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Periodic solutions of some forced Liénard differential equations at resonance

By

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1. Introduction. The study of the existence of 2π-periodic solutions for Duffing equations of the form

$$x'' + cx' + g(x) = e(t) \equiv e(t + 2\pi)$$

where c is arbitrary, g and e are continuous, and g is asymptotically linear in some sense has been initiated by Lazer in [3] where he proved that sufficient conditions for the existence were given by

$$\int_0^{2\pi} e(t)\,dt = 0, \qquad \lim_{|x| \to \infty} x^{-1} g(x) = 0, \qquad x g(x) \geqq 0$$

for $|x|$ sufficiently large. Conditions of this type are referred to as resonance conditions because they reduce in the linear case ($g = 0$) to the necessary and sufficient conditions for the solvability of the resonant 2π-periodic problem

$$x'' + cx' = e(t), \qquad x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0.$$

Successive generalizations and extensions of Lazer's result have been given by Mawhin [6], Chang [1], Reissig [9, 10], Martelli [4], Martelli and Schuur [5], and Gupta [2]. They deal in the most general case with equations of Liénard type of the form

$$x'' + f(x)x' + g(t, x, x') = e(t)$$

and sharp results with respect to the linear situation are obtained only for Liénard equations of the form

$$x'' + f(x)x' + g(t, x) = e(t). \tag{1.1}$$

The best known results for equation (1.1) are those of Reissig [9, 10] which insure the existence of a 2π-periodic solution for (1.1) with f, g and e continuous and e having mean value zero when, for some $R > 0$, and all x with $|x| \geqq R$ either

$$x g(t, x) \leqq 0 \tag{1.2}$$

or

$$x g(t, x) \geqq 0 \tag{1.2'}$$

with in the second case the supplementary condition

$$\limsup_{|x|\to\infty} x^{-1} g(t,x) \leqq q < 1\,, \tag{1.3}$$

uniformly in $t \in [0, 2\pi]$.

In a recent paper of the authors [8] dealing with non-resonant situations for (1.1), uniform conditions of non-resonance with the second eigenvalue of type (1.3) have been replaced by more general non-uniform ones of the form

$$\limsup_{|x|\to\infty} x^{-1} g(t,x) \leqq \Gamma(t)\,, \tag{1.4}$$

uniformly in $t \in [0, 2\pi]$, with some restrictions to the interaction of $\Gamma(t)$ with the second eigenvalue 1 of the associated linear problem. We shall adapt in this paper the methodology of [8] to prove existence theorems for the 2π-periodic solutions of (1.1) which generalize the papers quoted above. Our main result (Theorem 1 of Section 3) proves the existence of a 2π-periodic solution for (1.1) with f continuous, g Caratheodory and e Lebesque integrable under assumptions containing, as special cases, the conditions (1.2′) and (1.4) where Γ is measurable and such that

$$\Gamma(t) \leqq 1 \tag{1.5}$$

for a.e. $t \in [0, 2\pi]$, with strict inequality on a subset of $[0, 2\pi]$ of positive measure. A natural question is whether condition (1.3) or (1.4) can be replaced by the assumption that the inequality holds when $x \to -\infty$ but not necessarily as $x \to +\infty$, or vice versa. In Theorem 2 we deal with this question, where it is shown that the answer is positive if $\Gamma(t) \leqq 1/4$ for a.e. $t \in [0, 2\pi]$, with strict inequality on a subset $[0, 2\pi]$ of positive measure. An example shows that this result is almost sharp. Finally some of the above mentioned papers contain extensions of the results to some systems of Duffing or Liénard equations. The corresponding generalizations involving conditions of non-uniform type will be considered in another paper.

Let us end this introduction by mentioning that besides the classical spaces $C([0, 2\pi])$, $C^k([0, 2\pi])$ and $L^k(0, 2\pi)$ of continuous, k-times continuously differentiable or measurable real functions whose k-th power of the absolute value is Lebesque integrable, we shall use the Sobolev spaces $W^{2,1}(0, 2\pi)$ and $H^1(0, 2\pi)$ respectively defined by

$$W^{2,1}(0, 2\pi) = \{x : [0, 2\pi] \to \mathbb{R} : x \text{ and } x' \text{ are absolutely continuous on } [0, 2\pi]\}\,,$$

and

$$H^1(0, 2\pi) = \{x : [0, 2\pi] \to \mathbb{R} : x \text{ is absolutely continuous on } [0, 2\pi] \text{ and } x' \in L^2(0, 2\pi)\}$$

with respective norms

$$|x|_{W^{2,1}} = \sum_{k=0}^{2} \int_0^{2\pi} |x^{(k)}(t)|\, dt$$

and

$$|x|_{H^1} = \left\{\left[(2\pi)^{-1}\int_0^{2\pi} x(t)\,dt\right]^2 + (2\pi)^{-1}\int_0^{2\pi}[x'(t)]^2\,dt\right\}^{1/2}.$$

