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Metrically Well-Set Minimization Problems

Ewa Bednarczuk[1] and Jean-Paul Penot[2]

[1] Systems Research Institute, Polish Academy of Sciences, Newelska 6, 01-447 Warsaw, Poland

[2] Faculte des Sciences, Universite de Pau, Av. de l'Universite, 64000 Pau, France

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Abstract. A concept of well-posedness, or more exactly of stability in a metric sense, is introduced for minimization problems on metric spaces generalizing the notion due to Tykhonov to situations in which there is no uniqueness of solutions. It is compared with other concepts, in particular to a variant of the notion after Hadamard reformulated via a metric semicontinuity approach. Concrete criteria of well-posedness are presented, e.g., for convex minimization problems.

Key Words. Sensitivity of solutions, Well-posedness, Mathematical programming in abstract spaces.

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1. Introduction

In [2] we observed that among the three requirements characterizing well-posed minimization problems, namely, existence and uniqueness of solutions and convergence of minimizing sequences, the uniqueness condition is not as essential as the other conditions. We introduced several variants of this notion and related them by proving, in particular, that Hadamard well-posedness is essentially equivalent to Tykhonov well-posedness.

It is the purpose of this paper to follow a similar line of thought in the special framework of metric spaces and to prove verifiable criteria for well-posedness in our generalized sense. In metric spaces (and, more generally, in uniform spaces, but we refrain from adopting such framework although only minor changes would be necessary) we dispose of the concept of Hausdorff (or metric) continuity for multifunctions. This notion differs from the classical notions of upper and lower semicontinuity of multifunctions, so that the notion of well-set minimization problem introduced here does not coincide with the notion studied in [2]. Recall that a multifunction $F: X \to Y$ from a topological space X to a metric space (Y, d) is said to be *metrically upper continuous* or *upper hemicontinuous* (u.h.c.) or *upper-Hausdorff-continuous* at x_0 if for each $\varepsilon > 0$ there exists a neighborhood V of x_0 in X such that for each $x \in V$ and each $y \in F(x)$ we have $d(y, F(x_0)) < \varepsilon$, or $F(x) \subset B(F(x_0), \varepsilon)$, where for $y \in Y$ and a subset Z of Y we set $d(y, Z) = \inf\{d(y, z): z \in Z\}$ and $B(Z, r) = \{y \in Y: d(y, Z) < r\}$ for $r \in \mathbb{R}$. The multifunction F is said to be *lower hemicontinuous* (l.h.c.) at x_0 if for each $\varepsilon > 0$ there exists a neighborhood V of x_0 such that $F(x_0) \subset B(F(x), \varepsilon)$ for each $x \in V$.

The minimization problem

$$(\mathrm{P}_0) \quad \text{minimize}\, f_0(x) \qquad \text{for} \quad x \in A_0,$$

where A_0 is a nonempty subset of a metric space (X, d) and f_0 is a lower semicontinuous function from X into $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ with $m := \inf f_0(A_0) > -\infty$, is said to be *metrically well-set* if the multifunction assigning to every ε the set S_ε of approximate solutions is u.h.c. at $\varepsilon = 0$, where $S_\varepsilon = \{x \in A_0: f_0(x) \leq m + \varepsilon\}$. In other words, (P_0) is *metrically well-set* if for any minimizing sequence (x_n) of (P_0) we have $\lim_{n\to\infty} d(x_n, S_0) = 0$. As kindly pointed out by the referee this condition is not new but corresponds to a concept often used in the Russian literature (see for instance [5] where it is exploited in connection with the regularization method of Tykhonov). This condition is easily seen to be weaker than the condition of well-posedness in the generalized sense imposed by Furi and Vignoli [8], [9], namely, that any minimizing sequence is compact (and weaker than the generalization to nets considered by Čoban *et al.* [4]). In particular it does not imply that the set of solutions is compact. We devote Section 2 to a characterization of this notion in terms of compactness concepts (Theorem 2.5). We relate this condition of well-posedness to the Hadamard requirement of stability of the set of solutions to problems with perturbed data f_w, A_w depending on a parameter w. We establish some connections with the notions introduced in [2]. In fact since here we deal with metric notions of upper semicontinuity and not topological ones our results seem to be more remote to [2] than to other contributions we have been acquainted with since we write the first version of this paper (i.e., [17], [4], [21], and [22]). However, here X is not supposed to be complete and since we do not suppose uniqueness we cannot make use of the argument that a decreasing sequence of closed subsets of a complete metric space whose diameters tend to zero converges to a singleton as in [8], [9], [17], [21], and [22]. Still our comparison with Hadamard well-posedness given in Section 3 is quite simple and natural (see Theorem 3.1).

Besides the examples we provide, the work of Chavent (see [3] and its references) for instance shows that dropping uniqueness is not a spurious generalization but has real-world applications.

2. Metrically Well-Set Problems and Their Characterizations

The following formal definition is a rephrasing of the notion given in the introduction.

Definition 2.1. The problem (P_0) is said to be *metrically well-set* or *M-well-set* if the multifunction $T: \mathbb{R} \to A_0$ given by $T(r) = f_0^{-1}(-\infty, r) \cap A_0$ is u.h.c. at $m = \inf f_0(A_0)$. Equivalently, (P_0) is M-well-set if for any minimizing sequence (x_n) we have $d(x_n, S_0) \to 0$.

