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Representations of Numbers and Finite Automata

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Abstract. Numeration systems, the basis of which is defined by a linear recurrence with integer coefficients, are considered. We give conditions on the recurrence under which the function of normalization which transforms any representation of an integer into the normal one—obtained by the usual algorithm—can be realized by a finite automaton. Addition is a particular case of normalization. The same questions are discussed for the representation of real numbers in basis θ, where θ is a real number >1, in connection with symbolic dynamics. In particular it is shown that if θ is a Pisot number, then the normalization and the addition in basis θ are computable by a finite automaton.

1. Introduction

Numbers are used through a symbolic expression and the way they are represented plays an important role in computer science, in arithmetics, and in coding theory. The research of numeration systems adequate to specific problems, and in which the arithmetical operations can be accelerated, is far from being achieved. The interest for parallel architectures has led to algorithms like the "weak addition" [1] where an integer has several representations.

In this paper we study numeration systems, the basis of which is not a geometric progression but a sequence of integers given by a linear recurrence relation, whose paradigm is the sequence of Fibonacci numbers. These numeration systems have also been considered in [8] and [16]. In the Fibonacci numeration system every integer can be represented using the digits 0 and 1. The representation

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is not unique, but one of them is distinguished, the one which does not contain two consecutive 1's (see [19] and [13]).

More generally, let U be a strictly increasing sequence of integers such that 1 belongs to U. By the greedy algorithm every integer has a representation in basis U, that we call the *normal* representation. *Normalization* is the function which transforms any representation on any alphabet into the normal one. The addition of two integers represented in basis U can be performed that way: just add the two representations digit by digit, which gives a word on the double alphabet. Then normalize this word to obtain the normal representation of the sum. Thus addition can be viewed as a particular case of normalization.

Our purpose is to study the process of normalization in numeration systems where the basis is defined by a linear recurrence relation with integer coefficients. We call these numeration systems *linear numeration systems.*

Finite automata are a "simple" model of computation, since only a finite memory is required. It is known that in the standard k-ary numeration system, where k is an integer ≥ 2, addition and more generally normalization on any alphabet are computable by a finite automata (see [7] and [11]). A function computable by a finite automation is usually called a *rational function.* In previous works we considered particular cases of linear numeration systems which generalize the Fibonacci numeration system and we showed that normalization is computable by a finite automaton [9], [10]. In this paper we use different.methods. First we prove that if the set of normal representations is recognizable by a finite automaton, then normalization is computable by a finite automaton if and only if the set of words having value 0 in basis U is recognizable by a finite automaton (Proposition 2.3). To every word we associate a polynomial. Then we consider words which can be associated to polynomials belonging to the ideal generated by the characteristic polynomial P of the linear recurrence. Obviously every word of this set is equal to 0 in basis U. We give a construction which links recognizability by a finite automaton and division of polynomials by P. We prove that the set of words associated to the ideal (P), on any alphabet, is recognizable by a finite automaton if and only if P has no root of modulus 1 (Theorem 2.1). If the set of all words equal to 0 is recognizable by a finite automaton, then the set of words associated to (P) is also recognizable (Proposition 2.6). Thus if P has one root of modulus 1, then there exist alphabets on which normalization is not computable by a finite automaton.

In a similar manner we discuss the representation of real numbers in basis θ were θ is a real number >1. The normal θ-representation of a real number is called *θ-development* or *θ-expansion* in the literature [18].

The notion of normalization is defined for θ-representation as above. If θ is an algebraic integer, then a construction similar to the one given for the integers links the recognizability of the set of infinite words equal to 0 to the property of the minimal polynomial of θ of having no root of modulus 1 (Theorem 3.1).

A *symbolic dynamical system* is a closed shift-invariant subset of $A^{\mathbb{N}}$, the set of infinite sequences on an alphabet A. The *θ-shift* is the closure of the set of infinite sequences which are θ-developments of numbers of [0, 1[. It is thus a symbolic dynamical system. A symbolic dynamical system is said to be of *finite* *type* if the set of its finite factors is defined by the interdiction of a finite set of words. It is said to be *sofic* if the set of its finite factors is recognized by a finite automaton.

The nature of the θ-shift is related to the arithmetical properties of θ. Let θ be an algebraic integer >1; θ is a *Pisot* number if its conjugates have modulus <1; θ is a *Salem* number if its conjugates have modulus ≤ 1, and it is not a Pisot number; θ is a *Perron* number if its conjugates have modulus $<\theta$. If θ is a Pisot number, then the θ-shift S_θ is sofic [15]. If S_θ is sofic, then θ is a Perron number (see [14]).

We prove that if the set of θ-representations of 1 is recognizable by a finite automaton, then the θ-shift is a sofic dynamical system (Theorem 3.2). Then normalization is computable by a finite automaton if and only if the set of infinite words equal to 0 in basis θ is recognizable by a finite automaton (Proposition 3.5).

Thus normalization in basis θ is computable by a finite automaton on any alphabet if and only if the minimal polynomial of θ has no root of modulus 1 and if $\sum_{n\geq 0} s_n\theta^{-n} = 0$ implies $\sum_{n\geq 0} s_n\alpha^{-n} = 0$ for every conjugate α of modulus >1 (Theorem 3.3). As a consequence, if θ is a Pisot number, then normalization in basis θ is computable by a finite automaton on any alphabet—and addition also—(Corollary 3.6). If θ is a Salem number, there exist alphabets on which normalization is not computable by a finite automaton.

The integers and the golden mean $(1+\sqrt{5})/2$ being Pisot numbers our results cover most standard numeration systems.

2. The Integers

2.1. Representation of Integers

Only positive numbers are considered. Let $U = (u_n)_{n\geq 0}$ be a strictly increasing sequence of integers with $u_0 = 1$. Every positive integer N can be written with respect to the basis U, i.e., it is possible to find $n \geq 0$ and integers $d_0, \ldots, d_n$ such that $N = d_0 u_n + \cdots + d_n u_0$ by the following algorithm (folklore): Given integers x and y let us denote by $q(x, y)$ and $r(x, y)$ the quotient and the remainder of the Euclidean division of x by y. Let $n \geq 0$ such that $u_n \leq N < u_{n+1}$ and let $d_0 = q(N, u_n)$ and $r_0 = r(N, u_n)$, $d_i = q(r_{i-1}, u_{n-i})$ and $r_i = r(r_{i-1}, u_{n-i})$ for $i = 1, \ldots, n$. Then $N = d_0 u_n + \cdots + d_n u_0$.

For $0 \leq i \leq n$, $d_i < u_{n-i+1}/u_{n-i}$; thus if the ratio u_{n+1}/u_n is bounded by a positive constant K for all $n \geq 0$ (K minimal), then $0 \leq d_i \leq K-1$. The set $A = \{0, 1, \ldots, K-1\}$ is called the *canonical alphabet* of digits associated to the basis U, and (U, A) is the *canonical numeration system* associated to U.

The word $d_0 \cdots d_n$ of A^* obtained by this algorithm is called the *normal representation* of the integer N in basis U. It is denoted by $\langle N \rangle = d_0 \cdots d_n$. The normal representation of 0 is the empty word ε.

More generally, a *numeration system* is given by a strictly increasing sequence $U = (u_n)_{n\geq 0}$ of positive integers, with $u_0 = 1$, called the *basis*, and a finite subset C of $\mathbb{N}$, the alphabet of *digits*. A *representation* of an integer N in the system (U, C) is a word $d_0 \cdots d_n$ of the free monoid C^* such that $N = d_0 u_n + \cdots + d_n u_0$.

