KLEPPNER, A.
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Multipliers on Abelian Groups

By
ADAM KLEPPNER in College Park, Maryland

Introduction

In the study of representation of locally compact groups, one finds that projective representations arise as a natural and inescapable generalization. A first step in the study of projective representations of a group is the determination of all possible multipliers (factor systems) on the group. In this paper we shall determine the multipliers on a large class of locally compact abelian groups (theorem 7.1). What they are is easily described: each multiplier is similar to a continuous bilinear function on the group with values in the group of complex numbers of absolute value 1. For the case of finite abelian groups this is a result of SCHUR [11], and for abelian Lie groups this result was no doubt known to WEYL (see [13]) and can be found explicitly in [1] and [4]. But here these particular results will be found as a special case of a more systematic and general treatment.

The method of determining these multipliers is in essence simple. If ω is a multiplier on the locally compact abelian group G, the function $\omega^{(2)}(x, y) = \omega(x, y)\,\omega(y, x)^{-1}$ is a continuous bilinear function on G with values in the group of complex numbers of absolute value 1 (shortly, $\omega^{(2)}$ is a bicharacter of G). Moreover, a multiplier is trivial if and only if it is symmetric. This implies that the map $\omega \to \omega^{(2)}$ defines an isomorphism of the group $H^2(G)$ into the group $A(G)$ of anti-symmetric bicharacters. In the case that $x \to x^2$ is an automorphism of G then every anti-symmetric bicharacter can be written in the form $\varphi(x, y)\,\varphi(y, x)^{-1}$ for some bicharacter φ and it easily follows from this that each multiplier on G is similar to a bicharacter. This also leads to the determination of the group $H^2(G)$. If $x \to x^2$ is not an automorphism the situation is more complicated, and most of this paper is devoted to an attempt to circumvent this difficulty.

Because of the fact that for each multiplier ω, $\omega^{(2)}$ is a bicharacter, it is natural to study the group of bicharacters. Thus the first 6 sections of this paper are devoted to a study of the elementary algebraic and topological properties of groups of bilinear functions, and to questions of extension. There are some interesting open questions about these groups which tend to indicate that locally compact abelian groups can be viewed properly only in the setting of some much larger (and as yet undetermined) class of topological abelian groups.

Multipliers are discussed first in § 7 where the relevant definitions will be found.

Because this paper is almost entirely concerned with abelian groups we shall write "group" for "abelian group;" any nonabelian groups will be specified as such. All the groups will be written multiplicatively, with the exception of such groups as R and Z which are customarily written additively. The identity will be denoted by e or 1, or 0 for Z and R, with e being used for the most part if the group is abstractly given and 1 for groups of functions. T is the group of complex numbers of modulus 1. For each group G, $G^\wedge$ is its character group, provided with the usual topology, with the character function being denoted by $\langle x, \gamma\rangle$ or $\langle \gamma, x\rangle$. All groups are assumed to be Hausdorff. The words "homomorphism," "isomorphism," etc. refer only to maps with the usual algebraic properties; any topological properties will be explicitely mentioned.

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1. Elementary facts

Let G_1, G_2 and H be locally compact groups. A map $\varphi: G_1 \times G_2 \to H$ is *bilinear* if $\varphi(x_1 x_2, y_1) = \varphi(x_1, y_1)\,\varphi(x_2, y_1)$ and $\varphi(x_1, y_1 y_2) = \varphi(x_1, y_1)\,\varphi(x_1, y_2)$ for all $x_1, x_2 \in G_1$ and $y_1, y_2 \in G_2$. Let $B(G_1, G_2)$ denote the set of all bilinear functions on $G_1 \times G_2$ with values in T which are continuous as functions on $G_1 \times G_2$. If $G_1 = G_2 = G$ put $B(G_1, G_2) = B(G)$; $B(G)$ is the group of bicharacters of G: $B(G_1, G_2)$ is a group under pointwise multiplication of its elements. Let $\mathrm{Hom}(G_1, G_2^\wedge)$ denote the group (under pointwise multiplication) of all continuous homomorphisms of G_1 into G_2. For $\varphi \in B(G_1, G_2)$ and $x \in G_1$ let $\varphi^*(x)$ be the map $y \to \varphi(x, y)$ of G_2 into T and let φ^* denote the map $x \to \varphi^*(x)$.

Proposition 1.1. *For each $x \in G_1$ and $\varphi \in B(G_1, G_2)$, $\varphi^*(x) \in G_2^\wedge$ and $\varphi \to \varphi^*$ is an isomorphism of $B(G_1, G_2)$ on* $\mathrm{Hom}(G_1, G_2^\wedge)$.

Proof: The continuity of $\varphi^*(x)$ follows from the continuity of φ; thus $\varphi^*(x) \in G_2^\wedge$. Because φ is bilinear $\varphi \to \varphi^*$ is a homomorphism of G_1 into $G_2^\wedge$. $\varphi^* = 1$ if and only if $\varphi^*(x) = 1$ for all $x \in G_1$ and this happens if and only if $\langle \varphi^*(x), y\rangle = 1 = \varphi(x, y)$ for all $x, y \in G_1 \times G_2$, that is, if and only if $\varphi = 1$. Hence $\varphi \to \varphi^*$ is an isomorphism. For the continuity of φ^* it is enough to consider neighborhoods of the identity in G_2. A basis for these neighborhoods is given by the sets $W(C, V)$ of all $\gamma \in G_2^\wedge$ such that $\langle \gamma, C\rangle \subset V$ where C is a compact subset of G_2 and V a neighborhood of the identity in T. It follows (cf. [6], lemma 5F, p. 12) that the set of all $x \in G_1$ such that $\varphi^*(x) \in W(C, V)$ is open and that φ^* is continuous. For $\theta \in \mathrm{Hom}(G_1, G_2)$ let $\theta^0(x, y) = \langle \theta(x), y\rangle$. θ^0 is bilinear and $\theta^{0*} = \theta$. Once it has been shown that θ^0 is continuous it will follow that $\varphi \to \varphi^*$ is an isomorphism of $B(G_1, G_2)$ on $\mathrm{Hom}(G_1, G_2^\wedge)$. For the continuity of θ^0 let x, y be an arbitrary point of $G_1 \times G_2$ and let V be an arbitrary neighborhood of $1 \in T$. Let V_1 be a neighborhood of $1 \in T$ such that $V_1^3 \subset V$. Because $\theta(x)$ is continuous there exists a compact neighborhood U_2 of the identity in G_2 such that $\langle \theta(x), U_2\rangle \subset V_1$. Because θ is continuous there exists a compact neighborhood U_1 of the identity in G_1 such that $\theta(U_1) \subset$

$\subset W(U_2, V_1) \cap W(\{y\}, V_1)$. We now have $\theta^0(xU_1, yU_2) = \langle \theta(xU_1), yU_2\rangle = \langle \theta(x), y\rangle \langle \theta(U_1), y\rangle \langle \theta(x), U_2\rangle \langle \theta(U_1), U_2\rangle \subset \theta^0(x, y)\, V_1^3 \subset \theta^0(x, y)\, V$ and θ^0 is continuous.

Frequent use will be made of this isomorphism. To avoid unduly complicated notation we shall identify the groups $B(G_1, G_2)$ and $\mathrm{Hom}(G_1, G_2^\wedge)$. However, if we wish to emphasize the fact that we are treating $\varphi \in B(G_1, G_2)$ as an element of $\mathrm{Hom}(G_1, G_2^\wedge)$ we shall write $\varphi(x)$ or $\varphi: G_1 \to G_2^\wedge$.

If $\varphi: G_1 \to G_2^\wedge$ then $\hat{\varphi}: G_2 \to G_1^\wedge$, where $\hat{\varphi}$ is the transpose of φ, and as a bilinear function $\hat{\varphi}(y, x) = \varphi(x, y)$. The map $\varphi \to \hat{\varphi}$ is clearly an isomorphism of $B(G_1, G_2)$ on $B(G_2, G_1)$. If $G_1 = G_2 = G$ the automorphism $\varphi \to \hat{\varphi}$ of $B(G)$ gives rise to two important subgroups. Let $A(G)$ denote the set of all $\varphi \in B(G)$ such that $\varphi(x, y) = \varphi(y, x)^{-1}$. $A(G)$ is a subgroup of $B(G)$ for it is the kernel of the endomorphism $\varphi \to \varphi\hat{\varphi}$ of $B(G)$. If $\varphi \in A(G)$ we say that φ is *anti-symmetric*. Similarly we define $S(G)$ to be the subgroup of *symmetric* bilinear functions — those φ such that $\varphi(x, y) = \varphi(y, x)$ for all $x, y \in G \times G$. If G happens to be real Euclidean space then $A(G)$ and $S(G)$ correspond to groups of anti-symmetric and symmetric bilinear forms. In analogy with the decomposition of such a form into symmetric and anti-symmetric parts we might try to write every element of $B(G)$ as a unique product of an element of $A(G)$ with an element of $S(G)$. The first difficulty is that these two groups may have a non-trivial intersection. In fact,

Proposition 1.2. *$A(G) \cap S(G) = A(G/\overline{G^2}) = S(G/\overline{G^2})$ where G^2 is the group of all x^2 for $x \in G$.*[1])

Proof: $\varphi \in A(G) \cap S(G)$ if and only if $\varphi(x, y) = \varphi(y, x)^{-1} = \varphi(y, x)$ for all $x, y \in G \times G$. Thus $\varphi(x, y) = \pm 1$ and $\varphi(x^2, y) = \varphi(x, y^2) = 1$ for all $x, y \in G \times G$. Because φ is continuous $\varphi(z, y) = 1 = \varphi(y, z)$ for all $z \in \overline{G^2}$. Thus φ defines a bilinear function $\varphi' \in A(G/\overline{G^2})$ such that $\varphi(x, y) = \varphi'(\bar{x}, \bar{y})$ where $x \to \bar{x}$ is the canonical map of G on $G/\overline{G^2}$. Conversely, if $\varphi' \in A(G/\overline{G^2})$ then φ defined by $\varphi(x, y) = \varphi'(\bar{x}, \bar{y})$ is in $A(G)$. For all $\varphi' \in B(G/\overline{G^2})$, $\varphi'(\bar{x}, \bar{y}) = \pm 1$. Thus $A(G/\overline{G^2}) = S(G/\overline{G^2})$ and φ defined above is also in $S(G)$.

Corollary 1. *$A(G) \cap S(G) = \{1\}$ if and only if G^2 is dense in G.*

Corollary 2. *If G is connected $A(G) \cap S(G) = \{1\}$.*

For if G is connected G is the direct product of vector group and a compact connected group. In both of these groups the map $x \to x^2$ is surjective. Indeed this is clearly so for the vector group while for compact connected groups this is shown by Braconnier ([3], prop. 1, p. 17).

Continuing with the analogy of bilinear forms on Euclidean space, for $\varphi \in B(G)$ define φ' by: $\varphi'(x, y) = (\varphi(x, y)\,\varphi(y, x))^{1/2}$ and φ'' by: $\varphi''(x, y) = (\varphi(x, y)\,\varphi(y, x)^{-1})^{1/2}$. Now $\varphi'(x, y) = \varphi'(y, x)$, $\varphi''(x, y) = \varphi''(y, x)^{-1}$ and $\varphi = \varphi'\varphi''$. Unfortunately neither φ' nor φ'' need be bilinear, for $\varphi'(xy, z) = (\varphi(xy, z)\,\varphi(z, xy))^{1/2} = (\varphi(x, z)\,\varphi(y, z)\,\varphi(z, x)\,\varphi(z, y))^{1/2} = (\varphi(x, z)\,\varphi(z, x))^{1/2}$

[1]) Let H be a closed subgroup of the locally compact group G. Lifting each bilinear function on G/H to G gives an isomorphism of $B(G/H)$ with a subgroup of $B(G)$ which we dentify with $B(G/H)$. This will be discussed in more detail in § 3.

$(\varphi(y, z)\, \varphi(z, y))^{1/2} = \pm\, \varphi'(x, z)\, \varphi'(y, z)$, the sign depending on how the square roots were chosen. It is in general not possible to choose the square roots in such a fashion that φ' is bilinear. Similar remarks hold for φ''.