Since T is u.h.c. whenever it is u.s.c., a T-well-set problem in the sense of [2] (i.e., T is u.s.c. at m) is M-well-set. When (P_0) has a unique solution the two definitions coincide. Moreover, in this case (P_0) is well-posed in the sense of [8] (i.e., the diameter $\delta(T(r))$ of $T(r)$ converges to 0 as r tends to m_+); the converse is true when (X, d) is complete. Then (P_0) is well-posed iff any minimizing sequence is a Cauchy sequence.

More generally, when the solution set $S_0 = A_0 \cap f_0^{-1}(m)$ is compact, (P_0) is M-well-set iff it is T-well-set. The following lemma gives a partial generalization of this observation (recall that in the following f_0 is l.s.c. and hence S_0 is closed in A_0).

Lemma 2.2. *Suppose the open balls of (A_0, d) are connected and the boundary ∂S_0 of S_0 in A_0 is compact. Then (P_0) is M-well-set iff (P_0) is T-well-set.*

Proof. When the balls of a metric space (Z, d) are connected, for any nonempty closed subset Y of Z and any $x \in Z \setminus Y$ we have

$$d(x, Y) = d(x, \partial Y)$$

since for each $r > d(x, Y)$ the ball $B(x, r)$ meets Y and $Z \setminus Y$, and hence meets ∂Y. Let us prove that when ∂Y is compact, for any open subset V of Z containing Y there exists $r > 0$ with $B(Y, r) \subset V$. Otherwise, we can find a sequence (x_n) in $Z \setminus V$ with $x_n \in B(Y, 1/n)$; then we have $d(x_n, \partial Y) < 1/n$. As ∂Y is compact (x_n) has a cluster point $\bar{x}$ in ∂Y. Since $Z \setminus V$ is closed we get $\bar{x} \in (Z \setminus V) \cap \partial Y$, a contradiction, Taking $Z = A_0$, $Y = S_0$ we get that, for any multifunction $T: \mathbb{R} \to A_0$ with $T(m) = S_0$, T is u.s.c. at m when it is u.h.c. at m. □

In general, the notion of an M-well-set minimization problem is less restrictive than the notion of a T-well-set minimization problem even for a strongly structured class of problems such as linear programming problems or quadratic programming problems.

Example 2.3. Let $X = \mathbb{R}^2$, $A_0 = \{(x_1, x_2) \in \mathbb{R}^2,\ 0 \leq x_1 \leq 1,\ x_2 \geq 0\}$, so that $f(x_1, x_2) = x_1$, so that $T(0) = S_0 = \{0\} \times \mathbb{R}_+$, and $T(r) = [0, r] \times \mathbb{R}_+$ for $r \in [0, 1]$ so that T is u.h.c. at 0 but not u.s.c. at 0.

Example 2.4. Let X be a Hilbert space with scalar product $(\cdot|\cdot)$ and let $Q: X \to X$ be a symmetric continuous linear operator with $f_0(x) = 1/2(Qx|x) \geq 0$ for each

$x \in X$. Using the Lax–Milgram lemma on $\ker Q^{\perp}$ and the fact that a positive semidefinite nondegenerate operator Q is coercive (or positive definite in the sense that there exists $\alpha > 0$ such that $(Qv|v) \geq \alpha\|v\|^2$ for all v in X) it is easy to show that the problem (P_0) of minimizing f_0 over $A_0 := X$ is M-well-set iff the range of Q is closed. It is T-well-set iff it is generalized well-posed in the sense of Furi and Vignoli [8], [9] iff it is well-posed in the sense of Tykhonov [23] iff Q is an isomorphism.

The regularization method of Tykhonov can be seen as a restiction to a class of perturbations which is sufficiently narrow to yield a continuous dependence of some solution on a parameter, even for the ill-posed problem of minimizing

$$f_0(x) = 1/2\|Qx - b\|^2,$$

where Q is as above and $b \in X$ (see, for instance, [23] and Theorem 46E of [25]).

The following characterization is parallel to the one presented in [2]; it is similar to the characterization of [8], the difference lying in the fact that here the solution set is not supposed to be compact. Let us recall that *Kuratowski's measure of noncompactness* of a subset Y of X is the infimum $\kappa(Y)$ of the family of $r \in \mathbb{R}_+$ such that Y can be covered by a finite family of subsets of diameter less than r. Following [6] we call *precompact* a filter base $\mathscr{B}$ on X such that for any $\varepsilon > 0$ there exists $B \in \mathscr{B}$ and a finite covering of B by balls of radius ε. The *punctured* (resp. *hollow*) *minimizing filter base* $\mathscr{P}_0$ (resp. $\mathscr{Q}_0$) is given by

$$\mathscr{P}_0 = \{T(r)\backslash \operatorname{int} S_0 \colon r > 0\}$$

(resp. $\mathscr{Q}_0 = \{T(r)\backslash S_0 \colon r > 0\}$), with $S_0 = \{x \in A_0 \colon f_0(x) = m\}$, $m = \inf f_0(A_0)$.