The normal representation of an integer N has maximal length among the representations of N not beginning by a 0. It is also the greatest (for the lexicographical ordering) of all the representations of N of this same length in basis U. Given (U, C), the mapping $\pi: C^* \to \mathbf{N}$ is defined by $\pi(d_0 \cdots d_n) = d_0 u_n + \cdots + d_n u_0$. The *normalization* ν_C is the mapping which associates to a word f of C^* the normal representation of the integer represented by f:

$$\nu_C: C^* \to A^*$$
$$f \mapsto \langle \pi(f) \rangle.$$

Normalization is linked to the problem of addition of two integers written in basis U. To add two integers N and P with representations $f = f_0 \cdots f_k$ and $g = g_0 \cdots g_j$ respectively in (U, A) we add f and g digit by digit from the right and without carry. Let $f \oplus g = f_0 \cdots f_{k-j-1}(f_{k-j} + g_0) \cdots (f_k + g_j)$ (if $k \geq j$). Then $f \oplus g$ is a word written on the alphabet $\{0, \ldots, 2K - 2\}$. The addition of N and P reduces to the normalization of $f \oplus g$.

In this paper we study numeration systems where the basis is defined by

$$u_{n+m} = a_1 u_{n+m-1} + \cdots + a_m u_n, \qquad a_i \in \mathbf{Z}, \quad 1 \leq i \leq m, \quad a_m \neq 0.$$

These systems are called *linear numeration systems*. The ratio u_{n+1}/u_n is bounded for all $n \geq 0$ and the canonical alphabet is included in $\{0, \ldots, K - 1\}$ with $K < \max(a_1 + \cdots + a_m, \max\{(u_{i+1}/u_i) | 0 \leq i \leq m - 2\})$. If $m = 1$ and $a_1 \geq 2$ the system is the standard a_1-ary numeration system with $A = \{0, \ldots, a_1 - 1\}$ for canonical alphabet.

Example 2.1. The Fibonacci numeration system $\mathscr{F}$ is defined by the sequence of Fibonacci numbers

$$u_{n+2} = u_{n+1} + u_n,$$
$$u_0 = 1, \qquad u_1 = 2.$$

The canonical alphabet is $\{0, 1\}$. The representations of the integer 24 in $\mathscr{F}$ on $\{0, 1\}$ are the following:

21	13	8	5	3	2	1
	1	0	1	1	1	1
	1	1	0	0	1	1
	1	1	0	1	0	0
1	0	0	0	0	1	1
1	0	0	0	1	0	0

The normal representation of 24 is 1000100. The normal representation of an integer in $\mathscr{F}$ is the one that does not contain two consecutive 1's (see [19]).

2.2. *Normalization of Finite Words*

In this section we first show that if the set of normal representations of integers is rational, then normalization in basis U is computable by a finite automaton if and only if the set of words equal to 0 is recognizable by a finite automaton (Proposition 2.3).

First let us give some definitions. More details can be found in [7] and [2]. Let M be a monoid. The family Rat M of *rational* subsets of M is the least family of subsets of M containing the finite subsets and closed under product, union, and the star operation.

A subset L of M is *recognizable* if there exists a finite monoid N, a morphism φ from M into N, and a subset P of N such that $L = \varphi^{-1}(P)$.

A *finite automaton* $\mathscr{A} = (E, Q, I, T)$ is a directed graph labeled by letters of the alphabet E, with a finite set Q of vertices called *states*. $I \subset Q$ is the set of *initial* states, and $T \subset Q$ is the set of *terminal* states. A path in $\mathscr{A}$ is said to be *successful* if it starts in I and terminates in T. The set of successful paths is the *behavior* of $\mathscr{A}$. A word w of E^* is *recognized* by $\mathscr{A}$ if it is the label of a successful path of $\mathscr{A}$. A subset of E^* is *recognizable* if it is the behavior of a finite automaton on E. The recognizable subsets of E^* are exactly the rational subsets of E^* by the Kleene theorem (see [7]), and we use both denotations.

Let E and F be two alphabets. A *transducer* $\mathscr{T}$ is a finite automaton with edges labeled by couples of $E^* \times F^*$. A relation $R \subset E^* \times F^*$ is *rational* if and only if it is the behavior of a transducer. The composition of two rational relations is again a rational relation. A function $\varphi: E^* \to F^*$ is *computable by a finite automaton* or *rational* if its graph $\hat{\varphi}$ is a rational relation. From now on we use the word *rational.*

A transducer *with initial function* is a transducer $\mathscr{T} = (E^* \times F^*, Q, \alpha, T)$ where α is a partial function from Q *into* $\mathscr{P}(E^* \times F^*)$. The behavior of a transducer of this kind is defined as follows. A pair $(f, g) \in E^* \times F^*$ is recognized by $\mathscr{T}$ if there exist $i \in Q$ and $t \in T$, such that $\alpha(i) = (u, v)$ is defined, $f = uf'$, $g = vg'$, and (f', g') is the label of a path from i to t. The behavior of a transducer with initial function is a rational relation.

We assume that the characteristic polynomial $P(X) = X^m - a_1X^{m-1} - \cdots - a_m$ of U has a real root $\theta > 1$ which dominates strictly the modulus of its conjugates. A is the canonical alphabet and $L(U) \subset A^*$ is the set of normal representations of the integers in basis U.

If $c > 0$ is an integer, let $C = \{0, \ldots, c\}$, $\tilde{C} = \{-c, \ldots, c\}$, and $Z(U, c) = \{f = f_0 \cdots f_n \in \tilde{C}^* \mid f_0 u_n + \cdots + f_n u_0 = 0\}$ be the set of words on $\tilde{C}$ equal to 0 in basis U. Let ν_C: $C^* \to A^*$ be the normalization function on C^*. The index C in ν_C is dropped whenever the context is clear.

Some technical results are needed.

Lemma 2.1. *If $L(U)$ and $Z(U, c)$ are rational, then the normalization ν is a rational function.*

Proof. Let $f = f_0 \cdots f_n$ and $g = g_0 \cdots g_k$ be two words of C^* with $n \geq k$. Denote $f \ominus g = f_0 \cdots f_{n-k-1}(f_{n-k} - g_0) \cdots (f_n - g_k) \in \tilde{C}^*$. Then we have $\hat{\nu} = \{(f, g) \in C^* \times A^* \mid g \in L(U), f \ominus g \in Z(U, c)\}$.

Let S be the graph of $\ominus$:

$$S = \left[\left(\bigcup_{a\in C}((a,\varepsilon),a)\right)^* \cup \left(\bigcup_{a\in C}((\varepsilon,a),-a)\right)^*\right]\left[\bigcup_{a,b\in C}((a,b),a-b)\right]^*,$$

S is a rational subset of $(C^* \times C^*) \times \tilde{C}^*$. Let us consider the set

$$S' = S \cap ((C^* \times L(U)) \times Z(U,c)) \subseteq (C^* \times A^*) \times \tilde{C}^*.$$

Then $\hat{v}$ is the projection of S' on $C^* \times A^*$. As $L(U)$ and $Z(U, c)$ are rational, $(C^* \times L(U)) \times Z(U, c)$ is a recognizable subset of $(C^* \times A^*) \times \tilde{C}^*$ as a cartesian product of rational sets (see [2]). Since S is rational, S' is a rational subset of $(C^* \times A^*) \times \tilde{C}^*$. So, $\hat{v}$ being the projection of S', $\hat{v}$ is a rational subset of $C^* \times A^*$, that is, v is a rational function. □

Note that if v is a rational function, then its image $L(U)$ is a rational set. To prove that if v is rational, then $Z(U, c)$ is rational, it is necessary to give a precise characterization of normalization.

Let E and F be two alphabets. The length of a word f is denoted by $|f|$. Recall that a relation $R \subseteq E^* \times F^*$ is *length-preserving* if, for every $(f,g) \in R$, $|f| = |g|$ (see [7]). This is equivalent to $R \subseteq (E \times F)^*$. Eilenberg and Schützenberger have shown that a length-preserving rational relation of $E^* \times F^*$ is a rational subset of $(E \times F)^*$ [7].

Definition 2.1. A relation $R \subseteq E^* \times F^*$ is said to have *bounded differences* if there exists $k \in \mathbf{N}$ such that, for every $(f, g) \in R$, $||f| - |g|| \leq k$.

The set of words on E of length $\leq k$ is denoted by $E^{\leq k}$.

