduction.

Theorem 1.

I) *If G is infinite then $\|\gamma_n(.)\|_2/\gamma^n$ tends to zero as $n \to \infty$.*

II) *If G is finite then $\lim\limits_{n\to\infty} \|\gamma_n(.)\|_2/|E_n| = \sqrt{p/|G|}$ where $p = 1$ in case* a) *and $p = 2$ in case* b) *from the introduction, resp. Lemma 1.*

$\|.\|_2$ denotes the l_2-norm on G. Observe that $\gamma = 2t - 1$ if G is amenable (in particular if G is finite): this was proved in [3] and [5]. In this case $|E_n| = \dfrac{2t}{2t-1}\gamma^n$ and it seems to be more natural to study $\|\gamma_n(.)\|_2/|E_n|$ instead of $\|\gamma_n(.)\|_2/\gamma^n$: $\gamma_n(\tilde{w})/|E_n|$ is the probability that a randomly chosen word of length n in F_t lies in the coset wN, where $\pi(w) = \tilde{w}$.

Theorem 2.

I) *If G is infinite then $\lim\limits_{n\to\infty} \gamma_n(\tilde{w})/\gamma^n = 0$ uniformly for $\tilde{w} \in G$.*

II) *If G is finite then either*

a) $\lim\limits_{n\to\infty} \gamma_n(\tilde{w})/|E_n| = 1/|G| \quad \forall \tilde{w} \in G$ *or*

b) $\lim\limits_{n\to\infty} \gamma_{2n+|w|}(\tilde{w})/|E_{2n+|w|}| = 2/|G| \quad \forall \tilde{w} \in G$

in accordance with the two cases from Lemma 1.

Theorem 1, I) implies Theorem 2, I) and, vice versa Theorem 2, II) implies Theorem 1, II), as in case b) the set of all $\tilde{w} = \pi(w) \in G$ for which $w \in F_t$ has even length is a subgroup of G of index 2.

By Theorem 2, $\gamma_{2n}/|E_{2n}|$ $(\gamma_n = \gamma_n(\tilde{e}))$ always has a limit, and this limit is positive if and only if G is finite.

Proofs. In the proofs the knowledge about random walks will be applied to the simple random walk on G defined by σ_1. Identities between certain generating functions will lead to the proposed results.

The convolution of two measures ν, τ on a group is denoted by $\nu * \tau$, $\nu^{(n)}$ is the n'th convolution power of ν, $\nu^{(0)}$ is the Dirac measure at the group identity.

Lemma 2. *For* $z \in \mathbb{C}$ *let*

$$f(z) = \frac{(2t-1)^2 - z^2}{2t(2t-1) + 2tz^2} \quad \textit{and} \quad \psi(z) = \frac{2tz}{2t-1+z^2}.$$

Then

$$\sum_{n=0}^{\infty} z^n \mu_n = \frac{1}{2t}\mu_0 + f(z)\sum_{n=0}^{\infty}(\psi(z))^n \mu_1^{(n)}.$$

Proof. The following formula is well known [2, 4]:

$$\mu_1 * \mu_n = \frac{1}{2t}\mu_{n-1} + \frac{2t-1}{2t}\mu_{n+1} \quad \text{for} \quad n = 1, 2, \ldots \tag{1}$$

Consider the Markov chain on the nonnegative integers, called "random walk with one reflecting barrier", with the following transition probabilities:

$$P_{0,1} = 1, \quad P_{k,k-1} = \frac{1}{2t} \quad \text{and} \quad P_{k,k+1} = \frac{2t-1}{2t} \quad \text{for} \quad k \geqq 1,$$

$P_{k,l} = 0$ in any other case.

If $q_{n,k} = P_{0,k}^{(n)}$ denotes the probability to reach the state k at the n'th step after having started in 0, then formula (1) yields

$$\mu_1^{(n)} = \sum_{k=0}^{n} q_{n,k}\mu_k \tag{2}$$

and the following relations are satisfied (compare [2]):

$$\begin{aligned} &q_{0,0} = 1, \\ &q_{n+1,0} = \frac{1}{2t}q_{n,1}, \qquad q_{n+1,1} = q_{n,0} + \frac{1}{2t}q_{n,2}, \\ &q_{n+1,k} = \frac{2t-1}{2t}q_{n,k-1} + \frac{1}{2t}q_{n,k+1}, \qquad k = 2, \ldots, n+1, \end{aligned} \tag{3}$$

$q_{n,k} > 0$ if and only if $0 \leqq k \leqq n$ and $n-k$ is even.