Theorem 2.5. *The following implications hold for the assertions listed below:*

$$(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftarrow (d) \Rightarrow (e).$$

When (A_0, d) is complete we have (c) $\Rightarrow$ (d). *When the closed (or open) balls of A_0 are connected we have* (e) $\Rightarrow$ (a); *if, moreover, (A_0, d) is complete all these assertions are equivalent*:

(a) *$\mathscr{P}_0$ is precompact.*
(b) *$\mathscr{Q}_0$ is precompact and the boundary ∂S_0 of S_0 in A_0 is precompact.*
(c) $\lim_{\varepsilon \to 0_+} \kappa(T(m + \varepsilon)\backslash S_0) = 0$ *and S_0 is precompact.*
(d) (P_0) *is T-well-set and ∂S_0 is compact.*
(e) (P_0) *is M-well-set and ∂S_0 is precompact.*

Proof. The implications (a) $\Rightarrow$ (b) and (b) $\Leftrightarrow$ (c) follow from the definitions since $\kappa(Y) \leq \kappa(Z)$ when $Y \subset Z$ and since for any $\varepsilon > 0$ the inclusion $S_0 \subset T(m + \varepsilon)$ holds, S_0 and $T(m + \varepsilon)$ are closed in A_0 so that we have

$$T(m + \varepsilon)\backslash \operatorname{int} S_0 = (T(m + \varepsilon)\backslash S_0) \cup \partial S_0.$$

Since $\kappa(Y \cup Z) = \max(\kappa(Y), \kappa(Z))$ and $\kappa(Z) = 0$ iff Z is precompact, (c) is equivalent to the fact that $\lim_{\varepsilon \to 0_+} \kappa(T(m + \varepsilon)\backslash \operatorname{int} S_0) = 0$ which is equivalent to (a).

By Theorem 1.4 of [2], (d) is equivalent to the compactness of $\mathscr{P}_0$. By using Proposition 7.1 of [6], this implies that $\mathscr{P}_0$ is precompact. The converse is true when X is complete.

Now let us suppose that (e) holds and the closed (or open) balls of A_0 are connected. Let us prove that (c) holds. Given $\alpha > 0$ we can find $\beta > 0$ such that $T(m+\beta) \subset B(S_0, \alpha/2)$; moreover, we can find a finite subset $\{x_1, \ldots, x_k\}$ of ∂S_0 such that the family $\{B(x_i, \alpha/2): i = 1, \ldots, k\}$ is a covering of ∂S_0. Then for any $\varepsilon \in [0, \beta]$ and any $x \in T(m+\varepsilon)\backslash S_0$ we can find $i \in \{1, \ldots, k\}$ such that $x \in B(x_i, \alpha)$ since $d(x, \partial S_0) = d(x, S_0) < \alpha/2$ so that there exists $y \in \partial S_0$ with $d(x, y) < \alpha/2$ and some $i \in \{1, \ldots, k\}$ with $d(y, x_i) < \alpha/2$. Thus $\kappa(T(m+\varepsilon)\backslash S_0) \leq 2\alpha$ for $\varepsilon \in [0, \beta]$ and (c) holds. The last claim follows from the preceding ones. □

3. Comparison with Hadamard Well-Posedness

The classical definition of well-posedness introduced after Hadamard pertains to the notion of perturbation of (P_0). A *perturbation* of (P_0) is a quadruple (W, w_0, F, f), or in short (F, f), where w_0 is a point of a topological space W, $F: W \to X$ is a multifunction, $f: W \times X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is an extended real-valued function such that $F(w_0) = A_0$, $f_{w_0} = f_0$, where f_w is given by $f_w(x) = f(w, x)$ for $x \in X$. The *performance function* is given by

$$p(w) := \inf\{f_w(x): x \in F(w)\} \qquad \text{for} \quad w \in W,$$

the *solution* and the *approximate solution multifunctions* are defined respectively by

$$S(w) := \{x \in F(w): f_w(x) = p(w)\},$$

$$\hat{S}(\varepsilon, w) := \hat{S}_\varepsilon(w) := \{x \in F(w): f_w(x) \leq p(w) \mathbin{\hat{+}} \varepsilon\}$$

for $w \in W$, $\varepsilon \in \mathbb{R}_+$, where for $r \in \overline{\mathbb{R}}$, $s \in \mathbb{R}_+\backslash\{0\}$, $r \mathbin{\hat{+}} s$ is defined by $r \mathbin{\hat{+}} s = r + s$ for $r \in \mathbb{R}$, $r \mathbin{\hat{+}} s = +\infty$ for $r = +\infty$, $r \mathbin{\hat{+}} s = -s^{-1}$ for $r = -\infty$, and $r \mathbin{\hat{+}} s = r$ for $r \in \overline{\mathbb{R}}$, $s = 0$. We assume throughout that $m = p(w_0)$ is finite and we set $S_0 = S(w_0)$ as before.