For $\varphi \in B(G)$ define $\varphi^{(2)}$ by: $\varphi^{(2)}(x, y) = \varphi(x, y)\, \varphi(y, x)^{-1} = \varphi\hat{\varphi}^{-1}(x, y)$. $\varphi^{(2)} \in A(G)$ for all $\varphi \in B(G)$ and if $\varphi \in A(G)$, $\varphi^{(2)} = \varphi^2$ while if $\varphi \in S(G)$, $\varphi^{(2)} = 1$. Let $B^{(2)}(G)$ denote the group of all $\varphi^{(2)}$.

Proposition 1.3. *A necessary and sufficient condition that $B(G) = A(G)\, S(G)$ is that every element of $B^{(2)}(G)$ have a square root in $A(G)$.*

Proof: Suppose $B(G) = A(G)\, S(G)$. If $\varphi \in B(G)$ set $\varphi = \theta_1\theta_2$ where $\theta_1 \in A(G)$ and $\theta_2 \in S(G)$. Then $\varphi^{(2)} = \theta_1^2$. Conversely, if $\varphi^{(2)} = \theta^2$ for some $\theta \in A(G)$ then $\varphi\theta^{-1} \in S(G)$, for $\varphi\theta^{-1}(x, y) = \varphi(x, y)\, \theta^{-1}(x, y) = \varphi(x, y)\, \varphi^{(2)-1}\, \theta(x, y) \times \theta(x, y) = \varphi(y, x)\, \theta(y, x)^{-1} = \varphi\theta^{-1}(y, x)$. The decomposition $\varphi = (\varphi\theta^{-1})\, \theta$ is the required decomposition.

Corollary. *If $x \to x^2$ is an automorphism of G then $B(G) = A(G) \times S(G)$.*

In this case $A(G) \cap S(G) = \{1\}$ and every element of $A(G)$ has a square root. In fact if $\varphi \in A(G)$ then θ defined by $\theta(x^2, y) = \varphi(x, y)$ is also in $A(G)$ and $\varphi = \theta^2$.

Similarly we could define $\varphi^{[2]}$ by: $\varphi^{[2]}(x, y) = \varphi(x, y)\, \varphi(y, x) = \varphi\hat{\varphi}(x, y)$. If $B^{[2]}(G)$ is the group of all $\varphi^{[2]}$ then $B^{[2]}(G)$ is a subgroup of $S(G)$. Another necessary and sufficient condition that $B(G) = A(G)\, S(G)$ is that every element of $B^{[2]}(G)$ have a square root in $S(G)$. $B^{[2]}(G)$ and $B^{(2)}(G)$ are respectively the ranges of the endomorphisms $\varphi \to \varphi\hat{\varphi}$ and $\varphi \to \varphi\hat{\varphi}^{-1}$; the kernels of these endomorphisms are the groups $A(G)$ and $S(G)$.

We next consider certain continuity properties of bilinear functions.

Proposition 1.4. *Let G_1 and G_2 be locally compact groups and let φ be a bilinear function on $G_1 \times G_2$ which is continuous in each variable separately and continuous, as a function of two variables, at the identity in $G_1 \times G_2$. Then φ is continuous on $G_1 \times G_2$.*

Proof: Let x, y be an arbitrary point of $G_1 \times G_2$ and let V be a neighborhood of $1 \in T$. Let V_1 be a neighborhood of $1 \in T$ such that $V_1^3 \subset V$. Because φ is continuous at the identity in $G_1 \times G_2$ there exist neighborhoods U_1 of $e \in G_1$ and U_2 of $e \in G_2$ such that $\varphi(U_1 \times U_2) \subset V_1$. We can choose U_2 small enough so that $\varphi(x, U_2) \subset V_1$ and U_1 small enough so that $\varphi(U_1, y) \subset V_1$. Now $\varphi(xU_1, yU_2) = \varphi(x, y)\, \varphi(x, U_2)\, \varphi(U_1, y)\, \varphi(U_1 \times U_2) \subset \varphi(x, y)\, V_1^3 \subset (x, y)\, V$ and φ is continuous at x, y.

Proposition 1.5. *Let G_1 and G_2 be locally compact groups and let $\varphi: G_1 \times G_2 \to T$ be a Borel bilinear function. Then φ is continuous in each variable separately.*

Proof: Let E be a Borel subset of T. For each $x \in G_1$, $\varphi^{-1}(E) \cap \{x\} \times G_2$ is a Borel subset of $G_1 \times G_2$. The set of all $y \in G_2$ such that $x, y \in \varphi^{-1}(E) \cap \{x\} \times G_2$ is therefore a Borel set of G_2 and $y \to \varphi(x, y)$ is measurable in y for each x. Because measurable characters are continuous, $\varphi(x, y)$ is continuous in y for fixed x. Similarly, $\varphi(x, y)$ is continuous in x for fixed y.

It should be noted that the hypotheses of these last two propositions are stronger than needed, in that local compactness was not used in the proof of proposition 1.4 while all that is needed for the proof of proposition 1.5 is the measurability of φ in each variable separately. However, it is in the form given here that these results will be used.

It will follow from these two propositions and the results of § 7 that whenever φ is a Borel function $\varphi^{(2)}$ is continuous. Whether φ itself is continuous is an open question. Of course, asking whether every Borel bilinear function is continuous is the same as asking whether every bilinear function continuous in each variable separately is actually continuous. This is certainly not the case for all groups. For example, let E be a non-normable locally convex vector space and let $x', x \to x'(x)$ be the canonical bilinear form on $E' \times E$. Then $x', x \to \exp x'(x)$ is only separately continuous. For locally compact groups Professor Mackey has kindly pointed out that this question has an affirmative answer in an important case.

Proposition 1.6. *Let G_1 and G_2 be locally compact groups such that $G_2^\wedge$ is σ-compact. If $\varphi : G_1 \times G_2 \to T$ is bilinear and continuous in each variable separately, then φ is continuous.*

Proof: Let $W(C, U)$ be an arbitrary neighborhood of the identity in $G_2^\wedge$. Let U_1 be a closed neighborhood of $1 \in T$ such that $U_1 U_1^{-1} \subset U$. Then $W(C, U_1)\, W(C, U_1)^{-1} \subset W(C, U)$. For each $y \in G_2$ let V_y denote the set of all $x \in G_1$ such that $\varphi(x, y) \in U_1$. Because φ is separately continuous, each V_y is closed in G_1. Then $V = \bigcap_{y \in C} V_y$ is also closed and $V = \varphi^{-1}(W(C, U_1))$. Because $G_2^\wedge$ is σ-compact a countable number of translates of $W(C, U_1)$ covers $G_2^\wedge$. Hence a countable number of translates of V covers G_1 and V has positive Haar measure. Hence $V V^{-1}$ contains a neighborhood N of $e \in G_1$ ([12], p. 50). If $x \in N$ then $\varphi(x) \in W(C, U_1)\, W(C, U_1)^{-1} \subset W(C, U)$ and $\varphi : G_1 \to G_2^\wedge$ is continuous at the identity in G_1. Thus φ is a continuous homomorphism and by proposition 1.1 it is a continuous bilinear function.

2. A Topology for $B(G_1, G_2)$

Let G_1 and G_2 be locally compact groups. If A is a subset of G_1, B a subset of G_2 and C a subset of T, let $W(A, B, C)$ denote the set of all $\varphi \in B(G_1, G_2)$ such that $\varphi(A \times B) \subset C$. As a basis for the neighborhoods of the identity in $B(G_1, G_2)$ we take all sets of the form $W(C_1, C_2, U)$ where C_1 and C_2 are compact subsets of G_1 and G_2 respectively and U is a neighborhood of the identity in T. It is easily verified that the topology determined by this neighborhood system is compatible with the group structure.

This topology is equivalent to the compact-open topology and if we regard each $\varphi \in B(G_1, G_2)$ as being an element of $\mathrm{Hom}(G_1, G_2^\wedge)$ then $W(C_1, C_2, U)$ is just the set of all $\varphi \in \mathrm{Hom}(G_1, G_2^\wedge)$ such that $\varphi(C_1)$ is contained in the neighborhood $W(C_2, U)$ in $G_2^\wedge$. This topology is always complete ([2], Chapt. X, § 1, théorème 1). It will be assumed from now on that $B(G_1, G_2)$ is provided with the topology just described.

The determination of the topological structure of $B(G_1, G_2)$ is complicated by the fact that this structure depends heavily on the algebraic structures of G_1 and G_2 as well as on their topological structures. There are, however, some general facts which can be noted. In the first place the map $\varphi \to \tilde{\varphi}$ is clearly a homeomorphism as well as an isomorphism of $B(G_1, G_2)$ on $B(G_2, G_1)$. This observation quickly yields the following fact.

Proposition 2.1. *Let G be a locally compact group. Then $A(G)$ and $S(G)$ are closed subgroups of $B(G)$ and there is a continuous isomorphism of $B(G)/S(G)$ (resp. $B(G)/A(G)$) on $B^{(2)}(G)$ (resp. $B^{[2]}(G)$).*

Proof: The map $\varphi \to \varphi\hat{\varphi}^{-1}$ is a continuous homomorphism of $B(G)$ on $B^{(2)}(G)$. The kernel of this map is just $S(G)$, which is therefore closed. Similarly, the map $\varphi \to \varphi\hat{\varphi}$ defines a continuous isomorphism of $B(G)/A(G)$ on $B^{[2]}(G)$.

Proposition 2.2. *If G_1 and G_2 are σ-compact then $B(G_1, G_2)$ is metrizable.*

Proof: Suppose $G_1 = \bigcup_{j=1}^{\infty} C'_j$ and $G_2 = \bigcup_{k=1}^{\infty} C''_k$ where each C'_j and C''_k is a compact set. If U_n is a basis for the neighborhoods of $1 \in T$ then the countable collection $\{W(C'_j, C''_k, U_n)\}$ forms a basis for the neighborhoods of the identity in $B(G_1, G_2)$ which is therefore metrizable.

If either G_1 or G_2 is not σ-compact then $B(G_1, G_2)$ need not be metrizable.

Theorem 2.1. *a) If G_1 and G_2 are discrete then $B(G_1, G_2)$ is compact.*

b) If G_1 and G_2 are compact then $B(G_1, G_2)$ is discrete.

c) If G_1 and G_2 are generated by compact neighborhoods of their identities, then $B(G_1, G_2)$ is locally compact.

Proof: a) For each $x, y \in G_1 \times G_2$ set $T_{x,y} = T$ and $P = \prod_{x,y \in G_1 \times G_2} T_{x,y}$. Then $B(G_1, G_2)$ is a subgroup of P and the topology on $B(G_1, G_2)$ is the inherited product topology. $B(G_1, G_2)$ is closed in P for if $\varphi \in \overline{B(G_1, G_2)}$, let U be an arbitrary neighborhood of $1 \in T$ and let x_1, x_2 be arbitrary points in G_1 and y_1, y_2 arbitrary points in G_2. There exists $\theta \in B(G_1, G_2)$ such that $\varphi\theta^{-1}(x_1, y_1)$, $\varphi\theta^{-1}(x_2, y_2)$, $\varphi\theta^{-1}(x_1, y_2)$, $\varphi\theta^{-1}(x_2, y_1)$ and $\varphi\theta^{-1}(x_1x_2, y_1y_2)^{-1}$ are all in U. The product of these five numbers is in U^5. Thus φ is bilinear and $B(G_1, G_2)$ is closed in P. Because P is compact, $B(G_1, G_2)$ is compact.

b) Let U be a neighborhood of $1 \in T$ which contains no subgroups other than $\{1\}$. Then $W(G_1, G_2, U)$ is a neighborhood of the identity in $B(G_1, G_2)$. If $\varphi \in W(G_1, G_2, U)$ then for each $x \in G_1$, $\varphi(x, G_2)$ is a subgroup contained in U. Hence $\varphi(x, G_2) = \{1\}$ for each $x \in G_1$. Thus $\varphi = 1$, $W(C_1, C_2, U) = \{1\}$ and $B(G_1, G_2)$ is discrete.

c) If G_1 and G_2 are generated by compact neighborhoods of their identities then $G_1 = G'_1 \times R^{n_1} \times Z^{m_1}$ and $G_2 = G'_2 \times R^{n_2} \times Z^{m_2}$ where R is the real line, Z is the integers and G'_1 and G'_2 are compact groups ([12], p. 110). It will be proved in § 4 that B is "bilinear;" thus $B(G_1, G_2) = B(G'_1, G'_2) \times B(G'_1, R^n) \times \times B(G'_1, Z^{m_1}) \times B(R^{n_1}, G'_2) \times B(R^{n_1}, R^{n_2}) \times B(R^{n_1}, Z^{m_2}) \times B(Z^{m_1}, G'_2) \times B(Z^{m_1}, Z^{m_2}) \times \times B(Z^{m_1}, R^{n_2})$. $B(G'_1, G'_2)$ is discrete, as just shown. $B(G'_1, R^{n_2}) = \mathrm{Hom}(G'_1, R^{n_2}) = \{1\}$, since R^n contains no non-trivial compact subgroups; similarly, $B(R^{n_1}, G'_2) = \{1\}$. $B(G'_1, Z^{m_2})$ is the product of m_2 copies of $G'_1{}^{\wedge}$ and $B(Z^{m_1}, G'_2)$ is the product of m_1 copies of $G'_2{}^{\wedge}$. Also, $B(R^{n_1}, R^{n_2}) = R^{n_1 n_2}$, $B(R^{n_1}, Z^{m_2}) = R^{n_1 m_2}$, $B(Z^{m_1}, R^{n_2}) = R^{m_1 n_2}$ and $B(Z^{m_1}, Z^{m_2}) = T^{m_1 m_2}$. Thus $B(G_1, G_2)$ is a product of a finite number of locally compact groups and is therefore locally compact.