In any normed space, the strong and the weak convergence of sequences will respectively be denoted by $\to$ and $\rightharpoonup$, and we shall use the fact that $H^1(0, 2\pi)$ is compactly imbedded into $C([0, 2\pi])$ and is a Hilbert space with inner product defined by

$$(x, y)_{H^1} = ((2\pi)^{-1}\int_0^{2\pi} x(t)\,dt)\left((2\pi)^{-1}\int_0^{2\pi} y(t)\,dt\right) + (2\pi)^{-1}\int_0^{2\pi} x'(t)\,y'(t)\,dt.$$

2. An inequality for some Liénard operators with periodic boundary conditions. For $x \in L^1(0, 2\pi)$, let us write

$$\bar{x} = (2\pi)^{-1}\int_0^{2\pi} x(t)\,dt, \qquad \tilde{x}(t) = x(t) - \bar{x},$$

so that

$$\int_0^{2\pi} \tilde{x}(t)\,dt = 0.$$

Let $\tilde{H}^1(0, 2\pi) = \{x \in H^1(0, 2\pi) : \bar{x} = 0\}$.

The following results extend Lemmas 1 and 2 of [8] to some resonant situations, and may be of interest by themselves in some uniqueness questions.

Lemma 1. *Let*

$$\Gamma \in L^1(0, 2\pi)$$

be such that, for a.e. $t \in [0, 2\pi]$, *one has*

(2.1) $$\Gamma(t) \leqq 1$$

with the strict inequality on a subset of $[0, 2\pi]$ *of positive measure. Then there exists* $\delta = \delta(\Gamma) > 0$ *such that for all* $\tilde{x} \in \tilde{H}^1(0, 2\pi)$ *one has*

$$B_\Gamma(\tilde{x}) \equiv (2\pi)^{-1}\int_0^{2\pi}[(\tilde{x}'(t))^2 - \Gamma(t)\,\tilde{x}^2(t)]\,dt \geqq \delta\,|\tilde{x}|^2_{H^1}.$$

Proof. Using (2.1) and Wirtinger's inequality [7], we see that, for all $\tilde{x} \in \tilde{H}^1(0, 2\pi)$, we have

(2.2) $$B_\Gamma(\tilde{x}) \geqq (2\pi)^{-1}\int_0^{2\pi}[\tilde{x}'(t)]^2 - \tilde{x}^2(t)]\,dt \geqq 0$$

with moreover

(2.3) $$B_\Gamma(\tilde{x}) = 0$$

if and only if

$$\tilde{x}(t) = A\sin(t + \varphi)$$

for some $A \geqq 0$ and $\varphi \in \mathbb{R}$. But then by (2.2) and (2.3) we get

$$0 = \int_0^{2\pi} (1 - \Gamma(t))\, \tilde{x}^2(t)\, dt = A^2 \int_0^{2\pi} (1 - \Gamma(t)) \sin^2(t + \varphi)\, dt,$$

so that by our assumptions, $A = 0$ and hence $\tilde{x} = 0$. Assume now that the conclusion of the lemma is false. Then we can find a sequence $(\tilde{x}_n)$ in $\tilde{H}^1(0, 2\pi)$ and $\tilde{x} \in \tilde{H}^1(0, 2\pi)$ such that

$$(2.4) \qquad |\tilde{x}_n|_{H^1} = 1, \quad \tilde{x}_n \to \tilde{x} \text{ in } C([0, 2\pi]), \quad \tilde{x}_n \rightharpoonup \tilde{x} \text{ in } H^1(0, 2\pi),$$

and

$$(2.5) \qquad 0 \leqq B_\Gamma(\tilde{x}_n) \leqq 1/n, \quad n \in \mathbb{N}^*.$$

From Schwarz' inequality in $H^1(0, 2\pi)$, we deduce

$$[(\tilde{x}_n, \tilde{x})_{H^1}]^2 \leqq |\tilde{x}_n|_{H^1}^2 |\tilde{x}|_{H^1}^2, \quad n \in \mathbb{N}^*,$$

and hence

$$|\tilde{x}|_{H^1}^2 \leqq \liminf_{n \to \infty} |\tilde{x}_n|_{H^1}^2.$$

By (2.4) and (2.5), we obtain, for $n \to \infty$,

$$(2.6) \qquad |\tilde{x}_n|_{H^1}^2 \to (2\pi)^{-1} \int_0^{2\pi} \Gamma(t)\, \tilde{x}^2(t)\, dt$$

and hence

$$|\tilde{x}|_{H^1}^2 \leqq (2\pi)^{-1} \int_0^{2\pi} \Gamma(t)\, \tilde{x}^2(t)\, dt,$$

i.e.,

$$B_\Gamma(\tilde{x}) \leqq 0.$$

By the first part of the proof, $\tilde{x} = 0$, so that, by (2.6),

$$|\tilde{x}_n|_{H^1} \to 0,$$

a contradiction with the first equality in (2.4).