The family $(f_w)_{w \in W}$ is said to converge *uniformly* to f_0 as w tends to w_0 (and we write $(f_w) \xrightarrow{u} f_0$) if for each $\varepsilon > 0$ there exists W_ε in the family $\mathscr{N}(w_0)$ of neighborhoods of w_0 such that for any $w \in W_\varepsilon$, $x \in X$ we have

$$-(-f_0(x)) \mathbin{\hat{+}} \varepsilon \leq f_w(x) \leq f_0(x) \mathbin{\hat{+}} \varepsilon.$$

Let us call (P_0) *H_m-well-set* (or *metrically well-set in the sense of Hadamard*) if for any perturbation (F, f) of (P_o) with F u.h.c. and l.h.c. at w_0, and (f_w) converging uniformly to f_0, $(f_w) \xrightarrow{u} f_0$, the corresponding solution multifunction S is u.h.c. at w_0. The problem (P_0) is called *H_{ms}-well-set* if for any perturbation (F, f) of (P_0) with F u.h.c. and l.s.c., and (f_w) converging uniformly to f_0, $(f_w) \xrightarrow{u} f_0$, the solution multifunction S is u.h.c. at w_0. It will be said to be *O_m-well-set* if S is u.h.c. at w_0 whenever F is constant and $(f_w) \xrightarrow{u} f_0$. The following result extends Proposition 1 of [22] as here uniqueness is omitted; moreover, convergence of the constraint is different.

Theorem 3.1. *We have the following implications for* (P_0):

$$H_{ms}\text{-well-set} \;\Rightarrow\; H_m\text{-well-set} \;\Rightarrow\; O_m\text{-well-set} \;\Rightarrow\; M\text{-well-set}.$$

If f_0 *is uniformly continuous around* A_0, *then all these assertions are equivalent.*

Proof. The first implication follows from the fact that any l.h.c. multifunction is l.s.c. The second implication is obvious. Let us prove the third by setting, for $W = \mathbb{R}_+$, $w_0 = 0$, $F(w) = A_0$ for each $w \in W$, and for $(w, x) \in W \times X$,

$$f(w, x) = \max\{f_0(x), m + w\},$$

where $m = \inf f_0(A_0)$. Then, for $w \in W$, $S(w)$ is the set of w-approximate solutions of (P_0). Moreover, $f(0, x) = f_0(x)$ and, for any $(w, x) \in W \times X$,

$$f_0(x) \leq f(w, x) \leq f_0(x) \hat{+} w,$$

so that $(f_w) \xrightarrow{u} f_0$. Since (P_0) is O_m-well-set, $S(w) = T(m + w)$ is u.h.c. at w_0 and (P_0) is M-well-set.

Now let us suppose f_0 is uniformly continuous around A_0 and (P_0) is M-well-set. Let us observe that, for any perturbation (F, f) such that $(f_w) \xrightarrow{u} f_0$ and F is u.h.c. and l.s.c., the perturbation function p is u.s.c. (in fact l.s.c. of F and u.s.c. of f at (w_0, x_0) for some $x_0 \in S(w_0)$ would suffice).

Given $\varepsilon > 0$, let us find $V \in \mathcal{N}(w_0)$ such that $S(w) \subset B(S(w_0), \varepsilon)$ for $w \in V$. As (P_0) is M-well-set we can find $\delta > 0$ such that

$$T(m + \delta) = \{x \in A_0 : f_0(x) \leq p(w_0) + \delta\} \subset B(S(w_0), \varepsilon/2).$$

As f_0 is uniformly continuous around A_0 we can find $\alpha \in [0, \varepsilon/2]$ such that

$$|f_0(x) - f_0(x')| \leq \delta/4$$

whenever $x \in A_0$, $x' \in B(x, \alpha)$. Let $V \in \mathcal{N}(w_0)$ be such that

$$|f_0(x) - f_w(x)| \leq \delta/4$$

for $w \in V$, $x \in X$ and such that

$$p(w) \leq p(w_0) + \delta/2, \qquad F(w) \subset B(F(w_0), \alpha/2)$$

for $w \in V$. Then for each $w \in V$ and each $x \in S(w)$ we can find some $x' \in F(w_0)$ such that $d(x, x') < \alpha$ so that

$$f_0(x') \leq f_0(x) + \delta/4 \leq f_w(x) + \delta/2 = p(w) + \delta/2 \leq p(w_0) + \delta.$$

By our choice of δ, we obtain $x' \in B(S(w_0), \varepsilon/2)$ and, as $d(x, x') < \varepsilon/2$, $x \in B(S(w_0), \varepsilon)$. □

Let us observe that the preceding proof shows that when f_0 is uniformly continuous around A_0 the problem (P_0) is M-well-set iff (P_0) is H_{mp}-*well-set* in the following sense: for any perturbation (F, f) such that $(f_w) \xrightarrow{u} f_0$, F is u.h.c. and p is u.s.c. at w_0, the multifunction S is u.h.c. at w_0.

In the following variants the strong requirements of the uniform continuity of f_0 or uniform convergence of (f_w) are relaxed but the continuity assumption of F is stronger. These result are more powerful than Theorems 3.1 and 3.4 of [11], respectively, since the multifunction S is u.s.c. at w_0 whenever S is u.h.c. at w_0 and $S(w_0)$ is compact. The solution set $S(w_0)$ is compact when (P_0) is well-posed in the sense of [8] and [11] which is stronger than M-well-setting. The proof of the first variant is similar to the proof of Theorem 3.1 of [11] and is omitted; the second one requires more adjustments.