Corollary. *If G is either discrete or compact $B^{(2)}(G)$ and $B^{[2]}(G)$ are closed subgroups of $B(G)$.*

There are other conditions under which $B(G_1, G_2)$ is locally compact; for instance, $B(G, Z) = G^\wedge$ is always locally compact. In general, though, $B(G_1, G_2)$ is not locally compact.

Let G_1 and G_2 be locally compact groups. For each $x, y \in G_1 \times G_2$ the map $\varphi \to \varphi(x, y)$ is a character of $B(G_1, G_2)$. The map which assigns to x, y this character is bilinear and thus gives rise to a homomorphism of $G_1 \otimes G_2$ into $(B(G_1, G_2))^\wedge$. If G_1 and G_2 are discrete then by the definition of the tensor product the correspondence between characters of $G_1 \otimes G_2$ and bilinear maps on $G_1 \times G_2$ gives rise to a canonical isomorphism of $(G_1 \otimes G_2)^\wedge$ and $B(G_1, G_2)$. Thus in this case the homomorphism mentioned above is an isomorphism of $G_1 \otimes G_2$ on $(B(G_1, G_2))^\wedge$. Using this isomorphism we can obtain further information about $B(G_1, G_2)$.

Proposition 2.3. *If G_1 and G_2 are discrete and torsion free $B(G_1, G_2)$ is compact and connected.*

Proof: That $B(G_1, G_2)$ is compact follows from thm. 2.1. In order that $B(G_1, G_2)$ be connected it is necessary and sufficient that $B(G_1, G_2)^\wedge = G_1 \otimes G_2$ be torsion free, and for this it is sufficient that G_1 and G_2 be torsion free.

A partial converse to this result is the following.

Proposition 2.4. *If either G_1 or G_2 is divisible (not assumed discrete) then $B(G_1, G_2)$ is torsion free.*

Proof: Suppose that G_1 is divisible and that $\varphi^n = 1$ for some $\varphi \in B(G_1, G_2)$. Choose $x, y \in G_1 \times G_2$ and choose $x_1 \in G_1$ such that $x_1^n = x$. Then $\varphi(x, y) = \varphi(x_1^n, y) = \varphi^n(x_1, y) = 1$. Thus $\varphi = 1$.

3. Extensions

Proposition 3.1. *Let G' be a closed subgroup of the locally compact group G and set $G'' = G/G'$. Let H be an arbitrary locally compact group. Then there exists an exact sequence*

$$1 \to B(G'', H) \to B(G, H) \to B(G', H)$$

where the injection is a homeomorphism and the homomorphism on the right is continuous. Moreover, the image of $B(G'', H)$ in $B(G, H)$ is closed.

Proof: Since $B(G'', H) = \mathrm{Hom}(G'', H^\wedge)$, etc. the existence of the sequence follows from the well-known properties of Hom. To verify the topological properties of the maps involved, denote the bases for the neighborhoods of the identities in $B(G'', H)$, $B(G, H)$ and $B(G', H)$ by W'', W and W' respectively. Let p be the canonical map of G on G''. Let $W(C_1, C_2, U)$ be a neighborhood of the identity in $B(G, H)$. Then $W''(p(C_1), C_2, U)$ is a neighborhood of the identity in $B(G'', H)$ and if $\varphi \in W''(p(C_1), C_2, U)$ then $\varphi \circ p \in W(C_1, C_2, U)$ and $\varphi \to \varphi \circ p$ is continuous. Let $W''(C_1, C_2, U)$ be a neighborhood of the identity in $B(G'', H)$. There exists in G a compact set C_1' such that $C_1 = p(C_1')$ ([12], p. 19). The image in $B(G, H)$ of $W''(C_1, C_2, U)$ contains the intersection of the image of $B(G'', H)$ with $W(C_1', C_2, U)$ and $\varphi \to \varphi \circ p$ is bicontinuous. If

$W'(C_1, C_2, U)$ is a neighborhood of $1 \in B(G', H)$ then $W(C_1, C_2, U)$ is a neighborhood of $1 \in B(G, H)$ and the restriction map carries $W(C_1, C_2, U)$ into $W'(C_1, C_2, U)$. That the image of $B(G'', H)$ in $B(G, H)$ is closed follows from the fact that this image is complete.

Recalling that for any two locally compact groups G and H, $B(G, H)$ is bicontinuously isomorphic to $B(H, G)$, we obtain the following diagram:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] \\
1 \arrow[r] & B(G'') \arrow[r] \arrow[d] & B(G'', G) \arrow[r] \arrow[d] & B(G'', G') \arrow[d] \\
1 \arrow[r] & B(G, G'') \arrow[r] \arrow[d] & B(G) \arrow[r] \arrow[d] & B(G, G') \arrow[d] \\
1 \arrow[r] & B(G', G'') \arrow[r] & B(G', G) \arrow[r] & B(G')
\end{tikzcd}$$

Diagram 1

which is commutative, and in which the rows and columns are exact as indicated, the injections are homeomorphisms and the remaining homomorphisms are continuous. This diagram is symmetric about its main diagonal (under the map $\varphi \rightarrow \hat{\varphi}$). It will be convenient to identify the groups in the first row (resp. column) with their images in the second row (resp. column). From this diagram we can extract the sequence

$$1 \rightarrow B(G'') \rightarrow B(G) \xrightarrow{r} B(G')$$

in which $B(G'')$ can be identified with a closed subgroup of $B(G)$ and the map $r: B(G) \rightarrow B(G')$ is continuous. This sequence is in general not exact and questions of extension are essentially questions in the homology of this sequence.

Proposition 3.2. *If the diagram below is exact then*

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & B(G'') \arrow[r] \arrow[d] & B(G'', G) \arrow[r] \arrow[d] & B(G'', G') \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & B(G, G'') \arrow[r] \arrow[d] & B(G) \arrow[r] \arrow[d] & B(G, G') \arrow[d] & \\
1 \arrow[r] & B(G', G'') \arrow[r] \arrow[d] & B(G', G) \arrow[r] & B(G') & \\
 & 1 & & &
\end{tikzcd}$$

Diagram 2

$B(G'', G)\, B(G, G'')$ is a closed subgroup of $B(G)$ identical with the kernel of the map $r: B(G) \to B(G')$.

Proof: Suppose $\varphi \mid G' \times G' = 1$. Set $\varphi_{01} = \varphi \mid G \times G'$. Because $\varphi_{01} \mid G' \times G' = 1$ and by the exactness of the third column $\varphi_{01} \in B(G'', G')$. By the exactness of the first row φ_{01} can be extended to a map $\tau_1 \in B(G'', G) \subset B(G)$. Because φ and τ_1 both extend φ_{01}, $\varphi\tau_1^{-1} \in B(G, G'')$. Thus $\varphi = \tau_1 \varphi \tau_1^{-1} \in B(G'', G)\, B(G, G'')$ and $\ker(r) \subset B(G'', G)\, B(G, G'')$. Clearly $B(G'', G)\, B(G, G'') \subset \ker(r)$; thus these groups are identical. Because r is continuous, $\ker(r)$ is closed.

Proposition 3.3. *Assume diagram 2 is exact. If $\varphi \in A(G) \cap \ker(r)$ (resp. $\varphi \in S(G) \cap \ker(r)$ there exists $\tau \in B(G)$ such that $\varphi\tau^{(2)-1} \in A(G)$ (resp. $\varphi\tau^{[2]-1} \in S(G)$); τ may be chosen either in $B(G'', G)$ or $B(G, G'')$.*

Proof: As in the proof of the proposition above we set $\varphi_{01} = \varphi \mid G \times G'$ and find $\tau_1 \in B(G'', G)$ such that $\varphi\tau_1^{-1} \in B(G, G'')$. In the same way we set $\varphi_{10} = \varphi \mid G' \times G$ and find $\tau_2 \in B(G, G'')$ such that $\varphi\tau_2^{-1} \in B(G'', G)$. It now follows that $\varphi\tau_2^{-1}\tau_1^{-1} \in B(G'', G)\, B(G, G'')$; in fact, $(\varphi\tau_2^{-1})\tau_1^{-1} \in B(G'', G)\, B(G'', G) = B(G'', G)$ while $(\varphi\tau_1^{-1})\tau_2^{-1} \in B(G, G'')\, B(G, G'') = B(G, G'')$. Suppose $\varphi \in A(G)$. Then $\hat{\varphi}_{10}(x, y) = \varphi_{01}(x, y)^{-1}$ for $x, y \in G \times G'$. This means we can choose τ_2 so that $\hat{\tau}_2(x, y) = \tau_1(x, y)^{-1}$ for all $x, y \in G \times G$. Thus $\varphi\tau_1^{-1}\tau_2^{-1} = \varphi(\hat{\tau}_2\tau_1^{-1}) = \varphi\tau_2^{(2)-1} = \varphi\tau_1^{(2)-1}$. Because φ, τ_2, τ_1 are all in $A(G)$ it follows that $\varphi\tau_1^{(2)-1} \in A(G) \cap B(G'') = A(G'')$ and that $\varphi\tau_2^{(2)-1} \in A(G'')$. Similarly, if $\varphi \in S(G)$ then we can choose τ_2 so that $\hat{\tau}_2 = \tau_1$ and $\varphi\tau_1^{-1}\tau_2^{-1} = \varphi\tau_1^{[2]-1} = \varphi\tau_2^{[2]-1} \in S(G'')$.

These last two results describe, in a particular case, the possible different extensions of a bilinear function, when any at all exist. An extension need not always exist and determining which bilinear functions can be extended is difficult. There are, however, some cases in which a complete answer can be given.

Theorem 3.1. *Let G' be an open subgroup of the locally compact group G and let H be a locally compact group. If H is discrete and torsion free, or if both G and H are discrete and G/G' is torsion free, then each $\varphi \in B(G', H)$ can be extended to $G \times H$.*

Proof: If H is discrete and torsion free then $H^\wedge$ is compact and connected, hence divisible ([3], proposition 1, p. 17). By the lemma of Alexander each $\varphi: G' \to H^\wedge$ can be extended to a continuous homomorphism $\theta: G \to H^\wedge$ and $\theta \in B(G, H)$ is the desired extension of φ. If G and H are discrete then because $B(G, H)$ and $B(G', H)$ are compact the problem of showing that the restriction map $B(G, H) \to B(G', H)$ is onto is by duality the same as showing that the transposed map $B(G', H)^\wedge \to B(G, H)^\wedge$ is $1-1$. Using the fact mentioned in the last section that $B(G', H)^\wedge = G' \otimes H$ and $B(G, H)^\wedge = G \otimes H$ we see that this map is just the canonical map $G' \otimes H \to G \otimes H$ and because G/G' is torsion free this map is $1-1$ ([5], exp. 11, thm. 4).

Corollary. *If G/G' is discrete and torsion free then diagram 2 above is exact. If also G is discrete then diagram 3 below is exact.*

It can be shown, if only G/G' is discrete and torsion free and H is arbitrary, that all φ in a dense subgroup of $B(G', H)$ can be extended.