Proposition 2.1 [11], [12]. *A rational relation R of $E^* \times F^*$ which has differences bounded by k is equal to the behavior of a transducer $\mathcal{T} = (E \times F, Q, \alpha, T)$ with edges labeled by elements of $E \times F$, equipped with an initial function $\alpha: Q \to (E^{\leq k} \times \varepsilon) \cup (\varepsilon \times F^{\leq k})$.*

For the sake of completeness, we give here the proof of this result, which relies upon a key lemma of [7].

Lemma 2.2 [7]. *For every $S \in \mathrm{Rat}(E \times F)^*$ and every word $w \in E^*$ there exists a family $\{S_x | x \in E^{|w|}\}$ of rational sets of $(E \times F)^*$ such that*

$$S(w, 1) = \bigcup_{x\in E^{|w|}} (x, 1)S_x.$$

If $u = (f, g) \in E^* \times F^*$ we note $|u| = (|f|, |g|)$. Denote $D_k = (E^{\leq k} \times \varepsilon) \cup (\varepsilon \times F^{\leq k})$. Observe that an element u of $E^* \times F^*$ with length difference k can be uniquely written as a product $u = u'u''$ with $u' \in D_k$ and $u'' \in (E \times F)^*$.

Proof of Proposition 2.1. It mimics the proof of the theorem of Eilenberg and Schützenberger [7] and goes by induction on the star height of R. If R is of star height 0, it is a finite union of pairs with length difference k, each one being the product of an element of D_k and of an element of $(E \times F)^*$. Assume now that the proposition holds for every rational relation of star height h, and let R be of star height $h+1$. R is a finite union $R = \bigcup u_0 R_1^* u_1 \cdots u_{r-1} R_r^* u_r$, with the u_i's in $E^* \times F^*$ and the R_i's in $\mathrm{Rat}(E^* \times F^*)$ of star height at most h. Elementary length considerations show that every R_i has to be length-preserving, and thus $R_i \in \mathrm{Rat}(E \times F)^*$ and therefore $R_i^* \in \mathrm{Rat}(E \times F)^*$. From Lemma 2.2, $R_r^* u_r$ is a finite union of elements of the form vH, with $H \in \mathrm{Rat}(E \times F)^*$ and $v \in E^* \times F^*$, and $|v| = |u_r|$. Iterating this procedure for i going from r to 1 shows that the monomial $u_0 R_r^* u_1 \cdots u_{r-1} R_r^* u_r$ and thus R is a finite union of elements of the form wL, with $L \in \mathrm{Rat}(E \times F)^*$ and $|w| = |u_0 \cdots u_r|$. Since w has length difference k and may be written as $w = w'w''$, with $w' \in D_k$ and $w'' \in (E \times F)^*$, R is a finite union of elements of the form $w'L'$, $w' \in D_k$ and $L' \in \mathrm{Rat}(E \times F)^*$. Since L' is the behavior of a transducer with edges labeled by elements of $E \times F$, the result follows. □

Returning to the linear numeration systems we have

Proposition 2.2. *Normalization in basis U, restricted to words not beginning by* 0, *has bounded differences.*

Proof. Let f be a word of $(C \backslash \{0\})C^*$ of length $n+1$. Then $\pi(f) \leq cu_n + \cdots + cu_0$. Let $\theta_1 = \theta, \theta_2, \ldots, \theta_l$ be the roots of P, with multiplicity $\mu_1 = 1, \mu_2, \ldots, \mu_l$. So

$$u_n = \theta^n P_1(n) + \theta_2^n P_2(n) + \cdots + \theta_l^n P_l(n),$$

where $P_i(n)$ is a polynomial of degree $< \mu_i$, $1 \leq i \leq l$. As θ is the dominant root of P,

$$\lim_{n \to \infty} \frac{u_n}{\theta^n} = P_1(n) = \text{constant } \lambda.$$

So, for every $\varepsilon > 0$, $\exists N \geq 0$ such that $n \geq N \Rightarrow u_n < \lambda\theta^n(1+\varepsilon)$ and $\lambda\theta^n < u_n(1+\varepsilon)$. Let

$$x = c(u_0 + \cdots + u_n) = c(u_0 + \cdots + u_{N-1}) + c(u_N + \cdots + u_n).$$

Then

$$\begin{aligned} c(u_N + \cdots + u_n) &< \lambda c(1+\varepsilon)(\theta^N + \cdots + \theta^n) \\ &< \lambda c(1+\varepsilon)(1 + \cdots + \theta^n) \\ &< \lambda c(1+\varepsilon)\frac{\theta^{n+1}}{\theta - 1} \\ &< \lambda \theta^{n+j} \end{aligned}$$

with

$$j = \left[\log_\theta \frac{c(1+\varepsilon)}{\theta - 1}\right] + 2.$$

So

$$c(u_N + \cdots + u_n) < u_{n+j}(1 + \varepsilon).$$

Since $c(u_0 + \cdots + u_{N-1})$ is constant, there exists a constant $p > 0$ such that $x < u_{n+p}$. Thus, for every word f of $(C\backslash\{0\})C^*$, $|\nu(f)| - |f| \leq p - 1$. □

We are now able to prove

Lemma 2.3. *If the normalization* ν: $C^* \to A^*$ *in basis* U *is rational, then* $Z(U, c)$ *is rational.*

Proof. If ν: $C^* \to A^*$ is a rational function, then $\widehat{\nu^{-1} \circ \nu} = \{(f, g) \in C^* \times C^* \mid \pi(f) = \pi(g)\}$ is a rational subset of $C^* \times C^*$ and $Z(U, c) = \{h \in \tilde{C}^* \mid h = f \ominus g, (f, g) \in \widehat{\nu^{-1} \circ \nu}\}$. Denote $R = \widehat{\nu^{-1} \circ \nu} \cap (C\backslash\{0\}C^*) \times (C\backslash\{0\}C^*)$. As R is the intersection of a rational and of a recognizable subset of $C^* \times C^*$, R is a rational relation. Since ν has bounded differences, R also has bounded differences (by k).

Let $\mathcal{T}$ be the transducer defined in Proposition 2.1. From $\mathcal{T}$ a finite automaton $\mathcal{S}$ is constructed as follows. If $p \xrightarrow{(a,b)} q$, with a and b in C, is in $\mathcal{T}$, an edge $p \xrightarrow{a-b} q$ is created in $\mathcal{S}$. If $\alpha(i) = (u, \varepsilon)$ is defined in $\mathcal{T}$, an edge $e_0 \xrightarrow{u} i$ is created in $\mathcal{S}$ and e_0 is an initial state of $\mathcal{S}$. If $\alpha(i) = (\varepsilon, v)$ is defined in $\mathcal{T}$, an edge $e_1 \xrightarrow{-v} i$ is created in $\mathcal{S}$ and e_1 is an initial state of $\mathcal{S}$. The terminal states of $\mathcal{S}$ are those of $\mathcal{T}$. Then $(v, w) \in R$ if and only if $v \ominus w$ is recognized by $\mathcal{S}$. Thus, $Z(U, c)$ is rational. □

From Lemmas 2.1 and 2.3, we obtain the following.

Proposition 2.3. *If the set of normal representations* $L(U)$ *is rational, then the normalization* ν_C: $C^* \to A^*$ *is a rational function if and only if the set* $Z(U, c)$ *of words of* $\tilde{C}^*$ *equal to* 0 *in basis* U *is rational.*

It is noteworthy that there exist functions ν which are rational and such that the set $Z = \{f \ominus g \mid (f, g) \in \widehat{\nu^{-1} \circ \nu}\}$ is not rational, as shown by the following example.

Example 2.2. Let ν: $\{0, 1\}^* \to \{0, 1\}^*$ be the morphism defined by $\nu(0) = \varepsilon$ and $\nu(1) = 1$. Denote by $|f|_1$ the number of 1 in f. Then

$$Z = \{h = f \ominus g \in \{-1, 0, 1\}^* \mid |f|_1 = |g|_1\} = \{h \in \{-1, 0, 1\}^* \mid |h|_1 = |h|_{-1}\}$$

which is not a rational subset of $\{-1, 0, 1\}^*$.