Theorem 3.2. *Let (F, f) be a perturbation of (P_0) such that F and p are u.s.c. at w_0. Suppose (f_w) converges uniformly to f_0 and f_0 is uniformly continuous on bounded subsets of X. Then, if (P_0) is M-well-set, the multifunction S is u.h.c. at w_0.*

Theorem 3.3. *Let (F, f) be a perturbation of (P_0) such that F and p are u.s.c. at w_0 and S_0 is bounded. Suppose that for each $w \in W$ the level sets of $f_w|_{A_0}$ are connected. Suppose that (f_w) converges uniformly to f_0 on bounded subsets of X and f_0 is uniformly continuous on bounded subsets of X. Then, if (P_0) is M-well-set, the multifunction S is u.h.c. at w_0.*

In practice the assumption on the level sets of f_w is checked by means of a convexity or quasi-convexity property.

Proof. Suppose this is not the case. Then we can find $\varepsilon > 0$ and a net $((w_i, x_i))_{i \in I}$ in the graph of S with $\lim_i w_i = w_0$, $d(x_i, S_0) \geq \varepsilon$. As F is u.s.c. at w_0, we may suppose $d(x_i, F(w_0)) < \varepsilon/2$ for each $i \in I$, so that there exists $x'_{ij} \in A_0 = F(w_0)$ with $d(x_i, x'_i) < \varepsilon/2$ and $\lim_i d(x_i, x'_i) = 0$. Then we have $d(x'_i, S_0) \geq \varepsilon/2$.

Let us first suppose there exists a cofinal subset J of I such that $(x_j)_{j \in J}$ is bounded. Then $(x'_j)_{j \in J}$ is also bounded and

$$\limsup_j f_0(x'_j) = \limsup_j f_0(x_j) = \limsup_j f(w_j, x_j) = \limsup_j p(w_j) \leq m.$$

As (P_0) is M-well-set we obtain $d(x'_j, S_0) \to 0$, hence $d(x_j, S_0) \to 0$, a contradiction.

As F is u.s.c. at w_0, using [7] we have

$$\lim_{w \to w_0} \kappa(F(w) \backslash F(w_0)) = 0,$$

so that if $J = \{i \in I : x_i \in F(W_0) \backslash F(w_0)\}$, where $W_0 \in \mathcal{N}(w_0)$ is such that

$$\kappa(F(W_0) \backslash F(w_0)) \leq 1,$$

$(x_j)_{j \in J}$ is bounded and J is not cofinal. Then $K = \{i \in I \backslash J : x_i \in A_0\}$ is cofinal. Let x_0 be any element of S_0 and for $k \in K$ let

$$r_k = \max\{p(w_k), f(w_k, x_0)\},$$

$$L_k = \{x \in A_0 : f(w_k, x) \leq r_k\}.$$

Then $L_k \backslash B(S_0, \varepsilon)$ contains x_k and is closed. Since L_k is connected the sets cl $B(S_0, \varepsilon/2)$ and $L_k \backslash B(S_0, \varepsilon)$ cannot cover L_k. Therefore, there exists

$$z_k \in L_k \cap B(S_0, \varepsilon) \backslash B(S_0, \varepsilon/2).$$

As S_0 is bounded, (z_k) is bounded so that

$$\limsup_k f_0(z_k) = \limsup_k f(w_k, z_k) \le \limsup_k r_k \le p(w_0)$$

as $(f(w_k, x_0))_{k \in K}$ converges to $f_0(x_0) = p(w_0)$ and $\limsup_k p(w_k) \le p(w_0)$. Now since (P_0) is M-well-set we get $d(z_k, S_0) \to 0$, a contradiction. □

Example 3.4. Let (E, d) be a metric space and let C be a closed subset of E. Suppose that for some $z_0 \in E$ the best approximation problem

$$(P_0) \quad \text{minimize}\{d(z_0, x): x \in C\}$$

is M-well-set and the sets $\{x \in C: d(z_0, x) \le r\}$ are connected for each $r \in \mathbb{R}_+$. Then for any topological space W, and $w_0 \in W$, and any continuous map $z: W \to E$ with $z(w_0) = z_0$, any u.h.c. and l.s.c. multifunction $F: W \to E$ with $F(w) = C$, the best approximation multifunction $S: W \to E$ given by

$$S(w) = \{x \in F(w): d(z(w), x) = d(z(w), F(w))\}$$

is u.h.c. since the performance function $p(w) = d(z(w), F(w))$ is u.s.c. and the objective function $f_0(\cdot) = d(z_0, \cdot)$ is uniformly continuous.

Let us now discuss the assumptions of Theorem 3.1 and present some comments.

Remark 3.5. We can supplement Theorem 3.1 with a result similar to Proposition 2.9 of [2]. Suppose $F: W \to X$ is such that $F(w_0) \neq \emptyset$ and for any uniformly continuous function $f_0: X \to \mathbb{R}$ such that the problem (P_0) is M-well-set the solution multifunction S is u.h.c. at w_0. Then F is u.h.c. and l.s.c. at w_0. To see that F is u.h.c. at w_0 it suffice to take for f_0 a constant function.