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & B(G'') & \to & B(G'', G) & \to & B(G'', G') & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & B(G, G'') & \to & B(G) & \to & B(G, G') & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & B(G', G'') & \to & B(G', G) & \to & B(G') & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

Diagram 3

It will be shown in § 6 that for many groups G' if G/G' is discrete and torsion free, then each $\varphi \in A(G')$ has an anti-symmetric extension.

The situation is entirely different if G/G' is not torsion free; extensions do not always exist and specifying which of them can be extended is very difficult. There are some special cases which can be noted, however. Let G_n denote the subgroup of G of all x such that $x^n = e$ and G^n denote the subgroup of all x^n.

Lemma 3.1. *Let G be a locally compact group of exponent 2 and let H be a closed subgroup. Let H^* be the conjugate of H in $G^\wedge$. Then G is bicontinuously isomorphic to $H \times (H^*)^\wedge$.*

Proof: First suppose G is a discrete group of exponent 2. G is a vector space over the field of 2 elements and H is a subspace. Thus H admits a supplement, i.e. there exists a subgroup K such that $G = H \times K$ and $H \cap K = \{e\}$. Next suppose G is compact. Since $G^\wedge$ is discrete, H^* admits a supplement K such that $G^\wedge = H^* \times K$. Because $K = G^\wedge/H^* = H^\wedge$, $G^\wedge$ is isomorphic to $H^* \times H^\wedge$; thus G is isomorphic to $H \times (H^*)^\wedge$. Finally let G be an arbitrary locally compact group of exponent 2 and let G_0 be a compact open subgroup. Let A be the set of all finite subsets of $G - G_0$ (set complement) and for each $\alpha \in A$ let G_α be the group generated by $G_0 \cup \alpha$. Each G_α is a compact open subgroup of G and $G = \varinjlim G_\alpha$, where the maps involved are the inclusion maps. For each α set $H_\alpha = H \cap G_\alpha$. Each H_α is a closed subgroup of G_α and $H = \varinjlim H_\alpha$. By the previous results each G_α is isomorphic to $H_\alpha \times (H_\alpha^*)^\wedge$. If $\alpha < \beta$ the isomorphism which carries G_α onto $H_\alpha \times (H_\alpha^*)^\wedge$ is just the restriction to G_α of the isomorphism which carries G_β onto $H_\beta \times (H_\beta^*)^\wedge$. Thus $G = \varinjlim G_\alpha$ is isomorphic to $\varinjlim H_\alpha \times (H_\alpha^*)^\wedge = \varinjlim H_\alpha \times \varinjlim (H_\alpha^*) = H \times (H^*)^\wedge$.

Proposition 3.4. *Let G be a locally compact group, let H be an open subgroup and let $\varphi \in A(H) \cap S(H)$. In order that there exist $\varphi' \in A(G) \cap S(G)$ which extends φ it is necessary and sufficient that $\varphi(x, y) = 1$ for all $x, y \in (H \cap G^2) \times H$.*

Proof: If $\varphi' \in A(G) \cap S(G)$ then $\varphi'(x, y) = \pm 1$ for all x, y; in particular, $\varphi'(x^2, y) = 1$ for all x^2, y and the condition is necessary. For the sufficiency note first that by the continuity of φ and because H is open, $\varphi(x, y) = 1 = \varphi(y, x)$ for all $x, y \in (H \cap \overline{G^2}) \times H$. Thus φ can be identified with a function in $A(H/(H \cap \overline{G^2})$. Because H is open, $H\overline{G^2}$ is an open subgroup of G and under the homomorphism $G \to G/\overline{G^2}$, H is mapped onto the open subgroup $H\overline{G^2}/\overline{G^2}$. The algebraic isomorphism of $H/(H \cap \overline{G^2})$ onto $H\overline{G^2}/\overline{G^2}$ is continuous. $H/(H \cap \overline{G^2}) = K \times D$, where K is a compact group and D is discrete ([3], théorème 2, p. 41). Let K' be the inverse image of K in H. K' is open in H and hence in G. Under the map $G \to G/\overline{G^2}$, K' is mapped onto $K'\overline{G^2}/\overline{G^2}$. This is an open subgroup of $H\overline{G^2}/\overline{G^2}$. The restriction to K of the isomorphism of $H/(H \cap \overline{G^2})$ on $H\overline{G^2}/\overline{G^2}$ maps K on $K'\overline{G^2}/\overline{G^2}$; because this map is continuous and $1-1$, it is a homeomorphism. Thus the map of $H/(H \cap \overline{G^2})$ on $H\overline{G^2}/\overline{G^2}$ is a homeomorphism. This means we can identify $H/(H \cap \overline{G^2})$ with a subgroup of $G/\overline{G^2}$. $G/\overline{G^2}$ is a group of exponent 2 and by the preceding lemma $H/(H \cap \overline{G^2})$ is a direct factor of $G/\overline{G^2}$. Thus φ can be extended to an anti-symmetric function on all of $G/\overline{G^2}$ and then lifted to G to give the desired extension φ'.

Proposition 3.5. *Let G be a discrete p-group all elements of which have order $\leqq p^n$ and let H be a subgroup which contains all elements of order $\leqq p^{n-1}$. Let $\varphi \in B(H)$. In order that there exist an extension $\varphi' \in B(G)$ of φ it is necessary and sufficient that $\varphi((H \cap G^p) \times H_p) = \{1\} = \varphi(H_p \times (H \cap G^p))$.*

Proof: If $\varphi' \in B(G)$ and $y^p = e$ then $\varphi'(x^p, y) = 1 = \varphi'(x, y^p)$ and the condition is necessary. For the sufficiency let x_0 be any element of G not in H and let K be the group generated by H and x_0. Then $x_0^p \in H$ and $\varphi(x_0^p, H_p) = \{1\}$, i.e., $\varphi(x_0^p) \in H_p^* = ((H^\wedge)^p)^- = (H^\wedge)^p$, since $H^\wedge$ is compact. Thus there exists $\gamma_0 \in H^\wedge$ such that $\gamma_0^p = \varphi(x_0^p)$. Define $\varphi_1(x_0^m u, v) = \varphi(u, v)\langle\gamma_0^m, v\rangle$ for all $u, v \in H \times H$. Then φ_1 is an extension to $K \times H$ of φ. Note that $K \cap G^p = H \cap G^p$ and that $K_p = H_p$. Indeed for all $y \in G$, y^p has order $\leqq p^{n-1}$ and is in H while if $x_0 u \in K_p$ for some $u \in H$, then $x_0 u \in G_p \subset G_{p^{n-1}} \subset H$ and $u \in H$ implies that $x_0 \in H$ which is a contradiction. Every element of $K - H$ can be written in the form $x_0^b u$ with $u \in H$ and $1 \leqq b < p$. Because b and p^n are relatively prime there exists $u_1 \in H$ such that $u_1^b = u$ and $x_0^b u = (x_0 u_1)^b$. But $(x_0 u_1)^b$ has order p if and only if $x_0 u$ has order p. Thus we have found $\varphi_1 \in B(K, H)$ such that $\varphi_1((K \cap G^p) \times H_p) = \{1\} = \varphi_1(K_p \times (H \cap G^p))$. In just the same way we can extend $\varphi_1 \in B(H, K)$ to a map $\varphi_2 \in B(K)$ such that $\varphi_2((K \cap G^p) \times K_p) = \varphi_2(K_p \times (K \cap G^p))$. We can now apply Zorn's lemma to find an extension $\varphi' \in B(G)$.

4. Projective and injective limits

Let H_i be a closed subgroup of the locally compact group G_i, $i = 1, 2$. By the results of the last section it is possible to identify $B(G_i/H_i)$ with a closed subgroup of $B(G_i)$; more generally, it is possible to identify $B(G_1/H_1, G_2/H_2)$ with a closed subgroup of $B(G_1, G_2)$. This subgroup is the set of all $\varphi \in B(G_1, G_2)$ such that $\varphi(x, y) = 1$ if either $x \in H_1$ or $y \in H_2$. There is a continuous map (the restriction map) of $B(G_1, G_2)$ into $B(H_1, H_2)$. The image of

$B(G_1, G_2)$ under this map is the set of all $\varphi \in B(H_1, H_2)$ which can be extended to $G_1 \times G_2$.

Proposition 4.1. *Let G_i, $i = 1, \ldots, 4$ be locally compact groups and set $G' = G_1 \times G_2$, $G'' = G_3 \times G_4$. Then $B(G', G'') = B(G_1, G_3) \times B(G_1, G_4) \times$ $\times B(G_2, G_3) \times B(G_2, G_4)$.*

Proof: Let B_{ij} be the subgroup of all $\varphi \in B(G', G'')$ such that $\varphi(x, y) = 1$ if $x \in G_i$ or $y \in G_j$, $i = 1, 2$, $j = 3, 4$. Each B_{ij} is a closed subgroup and the intersection of any one of these groups with the product of the others is the identity subgroup. $B_{13} = B(G'/G_1, G''/G_3) = B(G_2, G_4)$ similarly for the others. If $\varphi \in B(G', G'')$ then $\varphi = \varphi_{13}\varphi_{23}\varphi_{14}\varphi_{24}$, where $\varphi_{ij} \in B_{ij}$. In fact, define $\varphi_{13}(x, y)$ $= \varphi(e, x; e, y)$ for $x, y \in G_2 \times G_4$ and similarly for the others. Thus $B(G', G'')$ is the direct product of the B_{ij}.

This result clearly extends to any finite number of factors. If we regard each element of G' and G'' as a 2-component vector, then each $\varphi \in B(G', G'')$ can be described by the matrix (φ_{ij}); if $x = (x_1, x_2)$ and $y = (y_1, y_2)$ then $\varphi(x, y) = \prod_{i,j} \varphi_{ij}(x_i, y_j)$. $\hat{\varphi}$ is described by the transposed matrix $(\hat{\varphi}_{ji})$

Corollary. *If $\varphi \in A(G')$ then $\varphi_{ii} \in A(G_i)$, $i = 1, 2$ and $\hat{\varphi}_{12} = \varphi_{21}^{-1}$, if $\varphi \in S(G')$ then $\varphi_{ii} \in S(G_i)$ and $\hat{\varphi}_{12} = \varphi_{21}$.*

Thus the anti-symmetric bicharacters are described by anti-symmetric matrices. This is in strict accord with the behaviour of bilinear forms over real vector spaces.

The locally compact group G is the projective limit of the groups $(G_\alpha)_{\alpha \in A}$ if each $G_\alpha = G/g_\alpha$ where g_α is a closed subgroup of G and $(g_\alpha)_{\alpha \in A}$ is a filtered decreasing family converging to $e \in G$ according to the filter of neighborhoods of e.

Suppose also that the locally compact group $H = \varprojlim H_i$, $i \in I$. The set $A \times I$ becomes directed by setting $\alpha, i < \beta, j$ if $\alpha < \beta$ and $i < j$. We identify the groups $B(G_\alpha, H_i)$ with subgroups of $B(G, H)$. If $\alpha, i < \beta, j$ then $B(G_\alpha, H_i) \subset$ $\subset B(G_\beta, H_j)$. If $\alpha, i < \gamma, k$ and $\beta, j < \gamma, k$ then $B(G_\alpha, H_i) \cup B(G_\beta, H_j) \subset$ $\subset B(G_\gamma, H_k)$.

Proposition 4.2. *If $G = \varprojlim G_\alpha$ and $H = \varprojlim H_i$ are compact then $B(G, H)$ $= \varinjlim B(G_\alpha, H_i)$.*

Proof: $B(G, H)$ and each of the groups $B(G_\alpha, H_i)$ are discrete. All that need be shown is that $B(G, H) = \bigcup_{\alpha, i} B(G_\alpha, H_i)$; that is, it must be shown that for each $\varphi \in B(G, H)$ there exists g_α and h_i, where $H_i = H/h_i$, such that φ is constant on the cosets of $g_\alpha \times h_i$ in $G \times H$. Let $\varphi \in B(G, H)$ and let U be an open neighborhood of $1 \in T$ which contains no subgroups other than the identity. For each $y \in H$ there exists a subgroup $g_{\alpha(y)}$ such that $\varphi(g_{\alpha(y)}, y) \subset U$. $g_{\alpha(y)}$ is compact; hence $\varphi(g_{\alpha(y)})$ is a compact and thus equicontinuous subset of $H^\wedge$. Thus there exists a neighborhood V_y of y such that $\varphi(g_{\alpha(y)}\ V_y) \subset U$. A finite number of these neighborhoods cover H. Let $g_{\alpha'}$ be the intersection of the corresponding $g_{\alpha(y)}$. For all $y \in H$, $\varphi(g_{\alpha'}, y) \subset U$ and because $\varphi(g_{\alpha'}, y)$ is a subgroup, $\varphi(g_{\alpha'}, y) = \{1\}$ for all $y \in H$. We can repeat this procedure with H and find a subgroup $h_{i'}$ such that $\varphi(x, h_{i'}) = \{1\}$ for all $x \in G$. Thus $\varphi \in B(G_{\alpha'}, H_{i'})$ and $B(G, H) = \varinjlim B(G_{\alpha'}, H_i)$.