Lemma 2. *Let Γ be like in Lemma 1, let $\delta > 0$ be associated to Γ by that lemma and let $\varepsilon > 0$. Then for all $p \in L^1(0, 2\pi)$ satisfying*

$$(2.7) \qquad \bar{p} \geqq 0 \quad \text{and} \quad p(t) \leqq \Gamma(t) + \varepsilon$$

a.e. on $[0, 2\pi]$, all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ and all $x \in W^{2,1}(0, 2\pi)$ with

$$(2.8) \qquad x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0,$$

one has

$$(2.9) \qquad (2\pi)^{-1} \int_0^{2\pi} (\bar{x} - \tilde{x}(t))\,(x''(t) + f(x(t))\, x'(t) + p(t)\, x(t))\, dt \geqq (\delta - \varepsilon)\, |\tilde{x}|_{H^1}^2.$$

Proof. If $x \in W^{2,1}(0, 2\pi)$ and satisfies (2.8), we obtain easily, integrating by parts,

$$(2\pi)^{-1} \int_0^{2\pi} (\bar{x} - \tilde{x}(t))(x''(t) + f(x(t))x'(t) + p(t)x(t))\,dt$$
$$= \bar{p}\bar{x}^2 + (2\pi)^{-1} \int_0^{2\pi} [(\tilde{x}'(t))^2 - p(t)\tilde{x}^2(t)]\,dt$$
$$\geqq B_\Gamma(\tilde{x}) - \varepsilon(2\pi)^{-1} \int_0^{2\pi} \tilde{x}^2(t)\,dt \geqq (\delta - \varepsilon)\,|\tilde{x}|^2_{H^1}.$$

3. Periodic solutions for a Liénard equation at resonance. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and let $g\colon [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$, $(t, x) \mapsto g(t, x)$ be such that $g(\cdot, x)$ is measurable on $[0, 2\pi]$ for each $x \in \mathbb{R}$ and $g(t, \cdot)$ is continuous on R for a.e. $t \in [0, 2\pi]$. Assume moreover that for each $r > 0$ there exists $\gamma_r \in L^1(0, 2\pi)$ such that

$$|g(t, x)| \leqq \gamma_r(t)$$

for a.e. $t \in [0, 2\pi]$ and all $x \in [-r, r]$. We consider the following periodic boundary-value problem for the Liénard equation

$$x''(t) + f(x(t))x'(t) + g(t, x(t)) = e(t), \quad t \in [0, 2\pi],$$
$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0. \tag{3.1}$$

We prove the following existence result for (3.1).

Theorem 1. *Assume that there exists $\Gamma \in L^1(0, 2\pi)$ such that*

$$\limsup_{|x| \to \infty} \frac{g(t, x)}{x} \leqq \Gamma(t) \tag{3.2}$$

uniformly a.e. in $t \in [0, 2\pi]$ and such that

$$\Gamma(t) \leqq 1$$

for a.e. $t \in [0, 2\pi]$, with strict inequality on a subset of $[0, 2\pi]$ of positive measure. Assume moreover that there exists real numbers a, A, r and R with $a \leqq A$ and $r < 0 < R$ such that

$$g(t, x) \geqq A \tag{3.3}$$

for a.e. $t \in [0, 2\pi]$ and all $x \geqq R$ and

$$g(t, x) \leqq a \tag{3.4}$$

for a.e. $t \in [0, 2\pi]$ and all $x \leqq r$. Then the problem (3.1) has at least one solution for each $e \in L^1(0, 2\pi)$ such that

$$a \leqq \bar{e} \leqq A. \tag{3.5}$$

Proof. Define g_1 on $[0, 2\pi] \times \mathbb{R}$ by

$$g_1(t, x) = g(t, x) - (1/2)(a + A)$$

and e_1 on $[0, 2\pi]$ by

$$e_1(t) = e(t) - (1/2)(a + A),$$

so that, for a.e. $t \in [0, 2\pi]$, using (3.3) to (3.5), we have

$$g_1(t, x) \geqq (1/2)(A - a) \geqq 0 \tag{3.6}$$

if $x \geqq R$,

$$g_1(t, x) \leqq (1/2)(a - A) \leqq 0 \tag{3.7}$$

if $x \leqq r$, and

$$(1/2)(a - A) \leqq \bar{e}_1 \leqq (1/2)(A - a). \tag{3.8}$$

Clearly, the equation in (3.1) is equivalent to

$$x''(t) + f(x(t))\,x'(t) + g_1(t, x(t)) = e_1(t). \tag{3.9}$$

Moreover, we have

$$\limsup_{|x| \to \infty} x^{-1} g_1(t, x) \leqq \Gamma(t)$$

uniformly a.e. in $t \in [0, 2\pi]$ and if $|x| \geqq \max(R, -r)$, then for a.e. $t \in [0, 2\pi]$ we have also