By (2) and a change of summation,

$$\sum_{n=0}^{\infty} y^n \mu_1^{(n)} = \sum_{k=0}^{\infty} Q_k(y)\mu_k, \quad \text{where} \quad Q_k(y) = \sum_{n=k}^{\infty} y^n q_{n,k} \quad \text{for} \quad y \in \mathbb{C}.$$

$Q_0(y)$ is the generating function of the sequence $(\mu_1^{(n)}(e))$, which has been calculated in [7]. Together with the relations (3) a recursion for $Q_i(y)$ is obtained:

$$Q_0(y) = \frac{2t-1}{t-1+\sqrt{t^2-(2t-1)y^2}},$$

$$Q_1(y) = \frac{2t}{y}(Q_0(y)-1), \qquad Q_2(y) = \frac{2t}{y}Q_1(y) - 2tQ_0(y),$$

$$\frac{1}{2t}yQ_{k+1}(y) - Q_k(y) + \frac{2t-1}{2t}yQ_{k-1}(y) = 0.$$

The solution of the recursion is

$$Q_k(y) = \frac{2t}{2t-1} Q_0(y)(\varphi(y))^k \quad \text{for} \quad k = 1, 2, \ldots$$

where

$$\varphi(y) = \frac{t - \sqrt{t^2 - (2t-1)y^2}}{y}.$$

Therefore

$$\sum_{n=0}^{\infty} y^n \mu_1^{(n)} = \frac{2t}{2t-1} Q_0(y) \sum_{n=0}^{\infty} (\varphi(y))^n \mu_n - \frac{1}{2t-1} Q_0(y) \mu_0.$$

If $\varphi(y) = z$ then $y = \psi(z) = \dfrac{2tz}{2t-1+z^2}$ and thus

$$\sum_{n=0}^{\infty} z^n \mu_n = \frac{2t-1}{2t\, Q_0(\psi(z))} \sum_{n=0}^{\infty} (\psi(z))^n \mu_1^{(n)} + \frac{1}{2t} \mu_0.$$

A short calculation yields $\dfrac{2t-1}{2t\, Q_0(\psi(z))} = f(z)$.

Remark. As $q_{n,k} = 0$ if $n - k$ is odd, also the following identity holds:

$$\sum_{n=0}^{\infty} z^{2n} \mu_{2n} = \frac{1}{2t} \mu_0 + f(z) \sum_{n=0}^{\infty} (\psi(z))^{2n} \mu_1^{(2n)}.$$

Lemma 3. *If* $z \in \mathbb{C}$ *and*

$$g(z) = \frac{(2t-1)^2 - z^2}{2t(2t-1) - 2tz^2}, \qquad h(z) = \frac{2t-1}{2t(2t-1) - 2tz^2}$$

then

$$\sum_{n=0}^{\infty} z^{2n} \mu_n^{(2)} = h(z)\mu_0 + g(z) \sum_{n=0}^{\infty} z^{2n} \mu_{2n}.$$

Proof. The following formula holds [4]:

$$\mu_n^{(2)} = \sum_{k=0}^{n} a_{n,k} \mu_{2k} \quad \text{where} \quad a_{n,0} = 1/|E_n| \quad \text{for} \quad n = 0, 1, \ldots \quad \text{and}$$

$$a_{n,k} = \frac{t-1}{t} \Big/ (2t-1)^{n-k} \quad \text{if} \quad 0 < k < n, \qquad a_{n,n} = \frac{2t-1}{2t}$$

for $n = 1, 2, \ldots$.

Then $\sum_{n=0}^{\infty} z^{2n} \mu_n^{(2)} = \sum_{k=0}^{\infty} A_k(z) \mu_{2k}$, where $A_k(z) = \sum_{n=k}^{\infty} z^{2n} a_{n,k}$ is easy to calculate:

$$A_0(z) = g(z) + h(z), \qquad A_k(z) = g(z) z^{2k}.$$

Proof of Lemma 1. As the formula of Lemma 2 remains invariant under the projection of F_t onto G, we have

$$\sum_{n=0}^{\infty} z^n \sigma_n(\tilde{w}) = \frac{1}{2t}\sigma_0(\tilde{w}) + f(z)\sum_{n=0}^{\infty}(\psi(z))^n \sigma_1^{(n)}(\tilde{w}) \quad \forall \tilde{w} \in G. \tag{4}$$

The simple random walk defined by σ_1 is an irreducible Markov chain on G, thus the number

$$\varrho = \limsup_{n\to\infty} \sigma_1^{(n)}(\tilde{w})^{1/n} \quad \text{is independent of } \tilde{w} \text{ and} \quad 0 < \varrho \leqq 1 \quad [9]. \tag{5}$$

$1/\varrho$ is the radius of convergence of $\sum_{n=0}^{\infty} y^n \sigma_1^{(n)}(\tilde{w})$. Therefore the radius of convergence of $\sum_{n=0}^{\infty} z^n \sigma_n(\tilde{w})$ does not depend on $\tilde{w}$. This means

$$\gamma = \limsup_{n\to\infty} \gamma_n(\tilde{w})^{1/n} \quad \forall \tilde{w} \in G. \tag{6}$$