Now let us suppose that F is not l.s.c. at w_0: there exist some $x_0 \in F(w_0)$, $r > 0$, and a net $(w_i)_{i \in I}$ with limit w_0 such that $d(x_0, F(w_i)) \ge r$ for each $i \in I$. Let $f_0(x) = \min(r, d(x_0, x))$ then the associated problem (P_0) is M-well-set and we get a contradiction with our assumption as $S(w_0) = \{x_0\}$, $S(w_i) = F(w_i)$ for $i \in I$.

The uniform continuity assumption of the objective function f_0 cannot be dropped in Theorem 3.1 as is shown by the following example.

Example 3.6. Let $W = \mathbb{R}_+$, $X = \mathbb{R}^2$, and $F(w) = w \times [0, 1 + 1/w]$ for $w > 0$ and $F(0) = \{0\} \times [0, +\infty)$, $f(w, x) = f_0(x)$ for $(w, x) \in W \times X$ with

$$f_0(x_1, x_2) = x_2 - (x_2 - 1)^+(x_1 + 1),$$

where $r^+ = \max(r, 0)$ for $r \in \mathbb{R}$. Then $p(w) = 0$ for any $w \in W$ and $S(0) = \{(0, 0)\}$, $S(w) = \{(w, 0), (w, 1 + 1/w)\}$ for $w > 0$ so that S is not u.h.c. at w_0. The problem (P_0) is M-well-set, F is u.h.c. and l.s.c. but f_0 is not uniformly continuous on X.

The fact that the solution multifunction S need not be u.h.c. if (P_0) is not M-well-set is illustrated by the following example.

Example 3.6. Let $W = [0, 1]$, $X = \mathbb{R}$, $F(0) = \mathbb{R}_+$, $F(w) = [w, 1/w]$ for $w \in (0, 1]$, $f(w, x) = f_0(x) = \min(x, 1/x)$. Then $S(0) = \{0\}$, $S(w) = \{w, 1/w\}$ for $w \in (0, 1]$ so that S is not u.h.c. although f_0 is uniformly continuous, F and p are continuous ($p(w) = w$). Here (P_0) is not M-well-set.

4. Firmness and Well-Posedness

A characterization of well-posed minimization problems in terms of firm functions has been given by Vainberg [24] and has been extended to sequences of minimization problems by Zolezzi [26]. Let us extend these characterizations to our framework in which uniqueness is dropped and compactness conditions are relaxed.

Recall that a function $c: \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *firm* (or forcing [15], or admissible [26]) if any sequence $(t_n) \subset \mathbb{R}_+$ such that $c(t_n) \to 0$ has limit 0. Let us call a family $(c_w)_{w \in W}$ of functions from $\mathbb{R}_+$ into $\mathbb{R}_+$ *firm* (at w_0) if any sequence (t_n) of $\mathbb{R}_+$ for which there exists a sequence (w_n) in W with $\lim_n w_n = w_0$, $\lim_n c_{w_n}(t_n) = 0$ has limit 0. Obviously, if $c_w = c$ for each $w \in W$, (c_w) is firm iff c is firm. Some criteria ensuring that a family (c_w) is firm will be presented later.

Given the parametrized optimization problem

$$(P_w) \quad \text{minimize } f_w(x) \qquad \text{for} \quad x \in F(w) \subset X,$$

where W, X, F, f_w are as above, let us set

$$c_w(t) = \inf\{|f_w(x) - p(w)|: x \in F(w),\ d(x, S(w_0)) = t\}.$$

The following is a variant of Theorem 1 of [26]; observe that in contrast with this result we have a complete characterization; moreover, assumptions (1) and (2) of [26] are dropped.

Theorem 4.1. *Let W, X, F, (f_w), and (c_w) be as above. Then the following assertions are equivalent:*

(a) *For any function ε: $W \to \mathbb{R}_+$ with $\lim_{w \to w_0} \varepsilon(w) = 0$ the multifunction $w \to \hat{S}_{\varepsilon(w)}(w)$ is u.h.c. at w_0.*
(b) *(c_w) is firm.*

Proof. (a) $\Rightarrow$ (b) Let (t_n) and w_n be sequences in $P = (0, +\infty)$ and W, respectively, such that $\lim_n w_n = w_0$, $\lim_n c_{w_n}(t_n) = 0$. By definition of c_{w_n} we can find $x_n \in F(w_n)$ such that $d(x_n, S(w_0)) = t_n$ and

$$|f_{w_n}(x_n) - p(w_n)| \leq c_{w_n}(t_n) + 2^{-n}.$$

Let ε: $W \to \mathbb{R}_+$ be given by $\varepsilon(w) = 0$ for $w \in W \backslash \{w_n: n \in N^*\}$, $w = w_0$, and

$$\varepsilon(w) = \{\sup c_{w_n}(t_n) + 2^{-n}: n \in N^*,\ w_n = w\}$$

for $w \in \{w_n\}$, $w \neq w_0$, so that $\lim_{w \to w_0} \varepsilon(w) = 0$ and $x_n \in \hat{S}_{\varepsilon(w_n)}(w_n)$ for each $n \in N^*$. Thus, the sequence $(t_n)_{n \in N^*} = (d(x_n, S(w_0)))_{n \in N^*}$ has limit 0 and (c_w) is firm.