Corollary 1. $B(G) = \varinjlim B(G_\alpha)$.

For the diagonal of $A \times A$ is cofinal in $A \times A$.

Corollary 2. *$A(G) = \varinjlim A(G_\alpha)$ and $S(G) = \varinjlim S(G_\alpha)$.*

Corollary 3. *If $G = \prod_\alpha G_\alpha$ and $H = \prod_i H_i$ are compact then $B(G, H) = \prod_\alpha \prod_i B(G_\alpha, H_i)$.*

If $G = \varinjlim G_\alpha$ is locally compact but not compact, then it is not generally true that $B(G) = \varinjlim B(G_\alpha)$. For instance, let $H = \varprojlim H_\alpha$ be a compact non-discrete group and set $G = H \times H^\wedge$. Then $G = \varprojlim H_\alpha \times H^\wedge$. The bilinear function $(x_1, \gamma_1), (x_2, \gamma_2) \to \langle \gamma_1, x_2 \rangle$, $x_i \in H$, $\gamma_i \in H^\wedge$, $i = 1, 2$, is not constant on the cosets of any of the subgroups $(h_\alpha \times H^\wedge) \times (h_\beta \times H^\wedge)$, where $H_\alpha = H/h_\alpha$. In this connection the following question arises. Suppose $G = \varprojlim G_\alpha$ is a union of compact open subgroups, for instance, this is so if G is a primary group. Then is it true that $\bigcup_\alpha B(G_\alpha)$ is dense in $B(G)$?

Suppose now that $G = \varinjlim G_\alpha$, where each G_α is an open subgroup of G, and that H is an arbitrary locally compact group. For each α let b_α denote the subgroup of all $\varphi \in B(G, H)$ such that $\varphi(G_\alpha \times H) = \{1\}$. Each b_α, being the kernel of a restriction map, is a closed subgroup. If $\alpha < \gamma$, $\beta < \gamma$, then $b_\gamma \subset b_\alpha \cap b_\beta$. Moreover, every neighborhood of the identity in $B(G, H)$ contains one of these groups. In fact, let $W(C_1, C_2, U)$ be a neighborhood of the identity in $B(G, H)$. Then $C_1 \subset G_\beta$ for some β. For if $x \in C_1$ then $x \in G_{\alpha(x)}$ for some index $\alpha(x)$. Because $G_{\alpha(x)}$ is open there exists a neighborhood V_x of x such that $V_x \subset G_{\alpha(x)}$. A finite number of these neighborhoods cover C_1. Thus $C_1 \subset \bigcap_1^n G_{\alpha_i} \subset G_\beta$ for certain indices $\alpha_1, \ldots, \alpha_n$ and $\beta > \alpha_i$ $i = 1, \ldots, n$. Now $b_\alpha \subset W(G_\beta, C_2, U) \subset \subset W(C_1, C_2, U)$. Taking into account the fact that $B(G, H)$ is complete we have proved

Proposition 4.3. *If $G = \varinjlim G_\alpha$, where each G_α is an open locally compact subgroup, and if H is an arbitrary locally compact group, then $B(G, H) = \varprojlim B(G,H)/b_\alpha$, where b_α is the group of all $\varphi \in B(G,H)$ such that $\varphi(G_\alpha \times H) = \{1\}$.*

Unfortunately, $B(G, H)/b_\alpha$ is in general isomorphic to only a proper subgroup of $B(G_\alpha, H)$, the discrepancy arising from the fact that not all $\varphi \in B(G_\alpha, H)$ can be extended to the whole of GH. There are, however, certain cases in which this difficulty does not occur.

Proposition 4.4. *If $G = \varinjlim G_\alpha$ is locally compact and if H is discrete and torsion free, or if G and H are discrete and each G/G_α is torsion free, then $B(G, H) = \varprojlim B(G_\alpha, H)$.*

Proof: What must be shown is that the map $\varphi \to \varphi | G_\alpha \times H$ of $B(G, H)$ into $B(G_\alpha, H)$ is surjective, that is, that every $\varphi \in B(G_\alpha, H)$ can be extended to all of $G \times H$. But this follows, under the conditions stated, from theorem 3.1.

Proposition 4.5. *If $G = \prod_{\alpha \in A} G_\alpha$ is discrete, then $B(G, H) = \prod_\alpha B(G_\alpha, H)$.*

Proof: We can write $G = \varinjlim G_f$, where the index set $\{f\}$ is the set of all finite subsets of A and $G_f = \prod_{\alpha \in f} G_\alpha$. Since G_f is a direct summand of G each $\varphi \in B(G_f, H)$ can be extended to $G \times H$. Thus $B(G, H) = \varprojlim B(G_f, H) = \prod_\alpha B(G_\alpha, H)$.

Using this result and proposition 4.2 it is easy to construct a group G such that $B(G)$ is not locally compact. For each α in an infinite set let H_α be a non-trivial cyclic group of finite order n and set $H = \coprod_\alpha H_\alpha$. $H^\wedge = \prod_\alpha H_\alpha^\wedge$ and $B(H, H^\wedge)$ is a closed subgroup of $B(H \times H^\wedge)$. It is easily computed that $B(H_\alpha, H_\beta)$ is also a cyclic group of order n. Now $B(H, H^\wedge) = \prod_\alpha \prod_\beta B(H_\alpha, H_\beta)$ is a full direct product of infinite discrete groups and is not locally compact. If $H = \coprod_\alpha H_\alpha$, where there are uncountably many groups H_α (this is just the case that $H \times H^\wedge$ is not σ-compact), then $B(H \times H^\wedge)$ contains as a closed subgroup $\prod_\alpha H_\alpha$ which is not metrizable, i.e. does not satisfy the first axiom of countability.

Suppose that $G = \prod_\alpha G_\alpha$ is compact. If we regard each element of G as a vector whose α^{th} component x_α is in G_α, then each $\varphi \in B(G)$ is described by a matrix $(\varphi_{\alpha\beta})$ where $\varphi_{\alpha\beta} \in B(G_\alpha, G_\beta)$ is such that $\varphi((x_\alpha), (y_\beta)) = \prod_\alpha \prod_\beta \varphi_{\alpha\beta}(x_\alpha, y_\beta)$. Almost every element in the matrix is the trivial bicharacter, that is, every element not in a certain finite submatrix is 1. Similarly, if $G = \coprod_\alpha G_\alpha$ is discrete, each $\varphi \in B(G)$ is described by a matrix $(\varphi_{\alpha\beta})$ such that $\varphi_{\alpha\beta} \in B(G_\alpha, G_\beta)$. In this case there is no other restriction on the matrix elements. If $\varphi = (\varphi_{\alpha\beta})$ then $\hat{\varphi} = (\hat{\varphi}_{\beta\alpha})$ is the transposed matrix. Thus $\varphi = (\varphi_{\alpha\beta}) \in A(G)$ (resp. $S(G)$) if and only if $\hat{\varphi}_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$ (resp. $\hat{\varphi}_{\alpha\beta} = \varphi_{\beta\alpha}$).

5. Some particular groups

We shall use the following notation:

Z — group of rational integers
R — additive group of real numbers
Q — additive group of rational numbers
Z_p — group of p-adic integers
$Z(p^\infty)$ — group of all $p^{n^{\text{th}}}$ roots of unity, p a prime, $n = 1, 2, \ldots$
$Z(r) = Z/rZ$ — cyclic group of order r

1. Connected groups. Let G be compact and connected. $G^\wedge$ is discrete and torsion free. If $\varphi \in B(G)$ then $\varphi(G)$ is a finite subgroup of $G^\wedge$; thus $\varphi(G) = \{1\}$ and $\varphi = 1$. Hence $B(G) = \{1\}$. The most general connected locally compact group is of the form $R^n \times G$, where G is compact and connected. $B(R^n \times G) = B(R^n) \times B(R^n, G) \times B(G, R^n) \times B(G)$. Because G is compact and $R = R^\wedge$ has no compact subgroups other than the identity, $B(G, R^n) = B(R^n, G) = \{1\}$. Thus $B(R^n \times G) = B(R^n)$.

$B(R)$ is isomorphic to R under the map $a \to \varphi_a$ where $\varphi_a(x, y) = \exp 2\pi i a x y$. By the results of the previous section $B(R^n) = R^{n^2}$ under the map which makes correspond to each $n \times n$ matrix $a = (a_{ij})$ the bilinear function φ_a, where

$$\varphi_a(x_1, \ldots, x_n; y_1, \ldots, y_n) = \exp 2\pi i \sum_{i,j} a_{ij} x_i y_j$$

φ_a is symmetric (anti-symmetric) if and only if a is symmetric (anti-symmetric).

Let G' be the connected component of the identity in G and assume that G is a union of compact open subgroups. This is equivalent to assuming that $G^\wedge$ is totally disconnected, for the connected component of the identity in $G^\wedge$ is the intersection of all the compact open subgroups of $G^\wedge$ and its conjugate in G is thus the union of all the compact open subgroups of G, i.e. G itself. If $G'' = G/G'$ then $G''^\wedge \subset G^\wedge$ is totally disconnected and $B(G', G'') = \mathrm{Hom}(G', G''^\wedge) = \{1\}$. Thus diagram 1 of § 3 becomes

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \\
1 & \to & B(G'') & \to & B(G'', G) & \to & 1 \\
 & & \downarrow & & \downarrow & & \\
1 & \to & B(G', G'') & \to & B(G) & \to & B(G, G') \\
 & & \downarrow & & \downarrow & & \\
 & & 1 & \to & B(G', G) & \to & 1
\end{array}$$

From this it follows that $B(G', G) = \{1\}$ and that $B(G'') = B(G, G'') = B(G)$. Moreover, $A(G) = A(G'')$, $S(G) = S(G'')$ and every element of $A(G)$ has a square root in $A(G)$ if and only if every element of $A(G'')$ has a square root.

2. Cyclic groups. every bicharacter of Z is of the form $n, m \to \exp 2\pi i \alpha n m$ where $0 \leqq \alpha \leqq 1$, i.e., $B(Z) = T$. Clearly $B(Z) = S(Z)$. Every bicharacter of $Z(r)$ is of the form $n, m \to \exp 2\pi i(\alpha m n/r)$ where α is an integer mod r. Thus $B(Z(r)) = Z(r)$. Again $B(Z(r)) = S(Z(r))$.

3. $Z(p^\infty)$. For these groups as well as others the following result is useful.

Proposition 5.1. *If G_1 is a divisible group and G_2 a torsion group, then* $B(G_1, G_2) = \{1\}$.

Proof: Let φ be any element of $B(G_1, G_2)$ and x, y an arbitrary point of $G_1 \times G_2$. If $y^n = e$ choose $x_1 \in G_1$ such that $x_1^n = x$. Then $\varphi(x, y) = \varphi(x_1^n, y) = \varphi(x_1, y^n) = 1$ *and* $\varphi = 1$.

Since all the groups $Z(p^\infty)$, $p = 2, 3, 5, \ldots$ are divisible and torsion, $B(Z(p^\infty), Z(q^\infty)) = \{1\}$ for any primes p and q. $Z(p^\infty)$ is an example of a group such that, except for the trivial bicharacter, no bicharacter on any subgroup can be extended to the whole group.