$$x^{-1} g_1(t, x) \geqq 0.$$

Let $\delta > 0$ be associated to the function Γ by Lemma 1. Then there exists $r_1 > 0$ such that for a.e. $t \in [0, 2\pi]$ and for all x with $|x| \geqq r_1$, one has

$$0 \leqq x^{-1} g_1(t, x) \leqq \Gamma(t) + \delta/2. \tag{3.10}$$

Define γ_1 on $[0, 2\pi] \times \mathbb{R}$ by

$$\gamma_1(t, x) = \begin{cases} x^{-1} g_1(t, x) & \text{if } |x| \geqq r_1 \\ r_1^{-1} g_1(t, r_1)x + (1 - x r_1^{-1})\,\Gamma(t) & \text{if } 0 \leqq x \leqq r_1 \\ r_1^{-1} g_1(t, -r_1)\,x + (1 + x r_1^{-1})\,\Gamma(t) & \text{if } -r_1 \leqq x \leqq 0. \end{cases}$$

Then, by (3.10), we have

$$0 \leqq \gamma_1(t, x) \leqq \Gamma(t) + \delta/2$$

for a.e. $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$. Moreover, the function

$$(t, x) \mapsto \gamma_1(t, x)\,x$$

satisfies the Caratheodory conditions, and the function h defined on $[0, 2\pi] \times \mathbb{R}$ by

$$h(t, x) = g_1(t, x) - \gamma_1(t, x)\,x$$

is such that, for some $\alpha \in L^1(0, 2\pi)$, a.e. $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$, we have

$$|h(t, x)| \leqq \alpha(t). \tag{3.11}$$

Finally, equation (3.9) can be written as

$$x''(t) + f(x(t))\,x'(t) + \gamma_1(t, x(t))\,x(t) + h(t, x(t)) = e_1(t). \tag{3.12}$$

To apply Theorem IV.5 of [7] to (3.12) with the periodic boundary conditions on $[0, 2\pi]$, we have to prove the existence of an a priori bound for the possible solutions of the family of equations

$$\begin{aligned} &x'' + \lambda f(x)x' + (1-\lambda)\Gamma(t)x + \lambda\gamma_1(t,x)x + \lambda h(t,x) = \lambda e_1, \\ &x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0, \quad \lambda \in [0, 1[. \end{aligned} \tag{3.13}$$

We refer to [8] for checking that (3.12) can be put into the setting of the abstract Theorem IV.5 of [7]. If x is a possible solution of (3.13) for some $\lambda \in [0, 1[$, then, using Lemma 2, we obtain

$$\begin{aligned} 0 &= (2\pi)^{-1}\int_0^{2\pi}[(\bar{x} - \tilde{x}(t))(x''(t) + \lambda f(x(t))x'(t) \\ &\quad + \lambda\gamma_1(t, x(t))x(t) + \lambda h(t, x(t)) - \lambda e_1(t))]\,dt \\ &\geqq (\delta/2)|\tilde{x}|_{H^1}^2 - (|\alpha|_{L^1} + |e_1|_{L^1})(|\bar{x}| + |\tilde{x}|_{C^0}) \\ &\geqq (\delta/2)|\tilde{x}|_{H^1}^2 - \beta(|\bar{x}| + |\tilde{x}|_{H^1}) \end{aligned}$$

for $\beta > 0$ depending only on α and e_1. Consequently,

$$|\tilde{x}|_{H^1}^2 \leqq (2\beta/\delta)(|\bar{x}| + |\tilde{x}|_{H^1}). \tag{3.14}$$

Integrating the differential equation in (3.13) over $[0, 2\pi]$, we obtain

$$(1-\lambda)(2\pi)^{-1}\int_0^{2\pi}\Gamma(t)x(t)\,dt + \lambda(2\pi)^{-1}\int_0^{2\pi}[g(t, x(t)) - e_1(t)]\,dt = 0 \tag{3.15}$$

and, without loss of generality, we can assume that

$$\bar{\Gamma} > 0. \tag{3.16}$$

If $x(t) \geqq R$ for all $t \in [0, 2\pi]$, then (3.6) and (3.8) imply that

$$(1-\lambda)\bar{\Gamma}R \leqq 0,$$

a contradiction with (3.16). Similarly, if $x(t) \leqq r$ for all $t \in [0, 2\pi]$, we reach a contradiction. Consequently, there exists $\tau \in [0, 2\pi]$ such that

$$r < x(\tau) < R,$$

and hence, if $\bar{\tau}$ is such that

$$x(\bar{\tau}) = \bar{x},$$

then

$$\bar{x} = x(\bar{\tau}) = x(\tau) + \int_\tau^{\bar{\tau}} x'(s)\,ds,$$

which implies

$$\begin{aligned} |\bar{x}| &\leqq \max(R, -r) + (2\pi)^{1/2}\left(\int_0^{2\pi}(x'(s))^2\,ds\right)^{1/2} \\ &= \max(R, -r) + 2\pi|\tilde{x}|_{H^1}. \end{aligned} \tag{3.17}$$