2.2.1. Recognizability and Division. Define a mapping between words of $\tilde{C}^*$ and polynomials of $\mathbf{Z}[X]$ by $f = f_0 \cdots f_n \in \tilde{C}^* \mapsto F(X) = f_0X^n + \cdots + f_n$, $f_i \in \tilde{C}$. The *Gaussian norm* of F is $\|F\| = \max_{i=0,\ldots,n} |f_i|$. This gives a correspondence between words of $\tilde{C}^*$ and polynomials of $\mathbf{Z}[X]$ of norm at most c.

Let us denote by (P) the ideal of $\mathbf{Z}[X]$ generated by P, and by $I(P, c)$ the trace on $\tilde{C}^*$ of (P), that is,

$$I(P, c) = \{f = f_0 \cdots f_n \in \tilde{C}^* \mid F(X) = f_0 X^n + \cdots + f_n \in (P)\}.$$

This set is strictly included in $Z(U, c)$.

In this section we give a construction which links the recognizability of $I(P, c)$ by an automaton to the Euclidean division of polynomials of (P) by P.

Let $f = uvw$. Then u is a *left factor*, v is a *factor*, and w is a *right factor* of f. The set of left factors of elements of a language L is denoted by $LF(L)$.

Proposition 2.4. *The set $I(P, c)$ is recognizable by a finite automaton if and only if the number of remainders of the Euclidean division by P of polynomials associated to words of $LF(I(P, c))$ is finite.*

To recognize a word f of $I(P, c)$ we divide by P the polynomials associated to longer and longer left factors of f. The remainders obtained at each step are considered as the states of the automaton, and thus the automaton is finite if and only if the number of these remainders is finite.

The remainder $R(F, P)$ of the Euclidean division of F by P is a polynomial of degree at most $m - 1$. To $R(F, P)$ is associated the word $r_P(f) = r_0(f) \cdots r_{m-1}(f)$, whose letters $r_i(f)$ are the coefficients (possibly equal to 0) of the polynomial $R(F, P)$. We say that $r_P(f)$ is the *remainder of the division* of f by P.

The *right congruence* modulo $I(P, c)$ is denoted by $\sim_{I(P,c)}$, that is, if f and g are two words of $\tilde{C}^*$, then

$$f \sim_{I(P,c)} g \quad \Leftrightarrow \quad \forall h \in \tilde{C}^*, \quad fh \in I(P, c) \quad \text{iff} \quad gh \in I(P, c).$$

Lemma 2.4. *If f and g belong to $LF(I(P, c))$, then*

$$f \sim_{I(P,c)} g \quad \Leftrightarrow \quad r_P(f) = r_P(g).$$

Proof. Let F and G be the polynomials associated to the words f and g

(i) If $r_P(f) = r_P(g)$ there exists a polynomial $H \in \mathbf{Z}[X]$ such that $F = G + PH$. Thus for every word y of $\tilde{C}^*$, $FX^{|y|} + Y = GX^{|y|} + PHX^{|y|} + Y$ belongs to (P) if and only if $GX^{|y|} + Y$ is in (P). Hence $f \sim_{I(P,c)} g$.

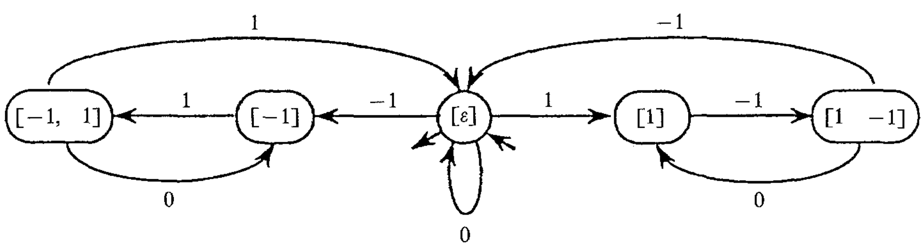
(ii) Since f is in $LF(I(P, c))$ and $g \sim_{I(P,c)} f$, there exists $y \in \tilde{C}^*$ such that fy and gy belong to $I(P, c)$. Then $FX^{|y|} + Y = PH$ and $GX^{|y|} + Y = PK$, with H and K in $\mathbf{Z}[X]$. We get $(F - G)X^{|y|} = P(H - K)$ so $F - G \in (P)$, because $a_m \neq 0$. Thus $r_P(f) = r_P(g)$. □

Proof of Proposition 2.4. It is a classical result that $I(P, c)$ is recognizable by a finite automaton if and only if the right congruence modulo $I(P, c)$ has finite index (see [7]). The words of $\tilde{C}^*$ which do not belong to $LF(I(P, c))$ are put in one single right class modulo $I(P, c)$. From Lemma 2.4 there exists a bijective correspondence between the right classes modulo $I(P, c)$ and the remainders of the division by P of the words of $LF(I(P, c))$. □

When the number of remainders by P of the words of $LF(I(P, c))$ is finite, the explicit construction of the minimal finite automaton $\mathscr{A} = (\tilde{C}, Q, i, i)$ which recognizes $I(P, c)$ follows from the above construction.

(i) The (finite) set of states Q is equal to the set of remainders by P of the elements of $LF(I(P, c))$, that is, the set of right classes modulo $I(P, c)$: $Q = \{[f]_{I(P,c)} = r_P(f) \mid f \in LF(I(P, c))\}$.
(ii) The initial state i is equal to $\{[\varepsilon]_{I(P,c)}\}$.
(iii) The terminal state is defined by: $\{[v]_{I(P,c)} \mid v \in I(P, c)\} = \{[\varepsilon]_{I(P,c)}\} = i$.
(iv) The transitions are of the form $[f]_{I(P,c)} \xrightarrow{a} [fa]_{I(P,c)}$ where $a \in \tilde{C}$.

Example 2.3. Let $P(X) = X^2 - X - 1$ be the characteristic polynomial of the Fibonacci sequence. Take $\tilde{C} = \{-1, 0, 1\}$. We denote $[\cdot]$ instead of $[\cdot]_{I(P,c)}$. The following finite automaton recognizes $I(P, c)$:



Since the polynomials considered belong to $\mathbf{Z}[X]$ we have

Corollary 2.1. *The set $I(P, c)$ is recognizable by a finite automaton if and only if the coefficients of the quotient by P of the words of $LF(I(P, c))$ are bounded.*

Proof. Let $f = f_0 \cdots f_n$ be a word of $I(P, c)$. Let $Q(X) = q_0 X^{n-m} + \cdots + q_{n-m}$ be the quotient of the Euclidean division of $F(X) = f_0 X^n + \cdots + f_n \in \mathbf{Z}[X]$ by P. For every $k \in [0, n]$, denote by $f^{(k)}$ the word $f_0 \cdots f_k$ and by $F^{(k)}$ the associated polynomial. Then

$$\begin{aligned} F^{(k)}(X) &= f_0 X^k + \cdots + f_k \\ &= P(X)(q_0 X^{k-m} + \cdots + q_{k-m}) + q_{k-m+1} X^{m-1} \\ &\quad + (-q_{k-m+1} a_1 + q_{k-m+2}) X^{m-2} \\ &\quad + (-q_{k-m+1} a_2 - q_{k-m+2} a_1 + q_{k-m+3}) X^{m-3} + \cdots \\ &\quad + (-q_{k-m+1} a_{m-1} - q_{k-m+2} a_{m-2} + \cdots + q_k) \end{aligned}$$

and thus the remainder of the Euclidean division of $F^{(k)}$ by P is

$$\begin{aligned} R^{(k)}(X) &= q_{k-m+1} X^{m-1} + (-q_{k-m+1} a_1 + q_{k-m+2}) X^{m-2} + \cdots \\ &\quad + (-q_{k-m+1} a_{m-1} - q_{k-m+2} a_{m-2} + \cdots + q_k). \end{aligned}$$

Let $r^{(k)} = r_0^{(k)} \cdots r_{m-1}^{(k)}$ be the word corresponding to the polynomial $R^{(k)}$. Let δ be the mapping $\delta\colon \mathbf{Z}^m \to \mathbf{Z}^m$,

$$\begin{pmatrix} q_{k-m+1} \\ q_{k-m+2} \\ \vdots \\ q_k \end{pmatrix} \mapsto \begin{pmatrix} r_0^{(k)} \\ r_1^{(k)} \\ \vdots \\ r_{m-1}^{(k)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -a_1 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ -a_{m-1} & -a_{m-2} & \cdots & -a_1 & 1 \end{pmatrix} \begin{pmatrix} q_{k-m+1} \\ q_{k-m+2} \\ \vdots \\ q_k \end{pmatrix}.$$

Since the matrix is invertible, δ is bijective.