4. Q and Q/Z. Q/Z also satisfies the conditions of the proposition; thus $B(Q/Z) = \{1\}$. Inserting this result and the fact that $B(Z) = T$ into diagram 1 of § 3 gives the diagram

$$\begin{array}{ccccc} & & 1 & & 1 \\ & & \downarrow & & \downarrow \\ 1 & \to & B(Q/Z, Q) & \to & B(Q/Z, Z) \\ \downarrow & & \downarrow & & \downarrow \\ 1 \to B(Q, Q/Z) & \to & B(Q) & \to & B(Q, Z) \\ \downarrow & & \downarrow & & \downarrow \\ 1 \to B(Z, Q/Z) & \to & B(Z, Q) & \to & T\,. \end{array}$$

It follows from the same proposition that $B(Q/Z, Q) = \{1\}$ and we can extract from this diagram the sequence

$$1 \to B(Q) \to B(Q, Z) = Q$$

$Q^\wedge$, being compact and connected, is divisible. By the lemma of ALEXANDER ([12], p. 94) each homomorphism of Z into $Q^\wedge$ can be extended to a homomorphism of Q into $Q^\wedge$. Thus $B(Q) = Q^\wedge$. The correspondence is the following: if $\gamma \in Q^\wedge$ then $\varphi_\gamma \in B(Q)$ is the map $r, s \to \exp 2\pi i \langle\gamma, r\rangle s$, where $\gamma : Q \to [0,1]$. $\varphi_\gamma \in A(Q)$ if and only if for all $r, s \in Q \times Q$, $\langle\gamma, r\rangle s = -\langle\gamma, s\rangle r$. Every element of $A(Q)$ has a square root in $A(Q)$.

5. Z_p. $Z_p = \varprojlim Z(p^n)$. Thus $B(Z_p) = \varinjlim B(Z(p^n)) = \varinjlim Z(p^n) = Z(p^\infty)$. Because $B(Z(p^n)) = S(Z(p^n))$, $B(Z_p) = S(Z_p)$.

6. $B^{(2)}(G)$ and $A(G)$

As we mentioned in the introduction, the favorable case for computing multipliers occurs when $x \to x^2$ is an automorphism, for in this case we are sure that $B^{(2)}(G) = A(G)$. In general, $B^{(2)}(G)$ is a proper subgroup of $A(G)$, but there is a weaker assertion, to which this section is devoted, which still permits the computation of multipliers.

For each locally compact group G denote by $Q(G)$ the assertion:

For each $\varphi \in A(G)$ *there exists* $\theta \in B(G)$ *such that* $\varphi\theta^{(2)-1} = \sigma \in S(G)$. This is just the assertion that $A(G) = B^{(2)}(G)\,(A(G) \cap S(G))$. It follows from the results of § 5 that $Q(G)$ is true if G is compact and connected, cyclic, or R^n. $Q(G)$ is also true if $x \to x^2$ is an automorphism of G.

Lemma 6.1. *If* $Q(G_1)$ *and* $Q(G_2)$ *are true, then* $Q(G_1 \times G_2)$ *is true.*

Proof: If $\varphi \in A(G)$ then $\varphi = (\varphi_{ij})$ where $\varphi_{ii} \in A(G_i)$, $i = 1, 2$, $\varphi_{12}\, B(G_1, G_2)$ and $\varphi_{21}^{-1} = \hat{\varphi}_{12}$. Choose $\theta_i \in B(G_i)$ such that $\varphi_{ii}\theta_i^{-1} = \sigma_i \in S(G_i)$, $i = 1, 2$. Let $\theta \in B(G_1 \times G_2)$ be described by the matrix

$$\theta = \begin{pmatrix} \theta_1 & \varphi_{12} \\ 1 & \theta_2 \end{pmatrix}.$$

Then

$$\varphi\, \theta^{-1} = \begin{pmatrix} \sigma_1 & 1 \\ 1 & \sigma_2 \end{pmatrix} \in S(G_1 G_2)$$

and $Q(G_1 \times G_2)$ is true.

Corollary. *If G is a compact Lie group then* $Q(G)$ *is true.*

For such a group has the form $T^n \times G_1$ where G_1 is a finite group; and G_1 is a direct product of cyclic groups.

Lemma 6.2 *If G is a compact group, $Q(G)$ is true.*

Proof: $G = \varprojlim G_\alpha$, where each G_α is a Lie group. If $\varphi \in A(G)$ then $\varphi \in A(G_\alpha)$ for some α. Hence there exists $\theta \in B(G_\alpha)$ such that $\varphi\theta^{(2)-1} \in S(G_\alpha) \subset S(G)$.

Corollary. *If G is generated by a compact neighborhood of the identity, $Q(G)$ is true.*

Such a group has the form $G_1 \times Z^m \times R^n$, where G_1 is a compact group.

We next turn our attention to discrete groups. Here we make use of the fact that if G is discrete, $B(G)^\wedge = G \otimes G$. Let j_G or j: $x \otimes y \to y \otimes x$ be the involution on $G \otimes G$. The transpose of j is the map $\varphi \to \hat{\varphi}$. Denote by $(G \otimes G)_a$ (resp. $(G \otimes G)_s$) the subgroup of anti-symmetric (resp. symmetric) tensors, that is the kernel of the map $u \to u(ju)$ (resp. $u \to u(ju)^{-1}$). Let $(G \otimes G)^{[2]}$ (resp. $(G \otimes G)^{(2)}$) be the range of $u \to u(ju)$ (resp. $u \to u(ju)^{-1}$).

Lemma 6.3. *If G is discrete,* $(G \otimes G)^{(2)*} = S(G)$, $(G \otimes G)^{[2]*} = A(G)$, $(G \otimes G)_s^* = B^{(2)}(G)$, $(G \otimes G)_a^* = B^{[2]}(G)$, $(G \otimes G)^{(2)}(G \otimes G)^2 = (G \otimes G)^{[2]}(G \otimes G)^2 = (A(G) \cap S(G))^*$.

Proof: $\varphi \in (G \otimes G)^{(2)*} \Leftrightarrow$ for all $u \in G \otimes G$, $\langle u(ju)^{-1}, \varphi\rangle = 1 \Leftrightarrow \langle u, \varphi\rangle = \langle ju, \varphi\rangle$, all $u \Leftrightarrow \varphi = \hat{\varphi}$. The proofs of the next three statements are similar. $\varphi \in A(G) \cap S(G) \Leftrightarrow \varphi \in A(G)$ and $\varphi^2 = 1 \Leftrightarrow \varphi \in S(G)$ and $\varphi^2 = 1 \Leftrightarrow \varphi \in A(G) \cap B(G)_2 \Leftrightarrow$ $\Leftrightarrow \varphi \in S(G) \cap B(G)_2$. Thus $(A(G) \cap S(G))^* = A(G)^*(B(G)_2)^* = S(G)^*(B(G)_2)^* = (G \otimes G)^{[2]}(G \otimes G)^2 = (G \otimes G)^{(2)}(G \otimes G)^2$.

Lemma 6.4. *If G is discrete, $Q(G)$ is true.*

Proof: Since always, $A(G) \supset B^{(2)}(G)(A(G) \cap S(G))$ we must show by duality that $(G \otimes G)^{[2]} \supset (G \otimes G)_s \cap (G \otimes G)^{(2)}(G \otimes G)^2$. If $u \in (G \otimes G)_s \cap (G \otimes G)^{(2)}(G \otimes G)^2$ then $u = v(j_G v)^{-1}w^2$, $v, w \in G \otimes G$. There exists a finitely generated subgroup H of G and $v_1, w_1 \in H \otimes H$ such that $v = \pi_G v_1$, $w = \pi_G w_1$, where π_G is the canonical map $H \otimes H \to G \otimes G$. If we set $u_1 = v_1(j_H v_1)^{-1}w_1^2$ then because $j_G\pi_G = \pi_G j_H$, $\pi_G u_1 = u$. $u \in (G \otimes G)_s$ implies that $\pi_G(j_H u_1)\, u_1^{-1}) = 0$. There exists a finitely generated subgroup $K \supset H$ such that $\pi_{HK}((j_H u_1)\, u_1^{-1}) = 0$, where $\pi_{HK}: H \otimes H \to$ $\to K \otimes K$ is the canonical map. Set $u_2 = \pi_{HK}u_1$. Then $(j_K u_2)u_2^{-1} = 0$ and also $u_2 = v_2(j_K v_2)^{-1}w_2^2$ where $v_2 = \pi_{HK}v_1$, $w_2 = \pi_{HK}w_2$. Further $u = \pi_K u_2$, $v = \pi_K v_2$, $w = \pi_K w_2$ where $\pi_K : K \otimes K \to G \otimes G$ is the canonical map. By the corollary to lemma 6.2, $Q(K)$ is true. Thus $(K \otimes K)^{[2]} = (K \otimes K)_s \cap (K \otimes K)^{(2)}(K \otimes K)^2$ and u_2 is of the form $t(j_K t)$ for some $t \in K \otimes K$. Hence $u = \pi_K u_2 = \pi_K(t)(\pi_K j_K t) = \pi_K t(j_G \pi_K t) \in (G \otimes G)^{[2]}$ and this proves the lemma.

It can be shown more generally that if G has a compact open subgroup then $B^{(2)}(G)\,(A(G) \cap S(G))$ is dense in $A(G)$.

Lemma 6.5. *Suppose that G is a union of compact open subgroups. Let K be the connected component of the identity and let P_2 be the 2-primary component of G/K ([3], ch. 3). If $Q(P_2)$ is true, then $Q(G)$ is true.*

Proof: By the results of § 5, $B(G) = B(G/K)$. G/K is a local direct product of its primary components ([3], ch. 3, théorème 1). In particular, it is a direct product of P_2 and a group in which $x \to x^2$ is an automorphism and for which Q is true. By lemma 8.1 $Q(G/K)$ is true. Thus $Q(G)$ is true.

Corollary. *With G and P_2 as above, $Q(G)$ is true if P_2 is a direct product of a compact group and a discrete group.*

Let G be an arbitrary locally compact group and let K be the connected component of the identity. Let U be the union of all the compact open subgroups in $G_0 = G/K$. Then G_0/U is a discrete torsion free group. If it were true that each $\varphi \in B(U)$ could be extended to all of $G_0 \times G_0$, then the results of § 3 could be used to show that $Q(G_0)$ and hence $Q(G)$ were true whenever $Q(U)$ were true. Thus to show that Q is true for all groups there remains the problem of showing Q is true for all 2-primary groups and for extensions by discrete torsion free groups. However, we are unable to answer these problems one way or the other.

For 2-primary groups the question of whether $B^{(2)}(G)$ is closed can be worded in the following way. Let $\{G_\alpha\}_{\alpha \in A}$ be the set of all compact open subgroups of G. Let $\{\theta_\alpha\}_{\alpha \in A}$ be a net such that $\theta_\alpha^{(2)} | G_\alpha \times G_\alpha = 1$. If for each α it is possible to choose θ'_α such that $\theta_\alpha'^{(2)} = \theta_\alpha^{(2)}$ and $\theta'_\alpha \to \theta \in S(G)$ then $B^{(2)}(G)$ will be closed. However, this direct approach seems to involve a question of extending symmetric bilinear functions, and this is difficult.

We sum up the results concerning Q in

Proposition 6.1. If either

a) G is discrete,

b) G/K is a union of compact open subgroups and the 2-primary component of G is a direct product of a compact group and a discrete group, where K is the connected component of the identity in G,

c) $x \to x^2$ is an automorphism of G,

d) G is a direct product of groups of the above 3 types,

then $A(G) = B^{(2)}(G) \times (A(G) \cap S(G))$.

It can also be shown that if G has a compact open subgroup, $G/\overline{G^2}$ is discrete and $B^{(2)}(G)$ is closed then $A(G) = B^{(2)}(G)\,(A(G) \cap S(G))$. In fact, if $G/\overline{G^2}$ is discrete, $A(G) \cap S(G)$ is compact so that $B^{(2)}(G)\,(A(G) \cap S(G))$ is closed if $B^{(2)}(G)$ is closed. If we use the assertion mentioned after lemma 6.4 that $B^{(2)}(G)\,(A(G) \cap S(G))$ is dense in $A(G)$, the result follows.

7. Multipliers

A *multiplier* ω on the locally compact group G is a Borel function $G \times G \to T$ satisfying:

(i) $\omega(e, x) = \omega(x, e) = 1$, all $x \in G$

(ii) $\omega(xy, z)\,\omega(x, y) = \omega(x, yz)\,\omega(y, z)$, all $x, y, z \in G \times G \times G$.