The following formula can be proved like (1.2) in [3]:

$$\gamma_m(\tilde{w})\,\gamma_n(\tilde{e}) \leqq \gamma_{m+n+2}(\tilde{w}).$$

Now, if in case a) $\lim_{n\to\infty} \gamma_n(\tilde{e})^{1/n} = \gamma$ then for $\pi(w) = \tilde{w}$

$$\gamma = \lim_{n\to\infty} (\gamma_{|w|}(\tilde{w})\,\gamma_{n-|w|-2}(\tilde{e}))^{1/n} \leqq \liminf_{n\to\infty} \gamma_n(\tilde{w})^{1/n}.$$

Together with (6) we obtain

$$\lim_{n\to\infty} \gamma_n(\tilde{w})^{1/n} = \gamma.$$

In case b), $\gamma_{2n-1}(\tilde{e}) = 0$ $\forall n \in \mathbb{N}$. If $\gamma_{2n+|w|-1}(\tilde{w}) > 0$ for some n then there is a word $v \in wN$ of length $2n + |w| - 1$, and $w^{-1}v \in N$ has odd length, a contradiction. Thus $\gamma_{2n+|w|-1}(\tilde{w}) = 0$ $\forall n \in \mathbb{N}$ and the same argument as above yields

$$\lim_{n\to\infty} \gamma_{2n+|w|}(\tilde{w})^{1/2n} = \gamma.$$

Proof of Theorem 1, I). The simple random walk on G with law σ_1 is either aperiodic or has period 2, because σ_1 is symmetric. In the first case (case a) from Lemma 1) the random walk defined by $\sigma_1^{(2)}$ is also an irreducible Markov chain on the whole of G, which is aperiodic. In the second case, the support of $\sigma_1^{(2)}$ generates a normal subgroup H of G of index 2, and the random walk on H defined by $\sigma_1^{(2)}$ is irreducible and aperiodic (cf. e.g. [10]).

We shall distinguish between two cases according to transience or recurrence of $\sigma_1^{(2)}$.

Case 1: $\sigma_1^{(2)}$ is transient. If ϱ is the number defined in (5) then it is known [6, p. 85] that

$$\sum_{n=0}^{\infty} \varrho^{-2n} \sigma_1^{(2n)}(\tilde{e}) < \infty. \tag{7}$$

Now the remark before Lemma 3 is applied in combination with Lemma 3:

$$\sum_{n=0}^{\infty} z^{2n} \sigma_n^{(2)}(\tilde{e}) = h(z) + g(z)/2t + f(z) g(z) \sum_{n=0}^{\infty} (\psi(z))^{2n} \sigma_1^{(2n)}(\tilde{e}) . \tag{8}$$

If $z = (2t - 1)/\gamma$ then $\psi(z) = 1/\varrho$ [3, 5].

As the coefficients of the power series in (8) are nonnegative, (7) and (8) yield

$$\sum_{n=0}^{\infty} \left(\frac{2t-1}{\gamma}\right)^{2n} \sigma_n^{(2)}(\tilde{e}) < \infty$$

implying

$$\| \gamma_n(.) \|_2^2 / \gamma^{2n} = \left(\frac{2t}{2t-1}\right)^2 \left(\frac{2t-1}{\gamma}\right)^{2n} \sigma_n^{(2)}(\tilde{e}) \to 0 \quad \text{as} \quad n \to \infty .$$

Case 2: $\sigma_n^{(2)}$ is recurrent. Then $\varrho = 1$ and $\gamma = 2t - 1$. If $\tilde{w} \in G$, resp. $\tilde{w} \in H$ according to a) and b) then it is known [8] that

$$\sum_{n=0}^{\infty} (\sigma_1^{(2n)}(\tilde{e}) - \sigma_1^{(2n)}(\tilde{w})) < \infty . \tag{9}$$

If $\tilde{w} \neq \tilde{e}$ then we obtain from (8) for $|z| < 1$:

$$\begin{aligned} &\sum_{n=0}^{\infty} z^{2n} (\sigma_n^{(2)}(\tilde{e}) - \sigma_n^{(2)}(\tilde{w})) \\ &= h(z) + g(z)/2t + f(z) g(z) \sum_{n=0}^{\infty} (\psi(z))^{2n} (\sigma_1^{(2n)}(\tilde{e}) - \sigma_1^{(2n)}(\tilde{w})) \end{aligned} \tag{10}$$