Thus the coefficient q_i's of the quotient are bounded if and only if the $r_j^{(k)}$'s are bounded for every k. As these are elements of $\mathbf{Z}$, the number of different remainders is finite if and only if the coefficients of the quotient are bounded. □

This result leads to the following question: What are the polynomials P such that the division by P of a polynomial F in the ideal (P) and of norm at most c, gives a polynomial with all coefficients bounded by a constant depending only on P and c?

We thus have

Definition 2.2. A polynomial P of $\mathbf{C}[X]$ satisfies the *bounded division property* (in short BD) if, for every $c > 0$, there exists a constant $\beta(P, c)$ such that, for every polynomial F of $\mathbf{C}[X]$, $F = PQ$, $Q \in \mathbf{C}[X]$, $\|F\| \leq c$, implies $\|Q\| \leq \beta(P, c)$.

Proposition 2.5 [4]. *The polynomials satisfying the BD are exactly the polynomials having no root of modulus* 1.

To prove this statement we first give a technical lemma about the Gaussian norm.

Lemma 2.5. *Let F and G be polynomials of* $\mathbf{C}[X]$, *with degree n and k respectively. Then* $\|FG\| \leq (1 + \min(n, k))\|F\| \cdot \|G\|$.

Proof. Let $F(X) = f_0 + f_1X + \cdots + f_nX^n$ and $G(X) = g_0 + g_1X + \cdots + g_kX^k$. Then $FG(X) = \sum_{i=0}^{n+k} h_iX^i$ with $h_i = \sum_{j=0}^{i} f_{i-j}g_j$, so $|h_i| \leq (1 + \min(n, k))\|F\| \cdot \|G\|$. □

The degree of a polynomial F is denoted by $d(F)$.

Proof of Proposition 2.5. (1) Let P_1 and P_2 be polynomials satisfying the BD. Then $P = P_1P_2$ satisfies the BD. To see that, let $c > 0$ and F be a polynomial equal to PQ, where $Q \in \mathbf{C}[X]$, such that $\|F\| \leq c$. Then $\|P_1P_2Q\| \leq c$. Since P_1 satisfies the BD, $\|P_2Q\| \leq \beta(P_1, c)$, and so $\|Q\| \leq \beta(P_2, \beta(P_1, c))$ since P_2 satisfies the BD.

(2) If $P = P_1P_2$ satisfies the BD, then P_1 and P_2 satisfy the BD. Let P_1 be a polynomial and $P = P_1P_2$ satisfying the BD. Let $c > 0$ and Q such that $\|P_1Q\| \leq c$. Let $d_2 =$ degree of P_2. Then

$$\|PQ\| = \|P_1P_2Q\| \leq (d_2 + 1)\|P_2\| \cdot \|P_1Q\| \leq c(d_2 + 1)\|P_2\|$$

(Lemma 2.5). Since P satisfies the BD, we have $\|Q\| \le \beta(P, c(d_2+1)\|P_2\|)$, which depends only on P_1 and c, since P and P_2 depend only on P_1. Thus P_1 satisfies the BD. Similarly P_2 satisfies the BD.

It follows from (1) and (2) that the set of polynomials satisfying the BD is composed of products $X-\alpha$, where $\alpha \in \mathbf{C}$ such that $X-\alpha$ satisfies the BD. So it is enough to consider $X-\alpha$.

(3) *Case* $|\alpha|>1$. Let $F(X)=f_0+f_1X+\cdots+f_nX^n$ such that $\|F\|\le c$ and $F=(X-\alpha)Q$. Then

$$\frac{F}{X-\alpha}=\frac{1}{\alpha}\frac{F}{1-X/\alpha}=-\frac{1}{\alpha}F\left(1+\frac{X}{\alpha}+\frac{X^2}{\alpha^2}+\cdots\right)=\sum_{k=0}^{n-1}q_kX^k$$

with

$$q_k=-\frac{1}{\alpha}\sum_{i=0}^{k}f_{k-i}\frac{1}{\alpha^i}.$$

Then

$$|q_k|<\frac{1}{|\alpha|}c\sum_{i\ge 0}\frac{1}{|\alpha|^i}=\frac{c}{|\alpha|-1}.$$

(4) *Case* $|\alpha|<1$. If $F=f_0+\cdots+f_nX^n$, its reciprocal polynomial is $\bar{F}=f_0X^n+\cdots+f_n$. First notice that if $F=PQ$, then $\bar{F}=\bar{P}\bar{Q}$. Take $P(X)=(X-\alpha)$. Then $\bar{F}(X)=(1-\alpha X)\bar{Q}(X)$. Since $\|\bar{F}\|=\|F\|$ and $|\alpha|<1$, we may use the previous case, and $\|Q\|=\|\bar{Q}\|<c/(1-|\alpha|)$.

(5) *Case* $|\alpha|=1$. First we show that this case reduces to the case $\alpha=1$. Let $X=\alpha u$ and $Q(X)=F(X)/(X-\alpha)$. Then $Q(\alpha u)=F(\alpha u)/(\alpha u-\alpha)=(1/\alpha)(F(\alpha u)/(u-1))$. Let H be the polynomial defined by $H(u)=F(\alpha u)$. Since $|\alpha|=1$, $\|H\|=\|F\|$. Let $V(u)=H(u)/(u-1)$. Then $\|V\|=\|Q\|$. So we consider the case $X-1$.

Let t be an integer ≥ 1 and

$$B_t(X)=-1-X-\cdots-X^{t-1}+X^t+\cdots+X^{2t-1}.$$

1 is a root of this polynomial, let $Q(X)$ be the quotient of B_t by $X-1$. The coefficient of X^t in $Q(X)$ is equal to $t-1$ and so $\|Q\|\ge t-1$. Thus $X-1$ does not satisfy the BD. □

From the characterization given in Proposition 2.5 we deduce

Theorem 2.1. *The set of words of* $\tilde{C}^*$, *the associated polynomial of which belongs to* (P), *is recognizable by a finite automaton for every positive integer* c *if and only if* P *has no root of modulus* 1.

Proof. For a fixed c, $I(P, c)$ is recognizable if and only if the coefficients of the division by P of words of $LF(I(P, c))$ are bounded (Corollary 2.1). Thus $I(P, c)$ is recognizable for every $c>0$ if and only if P satisfies the BD, that is, if and only if P has no root of modulus 1 (Proposition 2.5). □

Example 2.4. The Fibonacci polynomial $P(X) = X^2 - X - 1$ has no root of modulus 1, thus $I(P, c)$ is recognizable for every $c \geq 1$.

Example 2.5. Let $u_{n+2} = u_{n+1} + 2u_n$ and $P(X) = X^2 - X - 2 = (X + 1)(X - 2)$ be its characteristic polynomial. We can verify that

$$I(P, 3) \cap (-1)(3(-3))^*1(3(-3))^*2 = \{(-1)(3(-3))^p1(3(-3))^p2 \mid p \geq 0\}.$$

Since this set is not rational, $I(P, 3)$ is not rational either.

2.2.2. Rationality of the Set of Words Equal to 0 in Basis U. We now prove that if the set $Z(U, c)$ of words of $\tilde{C}^*$ equal to 0 in basis U is rational, then $I(P, c)$ is also rational. If $F(X) = f_0X^n + \cdots + f_n$, $F|_U$ denotes the value $f_0u_n + \cdots + f_nu_0 = \pi(f_0 \cdots f_n)$.

Theorem 2.2. *If P has one root of modulus* 1, *then there exists* $c_0 > 0$ *such that, for every* $c \geq c_0$, *the normalization* ν_C *is not rational.*

This is a consequence of the following.