(b) $\Rightarrow$ (a) Suppose $\varepsilon: W \to \mathbb{R}_+$ is such that $\lim_{w \to w_0} \varepsilon(w) = 0$ but $w \to \hat{S}_{\varepsilon(w)}(w)$ is not u.h.c. at w_0: we can find $\alpha > 0$ and sequences $(w_n)_{n \in N}$, $(x_n)_{n \in N}$ in W and X, respectively, with $\lim_n w_n = w_0$, $x_n \in \hat{S}_{\varepsilon(w_n)}(w_n)$ for each $n \in N$ and $d(x_n, S(w_0)) \geq \alpha$ for each n. Setting $t_n = d(x_n, S(w_0))$ we get

$$c_{w_n}(t_n) \leq |f_{w_n}(x_n) - p(w_n)| \leq \varepsilon(w_n)$$

so that (c_w) is not firm, a contradiction. □

Remark 4.2. When p is continuous at w_0 (this is supposed in assumption (2) of [26] for $W = N \cup \{+\infty\}$, $w_0 = +\infty$) the preceding conditions are equivalent to

(c) the family (d_w) defined as

$$d_w(t) = \inf\{|f_w(x) - p(w_0)|: x \in f(w), d(x, S(w_0)) = t\}$$

is firm.

The following lemma makes clear the connections of what precedes with Theorem 1 of [26].

Lemma 4.3. *Let $(c_w)_{w \in W}$ be a family of functions from $\mathbb{R}_+$ into $\mathbb{R}_+$ indexed by a topological space W. Suppose $w_0 \in W$ has a countable basis of neighborhoods $(W_n)_{n \in N}$.*

(a) *If (c_w) is firm, then the functions $\hat{c}$ and $\bar{c}$ defined as*

$$\hat{c}(t) = \lim \inf_{(s, w) \to (t, w_0)} c_w(s),$$

$$\bar{c}(t) = \lim \inf_{w \to w_0} c_w(t)$$

are firm.

(b) *If $\tilde{c} = \inf_{w \in W} c_w$ is firm, then $(c_w)_{w \in W}$ is firm.*

Note that $\hat{c}$ is epi-limit inferior of the family $(c_w)_{w \in W}$ as $w \to w_0$.

Proof. (a) As $\hat{c} \leq \bar{c}$ is suffices to prove that $\hat{c}$ is firm. Let (t_n) be a sequence of $\mathbb{R}_+$ such that $\lim_n \hat{c}(t_n) = 0$. Then for each $n \in N$ we can find $w_n \in W_n$ and s_n in $\mathbb{R}_+$ with $|s_n - t_n| \leq 1/n$ such that $c_{w_n}(s_n) \leq \hat{c}(t_n) + 1/n$. Therefore, $\lim_n c_{w_n}(s_n) = 0$ hence $s_n \to 0$ and $t_n \to 0$.

(b) Given $(t_n) \subset \mathbb{R}_+$ and $(w_n) \subset W$ with $\lim_n w_n = w_0$, $\lim_n c_{w_n}(t_n) = 0$ we have $\tilde{c}(t_n) \leq c_{w_n}(t_n)$, hence $\lim_n \tilde{c}(t_n) = 0$ and $t_n \to 0$. □

Under a weak equicoercivity assumption we have a converse of assertion (a) of the preceding lemma.

Lemma 4.4. *Let $(c_w)_{w \in W}$ be a family of functions from $\mathbb{R}_+$ into $\bar{\mathbb{R}}_+$ such that there exist $\alpha > 0$, $A > 0$, and a neighborhood W_0 of w_0 in W such that $c_w(t) \geq t$ for any $w \in W_0$, $t \geq A$. Then, if $\hat{c}$ given as in the preceding lemma is firm, the family (c_w) is firm.*

Proof. Let $(t_n) \subset \mathbb{R}_+$, $(w_n) \subset W$ be such that $\lim_n w_n = w_0$, $\lim_n c_{w_n}(t_n) = 0$. Let $n_0 \in N$ be such that $w_n \in W_0$ and $c_{w_n}(t_n) < \alpha$ for $n \geq n_0$. Then for $n \geq n_0$ we have $t_n \in [0, A]$ and we can find an infinite subset K of N such that $(t_k)_{k \in K}$ converges to some $\hat{t}$ in $[0, A]$. Then $\hat{c}(\hat{t}) \leq \liminf_{k \in K} c_{w_k}(t_k) = 0$, hence $\hat{c}(\hat{t}) = 0$. As $\hat{c}$ is firm we get $\hat{t} = 0$. As any subsequence of $(t_n)_{n \in N}$ can be substituted to (t_n) in what precedes we get $t_n \to 0$. □

Theorem 4.5. *The problem* (P_0) *is M-well-set iff there exists a nonempty closed subset B of* A_0 *and a nondecreasing firm function c on* $\mathbb{R}_+$ *such that* $c(0) = 0$ *and*

$$f_0(x) - m \geq c(d(x, B)) \qquad \textit{for each} \quad x \in A_0$$

with equality when $x \in B$. *Then B is the set of solutions to* (P_0).