Combining (3.14) and (3.17) we deduce the existence of some $\varrho_1 > 0$ such that

$$|x|_{H^1} < \varrho_1$$

for all the possible solutions of (3.13) with $\lambda \in [0, 1[$. Consequently, for those possible solutions, we have

$$|x|_{C^0} < \varrho_2, \quad |x'|_{L^1} < \varrho_3$$

for some ϱ_2, ϱ_3 depending only on ϱ_1, and hence (3.13) easily implies that

$$|x''|_{L^1} < \varrho_4$$

for some ϱ_4 independent of λ and x. This finally shows that

$$|x|_{C^1} < \varrho_5$$

for some ϱ_5 independent of $\lambda \in]0, 1[$ and all possible solutions x of (3.13). Taking

$$\Omega = \{x \in C^1([0, 2\pi]) : |x|_{C^1} < \varrho_5\}$$

in Theorem IV.5 of [7], the proof is complete.

Remark 1. Theorem 1 generalizes the earlier results in the following ways:

a) Lazer's theorem in [3] corresponds to f constant, e continuous, g continuous and independent of t, $a = A = 0$, $r = -R$ and

$$\lim_{|x| \to \infty} x^{-1} g(x) = 0.$$

b) Mawhin's theorem in [6] corresponds to Lazer's conditions except that f is an arbitrary continuous function.

c) Reissig's theorem in [9] corresponds to f and e continuous, g continuous and independent of t, $a = A = 0$, $r = -R$ and

$$\limsup_{|x| \to \infty} x^{-1} g(x) < 1.$$

d) Chang's theorem in [1] corresponds to f constant, e and g continuous, $a = A = 0$, $r = -R$ and

$$\lim_{|x| \to \infty} x^{-1} g(t, x) = 0$$

uniformly in $t \in [0, 2\pi]$.

e) Martelli's theorem in [4] corresponds to f, e, and g continuous, $a = A = 0$, $r = -R$ and

$$\limsup_{|x| \to \infty} x^{-1} g(t, x) < \frac{1}{2\pi + 1}$$

uniformly in $t \in [0, 2\pi]$.

f) Reissig's theorem in [10] corresponds to f, e, and g continuous, $a = A = 0$, $r = -R$ and

$$\limsup_{|x| \to \infty} x^{-1} g(t, x) \leqq q < 1, \tag{3.18}$$

uniformly in $t \in [0, 2\pi]$.

g) Gupta's theorem in [2] corresponds to f constant, e and g continuous, $a = A = 0$, $r = -R$ and (3.18).

Remark 2. We could have considered as well equations of the form

$$x'' + f(x)x' + g(t, x, x') = e(t) \tag{3.19}$$

at the expense of assuming that (3.2), (3.3), and (3.4) for $g(t, x, y)$ hold uniformly in $y \in \mathbb{R}$. Other sufficient conditions for the existence of a 2π-periodic solution of (3.19) can be found in the paper [5] of Martelli and Schuur.

Remark 3. An analysis of the proof of Theorem 1 shows that conditions (3.3) and (3.4) could have been replaced by the following more general one.

There exist γ, $\Gamma \in L^1[0, 2\pi]$ *such*

$$\gamma(t) \leqq \liminf_{|x| \to \infty} x^{-1} g(t, x) \leqq \limsup_{|x| \to \infty} x^{-1} g(t, x) \leqq \Gamma(t)$$

with $0 = \bar{\gamma} < \bar{\Gamma}$.

There exist real numbers a, A, r, R *with* $a \leqq A$ *and* $r < 0 < R$ *such that*

$$(2\pi)^{-1} \int_0^{2\pi} g(t, x(t))\, dt \geqq A$$

for all $x \in W^{2,1}(0, 2\pi)$ *satisfying* (2.8) *such that*

$$\min_{t \in [0, 2\pi]} x(t) \geqq R$$

and such that

$$(2\pi)^{-1} \int_0^{2\pi} g(t, x(t))\, dt \leqq a$$

for all $x \in W^{2,1}(0, 2\pi)$ *satisfying* (2.8) *such that*

$$\max_{t \in [0, 2\pi]} x(t) \leqq r.$$

Corollary 1. *Assume that there exists* $\Gamma \in L^1(0, 2\pi)$ *satisfying the conditions of Theorem 1 and that*

$$\liminf_{x \to -\infty} g(t, x) = +\infty, \qquad \limsup_{x \to -\infty} g(t, x) = -\infty$$

uniformly a.e. in $t \in [0, 2\pi]$. *Then the problem* (3.1) *has at least one solution for every* $e \in L^1(0, 2\pi)$.