σ_n is symmetric, therefore $\sigma_n^{(2)}$ is positive definite and $\sigma_n^{(2)}(\tilde{e}) \geqq \sigma_n^{(2)}(\tilde{w})$. Again using the nonnegativity of the coefficients, (9) and (10) yield for $z = 1$:

$$\sum_{n=0}^{\infty} (\sigma_n^{(2)}(\tilde{e}) - \sigma_n^{(2)}(\tilde{w})) < \infty$$

implying

$$\sigma_n^{(2)}(\tilde{e}) - \sigma_n^{(2)}(\tilde{w}) \to 0 \quad \text{as} \quad n \to \infty . \tag{11}$$

Suppose, $\lim_{k \to \infty} \sigma_{n_k}^{(2)}(\tilde{e}) = d > 0$ for a subsequence. Choose a finite subset B of G, resp. H having $m > 1/d$ elements. Then by (11),

$$1 \geqq \lim_{k \to \infty} \sum_{\tilde{w} \in B} \sigma_{n_k}^{(2)}(\tilde{w}) = m \cdot d > 1 ,$$

contradiction. Therefore,

$$\| \gamma_n(.) \|_2^2 / |E_n|^2 = \sigma_n^{(2)}(\tilde{e}) \to 0 \quad \text{as} \quad n \to \infty .$$

Proof of Theorem 2, II). In case a), the random walk on G with law σ_1 is aperiodic. Therefore [1] for $\tilde{w} \in G$, $\sigma_1^{(n)}(\tilde{w}) - 1/|G|$ tends to zero exponentially fast, and using Lemma 2 we see that

$$\sum_{n=0}^{\infty} z^n (\sigma_n(\tilde{e}) - \sigma_n(\tilde{w})) = \frac{1}{2t} + f(z) \sum_{n=0}^{\infty} (\psi(z))^n (\sigma_1^{(n)}(\tilde{e}) - \sigma_1^{(n)}(\tilde{w})) \tag{12}$$

has a radius of convergence greater than 1, as $\psi(z)$ is increasing for $z < \sqrt{2t-1}$ and $\psi(1) = 1$, and as all operations of the proof of Lemma 2 may be applied to the above series if $|\psi(z)|$ is smaller than the radius of convergence of the series on the right hand side of (12). Now $\sigma_n(\tilde{e}) - \sigma_n(\tilde{w}) \to 0$ as $n \to \infty$ for each $\tilde{w} \in G$, and an argument similar to the one used in case 2 of the proof of Theorem 1, I) leads to

$$\gamma_n(\tilde{w})/|E_n| = \sigma_n(\tilde{w}) \to 1/|G| \quad \text{for} \quad \tilde{w} \in G \quad \text{as} \quad n \to \infty.$$

In case b), the random walk on H with law $\sigma_1^{(2)}$ is aperiodic and $\sigma_1^{(2n)}(\tilde{w}) - 2/|G|$ tends to zero exponentially fast for $\tilde{w} \in H$, as $|H| = |G|/2$. $\sigma_{2n-1}(\tilde{w}) = 0 \;\forall n \in \mathbb{N}$ if $\tilde{w} \in H$, therefore (12) remains true when the sums are taken over all even n. Like above we obtain

$$\gamma_{2n}(\tilde{w})/|E_{2n}| \to 2/|G| \quad \text{for} \quad \tilde{w} \in H \quad \text{as} \quad n \to \infty.$$

Now choose $\tilde{w}_0 \in G - H$. Then $G - H = \tilde{w}_0 H$ and for $\tilde{v} \in H$ one has

$$\sigma_1^{(2n+1)}(\tilde{w}_0) - \sigma_1^{(2n+1)}(\tilde{w}_0 \tilde{v}) = \sum_{\tilde{u} \in H} \sigma_1(\tilde{w}_0 \tilde{u}^{-1}) (\sigma_1^{(2n)}(\tilde{u}) - \sigma_1^{(2n)}(\tilde{u}\tilde{v}))$$

which tends to zero exponentially fast.

If $\tilde{w} \in G - H$ then $\sigma_{2n}(\tilde{w}) = \sigma_1^{(2n)}(\tilde{w}) = 0$, and applying the same arguments as above to

$$\sum_{n=0}^{\infty} z^{2n+1}(\sigma_{2n+1}(\tilde{w}_0) - \sigma_{2n+1}(\tilde{w}))$$
$$= f(z) \sum_{n=0}^{\infty} (\psi(z))^{2n+1} (\sigma_1^{(2n+1)}(\tilde{w}_0) - \sigma_1^{(2n+1)}(\tilde{w}))$$

yields

$$\gamma_{2n+1}(\tilde{w})/|E_{2n+1}| \to 2/|G| \quad \text{for} \quad \tilde{w} \in G - H \quad \text{as} \quad n \to \infty.$$

Finally, observe that $w \in F_t$ has even length whenever $\pi(w) = \tilde{w} \in H$ and has odd length otherwise.

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Eingegangen am 29. 10. 1982

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