Two multipliers ω and τ are *similar*, $\omega \sim \tau$, if there exists a Borel function $\varrho : G \to T$ such that $\tau(x, y) = \varrho(x)\,\varrho(y)\,\varrho(xy)^{-1}\,\omega(x, y)$. If ω is the multiplier which is identically 1, τ is said to be *trivial*. Under pointwise multiplication the product of two multipliers is again a multiplier and with this operation the set of all multipliers on G forms a group $M(G)$. The trivial multipliers form a subgroup and the factor group of $M(G)$ by the group of trivial multipliers is denoted by $H^2(G)$. If $\omega \in M(G)$ its image in $H^2(G)$ will be denoted by $\tilde{\omega}$; thus

$\omega \sim \tau$ if and only if $\bar{\omega} = \bar{\tau}$. For each $\omega \in M(G)$ let $\hat{\omega}$ be the multiplier $\hat{\omega}(x, y) = \omega(y, x)$. Note that $\hat{\omega}^{-1} = (\omega^{-1})^\wedge$. A multiplier ω is *symmetric* if $\omega = \hat{\omega}$. For $\omega \in M(G)$ set $\omega^{(2)} = \omega\hat{\omega}^{-1}$.

Lemma 7.1. *For each* $\omega \in M(G)$, $\omega^{(2)} \in A(G)$.

Proof: The bilinearity of $\omega^{(2)}$ has been noted by Calabi ([4]) and Osima ([10]) and follows from the equation:

$$\begin{aligned}
\omega^{(2)}(xy, z) &= \omega(xy, z)\,\omega(z, xy)^{-1} = \omega(xy, z)\,\omega(x, y)\,\omega(x, y)^{-1}\,\omega(z, xy)^{-1} \\
&= \omega(x, yz)\,\omega(y, z)\,\omega(zx, y)^{-1}\,\omega(z, x)^{-1} \\
&= \omega(x, yz)\,\omega(y, z)\,\omega(x, z)\,\omega(x, z)^{-1}\,\omega(xz, y)^{-1}\,\omega(z, x)^{-1} \\
&= \omega(x, yz)\,\omega(y, z)\,\omega(x, z)\,\omega(x, zy)^{-1}\,\omega(z, y)^{-1}\,\omega(z, x)^{-1} \\
&= \omega^{(2)}(x, z)\,\omega^{(2)}(y, z)\,.
\end{aligned}$$

The anti-symmetry of $\omega^{(2)}$ is clear. Because ω is a Borel function, $\omega^{(2)}$ is also a Borel function. By proposition 1.5 $\omega^{(2)}$ is continuous in each variable separately. It follows from a theorem of Wigner ([1], theorem 1) that there exists a possibly non-measurable multiplier ω' (that is, a function satisfying (i) and (ii)) and a function $\varrho : G \to T$ such that ω' is continuous on a neighborhood of the identity in $G \times G$ and such that $\omega(x, y) = \varrho(x)\,\varrho(y)\,\varrho(xy)^{-1}\,\omega'(x, y)$. Now $\omega'(x, y)\,\omega'(y, x)^{-1} = \omega(x, y)\,\omega(y, x)^{-1}$ is continuous on a neighborhood of the identity. Thus by proposition 1.4, $\omega^{(2)}$ is continuous.

Lemma 7.2. $\omega^{(2)} = 1$ *if and only if* ω *is trivial.*

Proof: If ω is trivial then clearly $\omega^{(2)} = 1$. If $\omega^{(2)} = 1$ then ω is symmetric. As indicated by Mackey ([7], § 2) to each multiplier ω there corresponds a group G^ω (generally non-abelian) which is an extension of T by G. G^ω consists of all pairs (λ, x), $\lambda \in T$, $x \in G$, with multiplication defined by $(\lambda, x)\,(\mu, y) = (\lambda\mu\omega(x, y), xy)$. The product of Haar measures on T and G defines a measure on G^ω to which Weil's converse to Haar's theorem can be applied to give a locally bounded topology on G^ω. That this topology is actually locally compact was shown in the separable case by Mackey. In the general case this follows by noting that T is a closed normal subgroup of G^ω and G^ω/T is bicontinuously isomorphic to G. According to a theorem of Gleason ([9], Chap. II, theorem 2.2) every topological extension of a locally compact group by a locally compact group is itself locally compact. (The groups here are not necessarily abelian.) Thus G^ω is locally compact. The group G^ω is abelian if and only if ω is symmetric. By a result of Calabi ([4], proposition 18.4, corollary 3) every locally compact abelian extension of T splits and this implies that ω is trivial.

This lemma gives a useful criterion for determining when a multiplier is trivial: *it is trivial if and only if it is symmetric.*

The map $\omega \to \omega^{(2)}$ of $M(G)$ into $A(G)$ is clearly a homomorphism and by the lemma above $\omega \to \omega^{(2)}$ defines an isomorphism $\bar{\omega} \to \omega^{(2)}$ of $H^2(G)$ into $A(G)$.

Lemma 7.3. *The image of* $H^2(G)$ *under the map* $\omega \to \omega^{(2)}$ *contains the subgroup* $B^{(2)}(G)$.

Proof: This follows immediately by noting that each $\varphi \in B(G)$ is a multiplier, i.e. $B(G) \subset M(G)$. Under the map $\omega \to \omega^{(2)}$ $B(G)$ is carried onto the group $B^{(2)}(G)$.

There are several interesting consequences of this observation. If φ is a Borel measurable bilinear function, it is a multiplier; hence $\varphi^{(2)}$ is continuous. If $\varphi \in S(G)$ it is a symmetric and hence trivial multiplier. This means there exists a Borel function $\varrho: G \to T$ such that $\varphi(x, y) = \varrho(x)\,\varrho(y)\,\varrho(xy)^{-1}$ for all $x, y \in G \times G$. In particular, if $\varphi \in A(G) \cap S(G)$ then $\varphi(x, y)^2 = 1$ for all $x, y \in$ $\in G \times G$. This means that $\varrho^2(x)\,\varrho^2(y)\,\varrho^2(xy)^{-1} = 1$ for all $x, y \in G \times G$, i.e. ϱ^2 is a character of G. Thus each $\varphi \in A(G) \cap S(G)$ is the square root of a character; however, it is not true that the square root of every character can always be chosen to give a bilinear function.

The problem of computing multipliers reduces now to the question of identifying the subgroup of $A(G)$ onto which $H^2(G)$ is mapped. We consider first a special case.

Lemma 7.4. *Let G be a locally compact group of exponent 2. Then each multiplier on G is similar to a bicharacter.*

Proof: $G = G_1 \times G_2$, where the discrete group G_1 is a direct sum of groups of order 2 and the compact group G_2 is a direct product of groups of order 2. We can write $G_2 = \varprojlim G_\alpha$, where each G_α is a finite group of exponent 2. Repeated application of theorem 9.6 of [7] shows that each multiplier on a finite group is similar to a bicharacter. Let ω be a multiplier on G_2 and let L be a finite dimensional ω-representation, that is, L is a continuous map of G into the group of unitary operators of a Hilbert space $\mathscr{H}(L)$ such that $L_x L_y = \omega(x, y) L_{xy}$, all $x, y \in G$. L defines a continuous homomorphism L' of G into $\mathscr{U}^p\mathscr{H}(L)$ — the projective unitary group of $\mathscr{H}(L)$. This group is a (non-abelian) Lie group and there is a neighborhood of the identity in $\mathscr{U}^p\mathscr{H}(L)$ which contains no subgroups other than the identity. This implies that L' defines a homomorphism L'' of G_α into $\mathscr{U}^p\mathscr{H}(L)$ for some α, i.e. if $G_\alpha = G_2/g_\alpha$ then $L'(g_\alpha) = \{1\}$. By theorem 2.2 of [7], L'' defines a projective representation L^+ of G_α with a multiplier σ. The multiplier σ is similar to a bicharacter φ on G_α and φ lifted to G is similar to ω. This construction is due to C. MOORE and can be used to compute multipliers on any not necessarily abelian compact group.

Let $\{G_\alpha\}_{\alpha \in A}$ be the set of all finitely generated subgroups of G_1. Each G_α is a finite subgroup of exponent 2 and $G = \varinjlim G_\alpha$. The index set A is the set of all finite non-empty subsets of a basis of G_1 and is directed by setting $\beta < \alpha$ if and only if $\beta \subset \alpha$. Thus for each α there exist only finitely many indices $\beta < \alpha$, and we can write $A = \bigcup_1^\infty A_n$, where $\alpha \in A_n$ if and only if α is a set of n distinct elements. If $\alpha \in A_{n+1}$ and if $\beta_1, \beta_2 \in A_n$, $\beta_1 < \alpha$, $\beta_2 < \alpha$ then G_{β_1} and G_{β_2} generate G_α. Furthermore, if $\alpha > \beta \in A_n$ and $\alpha > \gamma \in A_1$ and $\beta \not> \gamma$ then $\gamma \cap \beta = \emptyset$, $\alpha = \beta \cap \gamma$ and this implies that $G_\alpha = G_\gamma \times G_\beta$. Since G_α is a finite group of exponent 2 every multiplier on G_α is similar to a bicharacter.

Let ω be a multiplier on G_1 and let ω_α denote its restriction to G_α. If $\alpha \in A_1$, G_α is a group of order 2 and ω is trivial. Thus for each $\alpha \in A_1$, there exists a function ϱ_α on G_α such that $\omega_\alpha(x, y) = \varrho_\alpha(x)\, \varrho_\alpha(y)\, \varrho_\alpha(xy)^{-1}$ for all $x, y \in G_\alpha \times G_\alpha$. We now make the following inductive hypothesis: For some $n > 1$ and all $\alpha \in \bigcup_1^n A_j$ there exists a function ϱ_α on G_α and $\varphi_\alpha \in B(G_\alpha)$ such that

1) $\omega_\alpha(x, y) = \varrho_\alpha(x)\, \varrho_\alpha(y)\, \varrho_\alpha(xy)^{-1}\, \varphi_\alpha(x, y)$ for all $x, y \in G_\alpha \times G_\alpha$

2) if $\beta < \alpha$ then $\varrho_\beta = \varrho_\alpha | G_\beta$.

This last condition implies that if $\alpha, \beta \in \bigcup_1^n A_j$ and if $x \in G_\alpha \cap G_\beta$ then $\varrho_\alpha(x) = \varrho_\beta(x)$ for there exists $\gamma < \alpha$, $\gamma < \beta$ such that $x \in G_\gamma$ and $\varrho_\alpha(x) = \varrho_\gamma(x) = \varrho_\beta(x)$.

Let $\alpha \in A_{n+1}$ and choose $\beta \in A_1$, $\gamma \in A_n$ such that $G_\alpha = G_\beta \times G_\gamma$, that is, $\beta < \alpha$, $\gamma < \alpha$, $\beta \cap \gamma = \emptyset$ and $\beta \cup \gamma = \alpha$. Define a function ϱ_α on G_α by: $\varrho_\alpha(x_1, x_2) = \varrho_\beta(x_1)\, \varrho_\gamma(x_2)$ if $x_1 \in G_\beta$, $x_2 \in G_\gamma$. The bilinear function φ_γ such that $\omega_\gamma(x, y) = \varrho_\gamma(x)\, \varrho_\gamma(y)\, \varrho_\gamma(xy)^{-1}\, \varphi_\gamma(x, y)$ can be extended to a bilinear function θ on G_α by defining $\theta(x_1 x_2, y_1 y_2) = \varphi_\gamma(x_2, y_2)$ where $x_1, y_1 \in G_\beta \times G_\beta$ and $x_2, y_2 \in G_\gamma \times G_\gamma$. The multiplier $x, y \to \omega_\alpha(x, y)\, \varrho_\alpha(x)^{-1}\, \varrho_\alpha(y)^{-1}\, \varrho_\alpha(xy)\, \theta(x, y)^{-1}$ reduces to 1 on both G_β and G_γ. By the corollary to theorem 9.5 of [7] the multiplier above is the extension φ to G_α of a bilinear function $\varphi' \in B(G_\beta, G_\gamma)$, where $\varphi'(x_1 x_2, y_1 y_2) = \varphi(x_1, y_2)$. Set $\varphi_\alpha = \varphi\theta$. Then $\omega_\alpha(x, y) = \varrho_\alpha(x)\, \varrho_\alpha(y)\, \varrho_\alpha(xy)^{-1}\, \varphi_\alpha(x, y)$ for all $x, y \in G_\alpha \times G_\alpha$. The functions φ_α and ϱ_α satisfy condition 1) of the inductive hypothesis. Suppose $\delta < \alpha$. Then either $\delta = \beta$ or $\delta < \gamma$. In the first case $\varrho_\alpha | G_\delta = \varrho_\alpha | G_\beta = \varrho_\beta$ (this follows from the definition of ϱ_α). In the second case, $\varrho_\alpha | G_\delta = \varrho_\gamma | G_\gamma$. By 2) of the inductive hypothesis, $\varrho_\gamma | G_\delta = \varrho_\delta$. We can now conclude that 1) and 2) hold for all n.