Proof. Let $e \in L^1(0, 2\pi)$ be given; then there exists $R > 0$ such that

$$g(t, x) \geqq \bar{e}$$

for a.e. $t \in [0, 2\pi]$ and all $x \geqq R$ and there exists $r < 0$ such that

$$g(t, x) \leqq \bar{e}$$

for a.e. $t \in [0, 2\pi]$ and all $x \leqq r$. The existence of a solution for (3.1) then follows from Theorem 1.

One might suppose that if (3.2) is weakened to hold only as $x \to -\infty$ or only as $x \to +\infty$ then the conclusions of Theorem 1 still hold. This, however, is false. If $1/2 < \alpha < 1$ then elementary calculations show that the differential equation

$$x'' - \alpha^2 x^- + \frac{\alpha^2}{(1-2\alpha)^2} x^+ = 0$$

has a non-trivial 2π-periodic solution, and the equation

$$x'' - \alpha^2 x^- + \frac{\alpha^2}{(1-2\alpha)^2} x^+ = \sin(t)$$

has no 2π-periodic solution. Here $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$, and $x = x^+ - x^-$.

The following result is thus almost sharp in that if in (3.20a) the number 1/4 is replaced by any number larger than 1/4, the assertion is false.

Theorem 2. *Assume that there exists $\Gamma \in L^1(0, 2\pi)$ such that*

(3.20) $$\limsup_{x \to -\infty} x^{-1} g(t, x) \leqq \Gamma(t)$$

uniformly a.e. in $t \in [0, 2\pi]$ and such that

(3.20a) $$\Gamma(t) \leqq 1/4$$

for a.e. $t \in [0, 2\pi]$, with strict inequality on a subset of $[0, 2\pi]$ of positive measure. Assume moreover that there exists real numbers a, A, r, and R with $a \leqq A$ and $r < 0 < R$ such that

(3.21) $$g(t, x) \geqq A$$

for a.e. $t \in [0, 2\pi]$ and all $x \geqq R$ and

(3.22) $$g(t, x) \leqq a$$

for a.e. $t \in [0, 2\pi]$ and all $x \leqq r$. Then the problem (3.1) has at least one solution for each $e \in L^1(0, 2\pi)$ such that

(3.23) $$a \leqq \tilde{e} \leqq A.$$

For use in the proof of Theorem 2 we have two lemmas.

Lemma 3. *Let $\alpha \in \mathbb{R}$ and*

$$\Gamma \in L^1(\alpha, \alpha + 2\pi)$$

be such that, for a.e. $t \in [\alpha, \alpha + 2\pi]$ one has

(3.24) $$\Gamma(t) \leqq 1/4$$

with strict inequality on a subset of $[0, 2\pi]$ of positive measure. Then there exists

One might suppose that if (3.2) is weakened to hold only as $x \to -\infty$ or only as $x \to +\infty$ then the conclusions of Theorem 1 still hold. This, however, is false. If $1/2 < \alpha < 1$ then elementary calculations show that the differential equation

$$x'' - \alpha^2 x^- + \frac{\alpha^2}{(1-2\alpha)^2} x^+ = 0$$

has a non-trivial 2π-periodic solution, and the equation

$$x'' - \alpha^2 x^- + \frac{\alpha^2}{(1-2\alpha)^2} x^+ = \sin(t)$$

has no 2π-periodic solution. Here $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$, and $x = x^+ - x^-$.

The following result is thus almost sharp in that if in (3.20a) the number 1/4 is replaced by any number larger than 1/4, the assertion is false.

Theorem 2. *Assume that there exists $\Gamma \in L^1(0, 2\pi)$ such that*

(3.20) $$\limsup_{x \to -\infty} x^{-1} g(t, x) \leqq \Gamma(t)$$

uniformly a.e. in $t \in [0, 2\pi]$ and such that

(3.20a) $$\Gamma(t) \leqq 1/4$$

for a.e. $t \in [0, 2\pi]$, with strict inequality on a subset of $[0, 2\pi]$ of positive measure. Assume moreover that there exists real numbers a, A, r, and R with $a \leqq A$ and $r < 0 < R$ such that

(3.21) $$g(t, x) \geqq A$$

for a.e. $t \in [0, 2\pi]$ and all $x \geqq R$ and

(3.22) $$g(t, x) \leqq a$$

for a.e. $t \in [0, 2\pi]$ and all $x \leqq r$. Then the problem (3.1) has at least one solution for each $e \in L^1(0, 2\pi)$ such that

(3.23) $$a \leqq \bar{e} \leqq A.$$

For use in the proof of Theorem 2 we have two lemmas.