Proof. Let

$$c(t) = \inf\{|f_0(x) - m|: x \in A_0, d(x, S_0) = t\}.$$

Suppose (P_0) is M-well-set. Take $B = S_0$. If (t_n) is a sequence of $\mathbb{R}_+$ such that $c(t_n) \to 0$, we can find (x_n) in A_0 with $d(x_n, S_0) = t_n, f(x_n) \to m$. As (P_0) is M-well-set, we get $t_n \to 0$, so that c is firm.

Conversely, when c is firm, for any minimizing sequence (x_n) we have

$$f_0(x) - m \geq c(t_n)$$

for $t_n = d(x_n, s_0)$, hence $t_n \to 0$ and (P_0) is M-well-set. □

In [12] a more precise study of such kinds of results is given in the case A_0 is a convex subset of a normed vector space X and f_0 is starshaped or convex. Let us turn to a criterion which uses similar assumptions (compare with Theorem 5.13 of [12] and Theorems 6 and 7 of [26]).

Proposition 4.6. *Suppose X is a Banach space,* A_0, f_0 *is convex and l.s.c. and its domain D has a nonempty interior. Suppose the set* S_0 *of minimizers of* f_0 *is nonempty and there exists a firm function* $c: \mathbb{R}_+ \to \mathbb{R}_+$ *such that for each x in the domain of the subdifferential* ∂f_0 *of* f_0 *and each* $z_0 \in S_0$ *we have*

$$\sup\{\langle x^*, x - z_0 \rangle: x^* \in \partial f_0(x)\} \geq c(d(x, S_0)). \qquad (*)$$

Then (P_0) *is M-well-set.*

Proof. Replacing c with c' given by $c'(r) = \inf\{c(s): s \geq r\}$ and c' with c'' given by $c''(r) = \sup\{c(s): s < r\}$ we may suppose c is nondecreasing and l.s.c. Let us prove that

$$f_0(x) - m \geq c(\tfrac{1}{2} \cdot d(x, S_0)) \qquad (**)$$

for each $x \in X$. We may suppose $x \in D$. Observing that the epigraph E of f_0 has a nonempty interior as f_0 is continuous on the interior int D of D we may even suppose that $x \in \text{int}\, D$ since for any $x \in D$ there exists a sequence (x_n, r_n) with limit (x, f_0) and $(x_n, r_n) \in E$ so that $(**)$ holds if it holds for x replaced with x_n. Given

$x \in \operatorname{int} D$ let (t_n) be a sequence in $(\frac{1}{2}, 1)$ with limit $\frac{1}{2}$; for each $n \in N$ we can find $z_n \in S_0$ with $\|x - z_n\| \leq 2t_n d(x, S_0)$. Let $y_n = t_n x + (1 - t_n)z_n$, so that

$$\|x - y_n\| = (1 - t_n)\|x - z_n\| \leq 2t_n(1 - t_n)d(x, S_0),$$

$$d(y_n, S_0) \geq d(x, S_0) - d(x, y_n) \geq (1 - 2t_n(1 - t_n))d(x, S_0) \geq \tfrac{1}{2} \cdot d(x, S_0). \qquad (***)$$

As $z_n \in S_0 \subset D$, $x \in \operatorname{int} D$ we have $y_n \in \operatorname{int} D$ and f_0 is continuous at y_0, hence subdifferentiable at y_n with $\partial f_0(y_n)$ weak*-compact.

Let $y_n^* \in \partial f_0(y_n)$ be such that

$$\langle y_n^*, y_n - z_n \rangle = \sup\{\langle y^*, y_n - z_n \rangle : y^* \in \partial f_0\}.$$

Then by (*), (**), and (***) we get

$$\begin{aligned} f_0(x) - m &\geq f_0(x) - f_0(y_n) \\ &\geq \langle y_n^*, x - y_n \rangle \\ &\geq t^{-1}(1 - t_n)\langle y_n^*, y_n - z_n \rangle \\ &\geq t^{-1}(1 - t_n)c(d(y_n, S_0)) \\ &\geq t^{-1}(1 - t_n)c(\tfrac{1}{2} \cdot d(x, S_0)). \end{aligned}$$

Taking the limit as $n \to +\infty$ we get (**). □

Example 4.7. Let f be as in Example 2.4, $f(x) = 1/2(Qx|x)$, where X is a Hilbert space and $Q: X \to X$ is a semidefinite positive symmetric continuous linear operator with closed range. Then there exists $\alpha > 0$ such that $(Qx|x) \geq \alpha d(x, S_0)^2$ for each $x \in X$, with $S_0 = \ker Q$, so that for each $x \in X$ and each $x_0 \in S_0$ we have, with $c(t) = \alpha t^2$ for $t \in \mathbb{R}_+$, $\langle f(x), x - x_0 \rangle = (Qx|x) \geq c(d(x, S_0))$. It follows that (P_0) is M-well-set.

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