Proposition 2.6. *Let* $c > 0$ *be a fixed integer. If* $Z(U, c)$ *is rational, then* $I(P, c)$ *is rational.*

We need two lemmas. It is assumed that P is the minimal polynomial of the linear recurrence, that is, U is not degenerate.

Lemma 2.6. *Let f be a word of* $LF(I(P, c))$. *Then* $[f]_{I(P,c)} \subset [f]_{Z(U,c)}$.

Proof. Clearly, $LF(I(P, c))$ is included in $LF(Z(U, c))$. Let f and g be two words of $LF(I(P, c))$ equivalent modulo $I(P, c)$, let F and G be the associated polynomials. Then there exists a polynomial Q such that $F = PQ + G$, and then, for every y of $\tilde{C}^*$, $FX^{|y|} + Y = GX^{|y|} + PQX^{|y|} + Y$. Thus $(FX^{|y|} + Y)|_U = 0$ if and only if $(GX^{|y|} + Y)|_U = 0$ so $f \sim_{Z(U,c)} g$.

Lemma 2.7. *An equivalence class modulo* $Z(U, c)$ *can contain only a finite number of classes modulo* $I(P, c)$.

Proof. Let us suppose that there exist infinitely many different classes modulo $I(P, c)$ with representatives $f_1, f_2, \ldots$, in the same class modulo $Z(U, c)$. Thus there exists an infinity of words $w_1, w_2, \ldots$ such that $f_iw_i \in I(P, c)$ and $f_jw_i \notin I(P, c)$ for $i \neq j$, else we would get f_iw_i and $f_jw_i \in I(P, c)$, which would imply $F_i - F_j \in (P)$, and then $f_i \sim_{I(P,c)} f_j$, contrary to the hypothesis. For every i and j we have $f_i \sim_{Z(U,c)} f_j$, so $\pi(f_iw_i) = 0$ and $\pi(f_jw_i) = 0$, thus $\pi((f_j - f_k)0^{|w_i|}) = 0$ for every i, j, k. So $f_j - f_k \in I(P, c)$ since P is the minimal polynomial of the linear recurrence, a contradiction. □

Proof of Proposition 2.6. If $Z(U, c)$ is rational, then the number of classes modulo $Z(U, c)$ is finite. From the above lemma, the number of classes modulo $I(P, c)$ is finite and $I(P, c)$ is rational. □

The proof of Theorem 2.2 easily follows from Propositions 2.3 and 2.6.

Example 2.6. Normalization in basis $u_{n+2} = u_{n+1} + 2u_n$ is not rational on any alphabet C containing $\{0, \ldots, 3\}$ (see Example 2.5).

Since we do not know whether the rationality of $I(P, c)$ implies the rationality of $Z(U, c)$, the question whether P has no root of modulus 1 implies that normalization in basis U is rational on any alphabet is still open.

3. The Real Numbers

3.1. Representation of Real Numbers

Let $\theta > 1$ and $x \geq 0$ be two real numbers. Every infinite sequence of positive integers $(z_n)_{n \geq 0}$ such that $x = \sum_{n \geq 0} z_n \theta^{-n}$ is a *θ-representation* of x. A particular θ-representation called the *θ-development* or the *θ-expansion* can be computed by the following algorithm (see [18]). Denote by $[y]$ and by $\{y\}$ the integer part and the fractional part of a number y. Let $x_0 = [x]$ and $r_0 = \{x\}$, and, for $i \geq 1$: $x_i = [\theta r_{i-1}]$ and $r_i = \{\theta r_{i-1}\}$. Then $x = \sum_{k \geq 0} x_k \theta^{-k}$.

For $i \geq 1$, $x_i < \theta$. If $\theta \in \mathbf{N}$, the canonical alphabet is $A = \{0, \ldots, \theta - 1\}$ and if $\theta \notin \mathbf{N}$, $A = \{0, \ldots, [\theta]\}$. We write $x = x_0.x_1x_2\cdots$ where x_0 is the integer and $.x_1x_2\cdots$ is the fractional part of x. The θ-development of x is the normal θ-representation of x and it is greater in the lexicographical ordering than any θ-representation of x.

It is clear that if $\theta = t_0.t_1t_2\cdots$ is the θ-development of θ, then $1 = 0.t_0t_1\cdots$. The sequence $t_0t_1\cdots$ is denoted by $d(1)$ and by extension is called the θ-development of 1. Let $x \in [0, 1[$ of θ-development $0.x_1x_2\cdots$. The sequence $x_1x_2\cdots \in A^{\mathbf{N}}$ is also said to be the θ-development of x.

Let C be any finite subset of the integers. As for the integers the *normalization* function ν_C: $C^{\mathbf{N}} \to A^{\mathbf{N}}$, where A is the canonical alphabet, maps a sequence $(y_n)_n$ of numerical value x in basis θ onto the θ-development of x.

Let D_θ be the set of θ-developments of numbers of $[0, 1[$, and let d: $[0, 1] \to D_\theta \cup \{d(1)\}$ be the function mapping $x \neq 1$ onto its θ-development $d(x)$ and 1 onto $d(1)$. The closure of D_θ is denoted by S_θ. Then S_θ is a subshift of $A^{\mathbf{N}}$ and $S_\theta = D_\theta \cup \{d(1)\}$. The subshift S_θ is called the *θ-shift* (see [3] and [5]). S_θ is a system of *finite type* if the set of finite factors $F(S_\theta)$ is defined by the interdiction of a finite set of words. S_θ is a *sofic* system if $F(S_\theta)$ is recognizable by a finite automaton. Recall that the θ-shift S_θ is a system of finite type if and only if $d(1)$ is finite [15]. S_θ is a sofic system if and only if $d(1)$ is eventually periodic [3].

Example 3.1. Let $\theta = (1 + \sqrt{5})/2$. Then $d(1) = 11$. Let $\theta = (3 + \sqrt{5})/2$. Then $d(1) = 21^\omega$.

3.2. *Normalization of Infinite Words*

In this section we characterize the numbers θ such that normalization in basis θ is rational on any alphabet.

Let us fix some definitions. An infinite path in a finite automaton $\mathscr{A} = (E, Q, I, T)$ is *successful* if it starts in I and goes infinitely often through T. The *infinite behavior* of an automaton is the set of all its successful paths. A subset of $E^{\mathbf{N}}$ is said to be *recognizable* if it is the infinite behavior of a finite automaton, that is, if it is Büchi-recognizable (see [7]).

A relation $R \subset E^{\mathbf{N}} \times F^{\mathbf{N}}$ is *rational* if it is the infinite behavior of a transducer.

A function φ: $E^{\mathbf{N}} \to F^{\mathbf{N}}$ is *rational* if its graph is a rational relation.

3.2.1. Recognizability and Division. As for the integers we first consider the set of infinite words on $\tilde{C}^{\mathbf{N}}$ equal to 0 in basis θ,

$$Z(\theta, c) = \{s = (s_n)_{n \geq 0} \in \tilde{C}^{\mathbf{N}} \mid \textstyle\sum_{n \geq 0} s_n \theta^{-n} = 0\}.$$

To every infinite word $s = (s_n)_{n \geq 0}$ of $\tilde{C}^{\mathbf{N}}$ is associated a formal power series $S(X) = \sum_{n \geq 0} s_n X^n$ in $\mathbf{Z}[[X]]$ whose *Gaussian norm* is $\|S\| = \sup_{n \geq 0} |s_n| \leq c$.

A construction similar to the one given in the case of finite words links the recognizability of $Z(\theta, c)$ and the division of polynomials by the polynomial $X - \theta$. Let us denote by $LF(Z(\theta, c))$ the set $\{w \in \tilde{C}^* \mid \exists s \in \tilde{C}^{\mathbf{N}}, ws \in Z(\theta, c)\}$. Let $f = f_0 \cdots f_n$ and $g = g_0 \cdots g_k \in \tilde{C}^*$. f and g are said to be *right congruent* modulo $Z(\theta, c)$ if, for every $s \in \tilde{C}^{\mathbf{N}}$, $fs \in Z(\theta, c)$ if and only if $gs \in Z(\theta, c)$. Let $F(X) = f_0 X^n + \cdots + f_n$ and $G(X) = g_0 X^k + \cdots + g_k$. Denote by $r_\theta(f)$ (resp. $r_\theta(g)$) the remainder of the Euclidean division of F (resp. G) by $X - \theta$.