It follows that for all $x \in G$, if $x \in G_\alpha \cap G_\beta$, then $\varrho_\alpha(x) = \varrho_\beta(x)$. Thus we can define a function ϱ on G by setting $\varrho(x) = \varrho_\alpha(x)$ if $x \in G_\alpha$ and be certain that ϱ is well defined. Finally we have that $\omega(x, y)\, \varrho(x)^{-1}\, \varrho(y)^{-1}\, \varrho(xy)$ is a bilinear function, for if $x \in G_\alpha$, $y \in G_\beta$, then both x and y are in some G_γ, $\gamma > \alpha, \beta$, and $\omega(x, y)\, \varrho(x)^{-1}\, \varrho(y)^{-1}\, \varrho(xy) = \omega_\gamma(x, y)\, \varrho_\gamma(x)^{-1}\, \varrho_\gamma(y)^{-1}\, \varrho_\gamma(xy) = \varphi_\gamma(x, y)$ and φ_γ is bilinear.

Thus each multiplier on G_1 and G_2 is similar to a bicharacter. It follows again from the corollary to theorem 9.5 of [7] that each multiplier on $G_1 \times G_2 = G$ is similar to a bicharacter.

The next step is reducing the study of multipliers on an arbitrary locally compact group to those on groups of exponent 2. This is accomplished in the following two lemmas.

Lemma 7.5. *Let ω be a multiplier on the locally compact group G such that $\omega^{(2)} \in A(G) \cap S(G)$. Let K be the connected component of the identity in G and assume either G/K is separable or $G/\overline{(G)^2}$ is discrete. Then there exists a closed subgroup H of G and a multiplier ω' on G/H such that G/H has exponent 2 and $\omega \sim \omega' \circ p$, where p is the natural map of $G \times G$ on $G/H \times G/H$.*

Proof: Because $\omega^{(2)} \in A(G) \cap S(G)$ there exists a Borel function $\varrho : G \to T$ such that $\omega^{(2)}(x, y) = \varrho(x)\, \varrho(y)\, \varrho(xy)^{-1}$, i.e. such that $\omega(x, y) = \varrho(x)\, \varrho(y)\, \varrho(xy)^{-1}\, \omega(y, x)$. Because the multiplier $x, y \to \omega(x, y)\, \omega(y, x)$ is symmetric, it is trivial. Hence there exists a Borel function $\tau : G \to T$ such that

$\omega(x, y)\,\omega(y, x) = \tau(x)\,\tau(y)\,\tau(xy)^{-1}$. Thus $\omega(x, y)^2 = \varrho\tau(x)\,\varrho\tau(y)\,\varrho\tau(xy)^{-1}$. This implies that ω is similar to a multiplier ω_1 such that $\omega_1(x, y)^2 = 1$ for all $x, y \in G \times G$. In fact, $\omega_1(x, y) = (\varrho\tau(xy)\,\varrho\tau(x)^{-1}\,\varrho\tau(y)^{-1})^{1/2}\,\omega(x, y)$, where the square root is chosen in any Borel measurable fashion. For this ω_1, $\omega_1(x, y) = \omega_1(y, x) = \omega_1(x, y)^{-1}$ and $\omega_1(x, y)\,\omega_1(y, x) = \omega_1(x, y)\,\omega_1(y, x)^{-1}$. The left side of this last equation is symmetric and hence trivial, while the right side is continuous and bilinear. Thus $\omega_1(x, y)\,\omega_1(y, x)^{-1} = \varrho_1(x)\,\varrho_1(y)\,\varrho_1(xy)^{-1}$, where $\varrho_1 : G \to T$ is a Borel function such that $x, y \to \varrho_1(x)\,\varrho_1(y)\,\varrho_1(xy)^{-1}$ is a continuous bilinear function assuming only the values ± 1. This implies that ϱ_1^2 is a character of G.

Let H denote the set of all $x \in G$ such that $\omega_1(x, y) = \omega_1(y, x)$ for all $y \in G$. H is also the set of all $x \in G$ such that $\varrho_1(x)\,\varrho_1(y) = \varrho_1(xy)$ for all $y \in G$. This means, because $x, y \to \varrho_1(x)\,\varrho_1(y)\,\varrho_1(xy)^{-1}$ is continuous and bilinear, that H is a closed subgroup of G. Because $\varrho_1(x^2)\,\varrho_1(y)\,\varrho_1(x^2 y)^{-1} = (\varrho_1(x)\,\varrho_1(y)\,\varrho_1(xy)^{-1})^2 = 1$ for all $y \in G$, $(G)^2 \subset H$ and G/H has exponent 2.

For all $x, y \in H \times H$, $\omega_1(x, y) = \omega_1(y, x)$; thus $\omega_1 | H$ is symmetric and hence trivial. Thus there exists a Borel function $\tau_1 : H \to T$ such that $\omega_1(x, y) = \tau_1(x)\,\tau_1(y)\,\tau_1(xy)^{-1}$ for all $x, y \in H \times H$. Because ω_1 assumes only the values ± 1, τ_1^2 is a character of H. By the theory of duality there exists a character τ' of G which extends τ_1^2. Define τ_2 by: $\tau_2(x) = \tau'(x)^{1/2}$, where the square root is chosen in any Borel measurable fashion, but with the restriction that $\tau_2(x) = \tau_1(x)$ for $x \in H$. Set $\omega_2(x, y) = \tau_2(xy)\,\tau_2(x)^{-1}\,\tau_2(y)^{-1}\,\omega_1(x, y)$. Then $\omega_2(x, y)^2 = 1$ for all $x, y \in G \times G$, $\omega_2(x, y) = \omega_2(y, x)$ for all $x, y \in H \times G$ and $\omega_2(x, y) = 1$ for all $x, y \in H \times H$. This implies $T \times H$ is in the center of G^ω.

Let L be an irreducible ω_2 representation of G. Because $\omega_2(x, y) = 1$ for all $x, y \in H \times H$, $L_x L_y = L_{xy}$ for $x, y \in H \times H$. Let $\{r_\alpha\}$ be a Borel set of coset representatives of H in G such that $e \in \{r_\alpha\}$ (such a set clearly exists if $G/\overline{(G)^2}$ is discrete, while if G/K is separable then $G/H = (G/K)\,(H/K)$ (because $H \supset \supset (G)^2 \supset K$) is separable and a Borel cross section exists by lemma 1.1 of [8]). Define a representation L^0 as follows: $L_x^0 = L_x$ for $x \in H$, $L_{r_\alpha}^0 = L_{r_\alpha}$, and $L_{x r_\alpha}^0 = L_x L_{r_\alpha} = L_x^0 L_{r_\alpha}$ for each α and all $x \in H$. If $x \in H$ and $y r_\alpha$, $y \in H$, is any element of G, then $L_x^0 L_{y r_\alpha}^0 = L_x^0 L_y^0 L_{r_\alpha}^0 = L_{xy}^0 L_{r_\alpha}^0 = L_{xy r_\alpha}$. Thus the multiplier ω_3 of the representation L^0 has the following properties: 1) $\omega_3 \sim \omega_2 \sim \omega_1 \sim \omega$, 2) $\omega_3(x, y) = \omega_3(y, x) = 1$ if $x \in H$. This last property implies that ω_3 is constant on the $H \times H$ cosets in $G \times G$. In fact, if $a, b \in H \times H$ then $\omega_3(ax, by) = \omega_3(a, x)\,\omega_3(ax, by) = \omega_3(a, xby)\,\omega_3(x, by) = \omega_3(x, by) = \omega_3(x, yb)\,\omega_3(y, b) = \omega_3(xy, b)\,\omega_3(x, y) = \omega_3(x, y)$. Thus there exists a multiplier ω' on G/H such that $\omega_3 = \omega' \circ p$ and this proves the lemma.

Corollary. *If $\omega^{(2)} \in A(G) \cap S(G)$ then ω is similar to a bicharacter.*

Lemma 7.6. *Let G be a locally compact group, let K be the connected component of the identity in G and assume either $G/\overline{(G)^2}$ is discrete or G/K is separable. If $Q(G)$ is true, i.e. if $A(G) = B^{(2)}(G)\,(A(G) \cap S(G))$, then every multiplier on G is similar to a bicharacter.*

Proof: Let ω be a multiplier on G. Because $\omega^{(2)} \in A(G)$ there exists $\varphi \in B(G)$ such that $(\omega \varphi^{-1}) \in S(G)$. $\omega \varphi^{-1}$ is also a multiplier on G and by the corollary above there exists $\theta \in B(G)$ such that $\omega \varphi^{-1} \sim \theta$. Thus $\omega \sim \varphi\theta$.

Corollary. *Under the conditions above,* $H^2(G) = B(G)/S(G)$.

Proof: The kernel of the map which sends each $\varphi \in B(G)$ into the similarity class $\bar{\varphi} \in H^2(G)$ is just $S(G)$. It follows from the preceding lemma that the image of $B(G)$ under this map is all of $H^2(G)$.

The results so far of this section together with proposition 6.1 yield the following theorem.

Theorem 7.1. *Let G be a locally compact abelian group and let K be the connected component of the identity in G. Assume either G/K is separable or that $G/(\overline{G^2})$ is discrete. If either*

a) *G is discrete,*

b) *G/K is a union of compact open subgroups and the 2-primary component of G/K is a direct product of a compact group and a discrete group*

c) *$x \to x^2$ is an automorphism of G,*

d) *G is a direct product of groups of the above 3 types,*

then every multiplier on G is similar to a bicharacter of G. Under these conditions, $H^2(G) = B(G)/S(G)$.

Recall that there is a continuous isomorphism of $B(G)/S(G)$ on $B^{(2)}(G)$ and that this isomorphism is bicontinuous if G is compact or discrete. Moreover, if $x \to x^2$ is an automorphism of G then $B(G) = A(G) \times S(G)$ and $H^2(G) = A(G)$; in particular, in this case every multiplier on G is similar to a unique anti-symmetric bicharacter. This theorem also gives a natural way of topologizing $H^2(G)$, by providing it with the quotient topology on $B(G)/S(G)$.

8. Concluding remarks

This theorem does not describe all multipliers on all locally compact groups. Undoubtedly the restrictions imposed in some cases that $G/\overline{G^2}$ be discrete, or G/K be separable which insures the existence of certain Borel cross sections, are inessential. The other restrictions, which insured that Q is true, may be necessary for the method of proof used here, or at least some restrictions may be necessary. However, it seems likely that a better method of proof should be available to show that every multiplier on every locally compact group is similar to a bicharacter. Possibly the simplest group to which the theorem does not apply and for which not all the multipliers are known is a local direct product of countably many cyclic groups of order 4 with respect to their open subgroups of order 2.

There are also some interesting problems about bilinear functions left unanswered here. For instance, if G' is an open subgroup of G such that G/G' is torsion free then can every bicharacter of G' be extended to G?

The group $B(G)$ is a topological group with sufficiently many characters. In the case G was discrete we made strong use of the fact that the characters of $B(G)$ could be identified to elements of $G \otimes G$, and, in fact, that $G \otimes G$ was the dual of $B(G)$. It would be desirable to have a theory of topological tensor products such that $B(G)$ is the dual, or possibly the completion of the dual, of a suitably topologized tensor product. If such a theory existed, in a workable form, perhaps the method used to show $Q(G)$ is true if G is discrete, could be extended to all locally compact groups. We have been able to construct such a theory, but in a crude and not too usable form. It seems likely that before a satisfactory theory of topological tensor products could be constructed, one would need an extension of the Pontrjagin duality theory to a class of groups which would contain all the groups $B(G)$.

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