Lemma 3. *Let $\alpha \in \mathbb{R}$ and*

$$\Gamma \in L^1(\alpha, \alpha + 2\pi)$$

be such that, for a.e. $t \in [\alpha, \alpha + 2\pi]$ one has

(3.24) $$\Gamma(t) \leqq 1/4$$

with strict inequality on a subset of $[0, 2\pi]$ of positive measure. Then there exists and e_1 by

$$e_1(t) = e(t) - (1/2)(a + A).$$

We have

$$\limsup_{x \to -\infty} x^{-1} g_1(t, x) \leqq \Gamma(t)$$

uniformly a.e. in $t \in [0, 2\pi]$. Let $\delta > 0$ be the number associated to Γ by Lemma 3. Then there exists $r_1 > 0$ such that for a.e. $t \in [0, 2\pi]$ and all x with $x \leqq -r_1$, one has

$$0 \leqq x^{-1} g_1(t, x) \leqq \Gamma(t) + \delta/2$$

and for $x \geqq r_1$ one has

$$0 \leqq x^{-1} g_1(t, x).$$

Define γ_1 and h on $[0, 2\pi] \times \mathbb{R}$ as in the proof of Theorem 1 so that the differential equation in (3.1) is equivalent to

$$x''(t) + f(x(t))\, x'(t) + \gamma_1(t, x(t))\, x(t) + h(t, x(t)) = e_1(t) \tag{3.29}$$

with

$$0 \leqq \gamma_1(t, x)$$

for a.e. $t \in [0, 2\pi]$ and all $x \in \mathbb{R}$ and

$$0 \leqq \gamma_1(t, x) \leqq \Gamma(t) + \delta/2 \tag{3.30}$$

for a.e. $t \in [0, 2\pi]$ and $x \leqq 0$. Moreover, for some $\alpha \in L^1(0, 2\pi)$, a.e. $t \in [0, 2\pi]$, and all $x \in \mathbb{R}$, we have

$$|h(t, x)| < \alpha(t). \tag{3.31}$$

We apply Theorem IV.5 of [7] to (3.29) with periodic boundary conditions on $[0, 2\pi]$. As in [8] we must prove the existence of an *a* priori bound in $C^1[0, 2\pi]$ for the possible solutions of the family of equations

$$\begin{gathered} x'' + \lambda f(x)\, x' + (1 - \lambda)\, \Gamma(t)\, x + \lambda \gamma_1(t, x)\, x + \lambda h(t, x) = \lambda e_1, \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0, \quad \lambda \in [0, 1[. \end{gathered} \tag{3.32}$$

Suppose x is a possible solution to (3.32) and write x as $x = x^+ - x^-$ where $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. We will first show that x^- is bounded in $H^1(0, 2\pi)$. To this end let us first extend $x(t)$, $\Gamma(t)$, $h(t, x)$, $\gamma_1(t, x)$, and $e(t)$ 2π-periodically in t to all of $\mathbb{R}$, so that $\Gamma(t + 2\pi) = \Gamma(t)$ for all $t \in \mathbb{R}$, $h(t + 2\pi, x) = h(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$, etc., using the same notation for the periodic extensions as for the original functions. Then $x(t)$ is a 2π-periodic solution on $\mathbb{R}$ to the differential equation

$$x'' + \lambda f(x)\, x' + (1 - \lambda)\, \Gamma(t)\, x + \lambda \gamma_1(t, x)\, x + \lambda h(t, x) = \lambda e_1(t).$$

Suppose first that x^- has a zero in $[0, 2\pi]$ and let

$$\alpha = \min\{t : x^-(t) = 0 \text{ and } 0 \leqq t \leqq 2\pi\}.$$

Clearly $|x^-|_{H^1(0, 2\pi)} = |x^-|_{H^1(\alpha, \alpha + 2\pi)}$.

Let $[c, d]$ be any component of the support of x^- in $[\alpha, \alpha + 2\pi]$. Then letting $y(t) = -x^-(t)$ we have that $y(t)$ solves

$$y'' + \lambda f(y) y' + (1 - \lambda) \Gamma(t) y + \lambda \gamma_1(t, y) y + \lambda h(t, y) = \lambda e_1(t) \tag{3.33}$$

on $[c, d]$ with $y(c) = y(d) = 0$.

Multiply each side of equation (3.33) by $-y(t)$ and integrate over $[c, d]$ using $y(c) = y(d) = 0$, obtaining

$$\begin{aligned}\int_c^d (y'(t))^2 - [(1 - \lambda)\Gamma(t) + \lambda \gamma_1(t, y(t))] y^2(t) \\ = -\int_c^d \lambda h(t, y(t)) y(t)\, dt + \lambda \int_c^d e_1(t) y(t)\, dt.\end{aligned}$$

Since $y(t) \leqq 0$ we have by (3.30)

$$\int_c^d (y'(t))^2 - (\Gamma(t) + \delta/2) y^2(t)\, dt \leqq -\lambda \int_c^d (h(t, y(t)) + e_1(t)) y(t)\, dt. \tag{3.34}$$