Lemma 3.1. *If f and g belong to $LF(Z(\theta, c))$, then*

$$f \sim_{Z(\theta,c)} g \quad \Leftrightarrow \quad r_\theta(f) = r_\theta(g).$$

Proof. Let $f = f_0 \cdots f_n$ and $g = g_0 \cdots g_k$ be two words of $\tilde{C}^*$. Suppose $n \geq k$. Let F and G be the associated polynomials. Then $F - G \in (X - \theta)$ if and only if $\bar{F} - X^{n-k}\bar{G} \in (1 - \theta X)$.

(i) If $r_\theta(f) = r_\theta(g)$, then $\bar{F} - X^{n-k}\bar{G} \in (1 - \theta X)$, and there exists $H \in \mathbf{Z}[X]$ such that $\bar{F} = X^{n-k}\bar{G} + (1 - \theta X)\bar{H}$. Thus, for every $s \in \tilde{C}^{\mathbf{N}}$, $s = (s_n)_{n \geq 0}$, we have

$$\begin{aligned} f_0 + \cdots + \frac{f_n}{\theta^n} + \frac{s_0}{\theta^{n+1}} + \frac{s_1}{\theta^{n+2}} + \cdots &= \frac{1}{\theta^{n-k}}\left(g_0 + \cdots + \frac{g_k}{\theta^k}\right) + \frac{s_0}{\theta^{n+1}} + \frac{s_1}{\theta^{n+2}} + \cdots \\ &= \frac{1}{\theta^{n-k}}\left(g_0 + \cdots + \frac{g_k}{\theta^k} + \frac{s_0}{\theta^{k+1}} + \frac{s_1}{\theta^{k+2}} + \cdots\right) \end{aligned}$$

and $fs \in Z(\theta, c)$ if and only if $gs \in Z(\theta, c)$.

(ii) Let $s \in \tilde{C}^{\mathbf{N}}$ such that fs and $gs \in Z(\theta, c)$. Then

$$f_0 + \cdots + \frac{f_n}{\theta^n} + \frac{s_0}{\theta^{n+1}} + \frac{s_1}{\theta^{n+2}} + \cdots = g_0 + \cdots + \frac{g_k}{\theta^k} + \frac{s_0}{\theta^{k+1}} + \frac{s_1}{\theta^{k+2}} + \cdots = 0$$

so $(\bar{F} - X^{n-k}\bar{G})(\theta^{-1}) = 0$ and $r_\theta(f) = r_\theta(g)$. □

Proposition 3.1. *Let θ be a real number >1. The set $Z(\theta, c)$ of words of $\tilde{C}^{\mathbb{N}}$ equal to 0 in basis θ is recognizable by a finite automaton if and only if the number of remainders of the Euclidean division by the polynomial $X-\theta$ of polynomials associated to words of $LF(Z(\theta, c))$ is finite.*

We need a lemma on recognizable sets of infinite words.

Lemma 3.2. *Let S be a recognizable subset of $E^{\mathbb{N}}$. The right congruence modulo S has finite index.*

Proof. Let $\mathscr{A}=(E, Q, I, T)$ be an automaton whose infinite behavior is equal to S. Let $f\in E^*$ and $I(f)=\{q\in Q \mid \exists i\in I\, i\xrightarrow{f} q\}$. An equivalence relation, denoted by $\equiv$, is defined on E^* by $f\equiv g$ if and only if $I(f)=I(g)$. Then $\equiv$ is finer than $\sim_S$: for every s of $E^{\mathbb{N}}$, $fs\in S$ if and only if there exist $i\in I$ and $q\in I(f)$ such that the path $i\xrightarrow{f} q\xrightarrow{s}\cdots$ goes infinitely often through T. If $f\equiv g$, then $q\in I(g)$, and so there exists $j\in I$ such that the infinite path $j\xrightarrow{g} q\xrightarrow{s}\cdots$ is successful if and only if the infinite path $i\xrightarrow{f} q\xrightarrow{s}\cdots$ is successful. As the relation $\equiv$ has finite index, $\sim_S$ has also finite index. □

Proof of Proposition 3.1. If $Z(\theta, c)$ is recognizable, then the equivalence $\sim_{Z(\theta,c)}$ has finite index. Conversely, if the number of remainders is finite, then the number of classes modulo $Z(\theta, c)$ is finite. We define, as in the previous section, a finite automaton $\mathscr{B}=(\tilde{C}, Q, i, Q)$ by:

(i) The (finite) set of states Q is $\{[f]_{Z(\theta,c)}=r_\theta(f) \mid f\in LF(Z(\theta, c))\}$.
(ii) The initial state i is equal to $\{[\varepsilon]_{Z(\theta,c)}\}=0$.
(iii) Every state is terminal.
(iv) The transitions are of the form $[f]_{Z(\theta,c)}\xrightarrow{a}[fa]_{Z(\theta,c)}$ where $a\in\tilde{C}$.

The infinite behavior of $\mathscr{B}$ is equal to $Z(\theta, c)$:

(a) Let $s=(s_n)\in Z(\theta, c)$. Then, for every $n\geq 0$, $0\xrightarrow{s_0\cdots s_n}[s_0\cdots s_n]_{Z(\theta,c)}$ is a path in $\mathscr{B}$ and s is the label of an infinite successful path in $\mathscr{B}$.
(b) Conversely let $s=(s_n)_{n\geq 0}$ be the label of an infinite successful path origining in the initial state 0 of $\mathscr{B}$. Thus, for every $n\geq 0$, $s_0+s_1X+\cdots+s_nX^n=(1-\theta X)(q_0+q_1X+\cdots+q_{n-1}X^{n-1})+e_nX^n$ where e_n is the remainder. So $e_n=\theta^n s_0+\cdots+s_n$. As the remainder is bounded, $\lim_{n\to\infty}|s_0+\cdots+s_n\theta^{-n}|=0$ and so $S(\theta^{-1})=0$. □

This construction also yields

Corollary 3.1. *If $c\geq[\theta]$ and $Z(\theta,c)$ is recognizable, then θ is an algebraic integer.*

Proof. Let $d(1) = (t_n)_{n \geq 1}$ be the θ-development of 1. Then $(-1)t_1t_2 \cdots \in Z(\theta, c)$. From the above construction there exist n and p such that the states $e_n = [(-1)t_1 \cdots t_n]_{Z(\theta,c)}$ and $e_{n+p} = [(-1)t_1 \cdots t_{n+p}]_{Z(\theta,c)}$ are the same. Since $e_n = -\theta^n + t_1\theta^{n-1} + \cdots + t_n$ and $e_{n+p} = -\theta^{n+p} + t_1\theta^{n+p-1} + \cdots + t_{n+p}$, θ is the root of a monic polynomial of $\mathbf{Z}[X]$. □

Thus we can restrict ourselves to the case where θ is an algebraic integer.

Proposition 3.2. *Let θ be an algebraic integer >1. The set $Z(\theta, c)$ is recognizable by a finite automaton if and only if the number of remainders of the division by the minimal polynomial M of θ of polynomials associated to words of $LF(Z(\theta, c))$ is finite.*

Proof. Two polynomials F and G of $\mathbf{Z}[X]$ have same remainder by M if and only if they have same remainder by $X - \theta$. □

From Proposition 3.2 we deduce

Corollary 3.2. *If $Z(\theta, c)$ is recognizable, then, for every $(s_n)_{n\geq 0} \in Z(\theta, c)$ and for every root α of modulus >1 of M, $\sum_{n\geq 0} s_n\alpha^{-n} = 0$.*

Proof. Let d be the degree of M. For every $n \geq 0$, $s_0X^n + \cdots + s_n = MQ + R$ where Q is a polynomial of degree $n - d$ and R is the remainder of Euclidean division by M. Thus $s_0 + s_1\alpha^{-1} + \cdots + s_n\alpha^{-n} = \alpha^{-n}R(\alpha)$. As the number of remainders is finite and $|\alpha| > 1$, $\sum_{n\geq 0} s_n\alpha^{-n} = \lim_{n\to\infty} |s_0 + s_1\alpha^{-1} + \cdots + s_n\alpha^{-n}| = 0$. □

As above, the number of remainders is finite if and only if the coefficients of the quotient are bounded since the polynomials belong to $\mathbf{Z}[X]$.