Thus (3.34) holds for all components $[c, d]$ of the support of y in $[\alpha, \alpha + 2\pi]$, and (3.34) also holds for components of the complement in $[\alpha, \alpha + 2\pi]$ of the support of y, since on those intervals y is identically zero. We thus have, by summing over all such inequalities in $[\alpha, \alpha + 2\pi]$, that for some $C > 0$

$$\int_\alpha^{\alpha+2\pi} (y'(t))^2 - (\Gamma(t) + \delta/2) y^2(t)\, dt \leqq C |y|_{C^0}$$

and by Lemma 3

$$\delta/2\, |y|^2_{H^1(\alpha, \alpha+2\pi)} \leqq C |y|_{C^0}$$

and hence there is a constant $m > 0$ such that

$$|y|_{H^1(\alpha, \alpha+2\pi)} = |x^-|_{H^1(0, 2\pi)} \leqq m. \tag{3.35}$$

If x^- has no zero in $[0, 2\pi]$ then $x = -x^-$ and $x(t) < 0$ for all $t \in [0, 2\pi]$. Since $\Gamma(t) \leqq 1/4 < 1$ and x is a 2π-periodic solution one may now proceed as in the proof of Theorem 1 to show the existence of a bound on $|x|_{H^1} = |x^-|_{H^1}$.

Using $g_1(t, x) = \gamma_1(t, x) x + h(t, x)$ it is now convenient to write the differential equation in (3.32) in the form

$$x'' + \lambda f(x) x' + (1 - \lambda) \Gamma(t) x + \lambda g_1(t, x) = \lambda e_1(t). \tag{3.36}$$

Integrating (3.36) over $[0, 2\pi]$ and using the periodic boundary conditions we have

$$(1 - \lambda) \int_0^{2\pi} \Gamma(t) x(t)\, dt + \lambda \int_0^{2\pi} g_1(t, x(t))\, dt = \lambda \bar{e}_1, \tag{3.37}$$

and hence

$$\begin{aligned}(1 - \lambda) \int_{x(t)>0} \Gamma(t) x(t)\, dt + \lambda \int_{x(t)>0} g_1(t, x(t))\, dt \\ = \lambda \bar{e}_1 - (1 - \lambda) \int_{x(t)\leqq 0} \Gamma(t) x(t)\, dt - \lambda \int_{x(t)\leqq 0} g_1(t, x(t))\, dt,\end{aligned}$$

and by (3.35) there is a number $m_1 > 0$ such that

$$(1-\lambda) \int_{x(t)>0} \Gamma(t)\, x(t)\, dt + \lambda \int_{x(t)>0} g_1(t, x(t))\, dt \leqq m_1 . \tag{3.38}$$

Now using (3.35) and (3.38) it is easy to show there is an $m_2 > 0$ with

$$\int_0^{2\pi} |(1-\lambda)\,\Gamma(t)\, x(t) + \lambda g_1(t, x(t))|\, dt \leqq m_2 , \tag{3.39}$$

independently of $0 \leqq \lambda < 1$.

Now since

$$x'' + \lambda f(x)\, x' = -(1-\lambda)\,\Gamma(t) - \lambda g_1(t, x) + \lambda e_1(t)$$

we have by (3.39) and Lemma 4

$$|\tilde{x}|_{H^1} \leqq k(m_2 + |e_1|_{L^1})$$

for some $k > 0$, independent of $\lambda \in [0, 1[$.

We may now use (3.37) as in the proof of Theorem 1 to show the existence of a $\tau \in [0, 2\pi]$ with $r < x(\tau) < R$ and hence that $|x|_{H^1} \leqq C_1$ for some constant $C_1 > 0$. Using (3.32) one can now show, again as in Theorem 1, the existence of a constant $C_2 > 0$ with

$$|x|_{C^1} < C_2$$

independent of x and λ. Taking

$$\Omega = \{x \in C^1([0, 2\pi]) : |x|_{C^1} < C_2\}$$

in Theorem IV.5 of [7], we may complete the proof.

Remark 4. It is clear that Theorem 2 remains true if the limit in (3.20) is taken as $x \to +\infty$. Theorem 2 generalizes the earlier results of Lazer [3], Mawhin [6], Chang [1], Reissig [11], Schmitt [12], and Ward [13], [14]. The conditions in Ward [13], [14] are similar to those in Theorem 2 except that in [14], for second order vector equations, and [13], for nth order scalar equations, the coefficient of x' (and of higher derivatives) must be a constant; here we allow an arbitrary continuous f. Moreover, the bound on $x^{-1} g(t, x)$ as $x \to -\infty$ is improved to the almost sharp result here; in [14] the bound was any $\alpha \geqq 0$ with $\alpha < (2\pi)^{-2}$. The result in [14] generalized those of [11] and [12]; the bound on $x^{-1} g(t, x)$ as $x \to -\infty$ in [14] is weaker than that needed to apply the result of [13] to second order scalar ordinary differential equations.

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