Corollary 3.3. *The set $Z(\theta, c)$ is recognizable if and only if the coefficients of the quotient by M of the words of $LF(Z(\theta, c))$ are bounded.*

This leads to

Definition 3.1. A polynomial P of $\mathbf{C}[X]$ satisfies the *bounded division property on series* (in short BDS) if, for every $c > 0$, there exists a constant $\beta(P, c)$ such that, for every $S \in \mathbf{C}[[X]]$, $\|S\| \leq c$, $S(\lambda) = 0$ for every root λ of modulus <1 of P, $S = PT$ with $T \in \mathbf{C}[[X]]$, imply $\|T\| \leq \beta(P, c)$.

Clearly, if a polynomial satisfies the BDS, then it satisfies the BD. The converse is also true.

Proposition 3.3. *The polynomials satisfying the BDS are exactly the polynomials having no root of modulus 1.*

Lemma 3.3. *Let F be a polynomial of* $\mathbf{C}[X]$ *and let S be a formal series of* $\mathbf{C}[[X]]$. *Then* $\|FS\| \le (d(F)+1)\|F\| \cdot \|S\|$.

Proof. Let $T = (t_n) = FS$. Then $t_i = \sum_{j=0}^{i} f_{i-j} s_j$. □

Proof of Proposition 3.3. As in the proof of Proposition 2.5, if P_1 and P_2 are two polynomials of $\mathbf{C}[X]$, P_1 and P_2 satisfy the BDS if and only if $P_1 P_2$ satisfies the BDS, with the help of Lemma 3.2.

So we consider polynomials of the form $X - \alpha$. Let $c > 0$, $S = (X - \alpha)T$, and $\|S\| \le c$. Put $S = \sum_{n\ge 0} s_n X^n$ and $T = \sum_{n \ge 0} t_n X^n$. So, for every $n \ge 0$, $t_n = -(1/\alpha)\sum_{i=0}^{n} s_{n-i}/\alpha^i$.

(1) $|\alpha| > 1$. Then $|t_n| \le c/(|\alpha| - 1)$.
(2) $|\alpha| < 1$. We have $t_n = -(1/\alpha^{n+1})(s_0 + s_1\alpha + \cdots + s_n\alpha^n)$. Since $S(\alpha) = \sum_{i\ge 0} s_i \alpha^i = 0$, $t_n = (1/\alpha^{n+1})(s_{n+1}\alpha^{n+1} + s_{n+2}\alpha^{n+2} + \cdots) = s_{n+1} + s_{n+2}\alpha + \cdots$ and so $|t_n| \le c\sum_{i\ge 0}|\alpha|^i = c/(1-|\alpha|)$ because $|\alpha| < 1$.
(3) $|\alpha| = 1$. The counterexample given in the proof of Proposition 2.5 leads to the conclusion. □

We thus get

Theorem 3.1. *Let θ be an algebraic integer >1 and let M be its minimal polynomial. The set $Z(\theta, c)$ is recognizable for every c if and only if M has no root of modulus 1, and if, for every infinite word $s = (s_n)_{n \ge 0}$ of $Z(\theta, c)$, we have $\sum_{n\ge 0} s_n \alpha^{-n} = 0$ for every root α of modulus >1 of M.*

Proof. (1) If, for every c, $Z(\theta, c)$ is recognizable, then, for every $s \in Z(\theta, c)$, the associated series S verifies $S(\alpha^{-1}) = 0$ for every root α of modulus >1 of M (Corollary 3.2) and thus $S(\beta) = 0$ for every root β of modulus <1 of $\bar{M}$. By Proposition 3.2 and Corollary 3.3, $\bar{M}$ satisfies the BDS and so M has no root of modulus 1.

(2) Conversely, the results follows from Proposition 3.2. □

Corollary 3.4. *If θ is a Pisot number, then, for every $c > 0$, $Z(\theta, c)$ is recognizable. If θ is a Salem number, then there exists $c_0 > 0$ such that, for every $c \ge c_0$, $Z(\theta, c)$ is not recognizable.*

3.2.2. θ-Representations of 1. Let θ be a real number >1, $c \ge [\theta]$, and let $E(\theta, c) = \{(s_n)_{n\ge 1} \in C^{\mathbf{N}} \mid 1 = \sum_{n\ge 1} s_n \theta^{-n}\}$ be the set of all θ-representations of 1 on the alphabet C. Clearly, $(-1)E(\theta, c) = Z(\theta, c) \cap (-1)C^{\mathbf{N}}$. So if $Z(\theta, c)$ is recognizable, then $E(\theta, c)$ is recognizable, which is true in particular if θ is Pisot (see [17] where this result is proved by different methods). Actually, the construction given above in Proposition 3.1 for the set $Z(\theta, c)$ is easily transferred to the set $(-1)E(\theta, c)$.

Proposition 3.4. *The set $E(\theta, c)$ is recognizable if and only if the number of remainders of the division by $X - \theta$ of polynomials associated to words of $LF((-1)E(\theta, c))$ is finite.*

Theorem 3.2. *Let $c \geq [\theta]$. If $E(\theta, c)$ is recognizable, the θ-shift is a sofic system.*

Proof. Let $\mathscr{R}$ be the automaton of remainders which recognizes $-1E(\theta, c)$. Since $(t_n)_{n\geq 1}$ is the θ-development of 1, $(-1)t_1t_2\cdots$ is a successful path in $\mathscr{R}$. Thus, for every $n \geq 1$, $0 \xrightarrow{(-1)t_1\cdots t_n} e_n$ where $e_n = -\theta^n + t_1\theta^{n-1} + \cdots + t_n$.

On the other hand, from the computation of the θ-development of 1 we obtain $t_1 = [\theta]$, $r_1 = \{\theta\}, \ldots, t_n = [\theta r_{n-1}]$, $r_n = \{\theta r_{n-1}\}$. Thus $r_n = \theta^n - t_1\theta^{n-1} - \cdots - t_n = -e_n$.

So the number of these r_n is finite. Let $N \geq 1$ and $p \geq 1$ be the smallest integers such that $r_N = r_{N+p}$. Thus $t_{N+1} = t_{N+p+1}$ and $r_{N+1} = r_{N+p+1}\ldots$ Hence $d(1) = t_1\cdots t_N(t_{N+1}\cdots t_{N+p})^\omega$, i.e., $d(1)$ is eventually periodic. From a result of [3] if follows that the θ-shift is a sofic system. □

It is known that if $d(1)$ is eventually periodic, then θ cannot have a real conjugate >1 (see [5]).

Corollary 3.5. *Let $c \geq [\theta]$. If $E(\theta, c)$ is recognizable, then θ is a Perron number with every conjugate of modulus <2, without a real conjugate >1 and such that if $1 = \sum_{n\geq 1} s_n\theta^{-n}$, with $s_n \leq c$, then $1 = \sum_{n\geq 1} s_n\alpha^{-n}$ for every conjugate α of modulus >1.*

Proof. If the θ-shift is sofic, then θ is a Perron number [14], with every conjugate of modulus <2 [15]. The same proof as for Corollary 3.2 implies that, for every conjugate α of modulus >1, if $(s_n)_{n\geq 1} \in E(\theta, c)$, then $1 = \sum_{n\geq 1} s_n\alpha^{-n}$. □

Remark 3.1. There exist Perron numbers θ which are not Pisot numbers such that E(θ, $[\theta]$) is recognizable, as it is shown in the examples below.

Example 3.2. Let θ be the dominant root of the polynomial $X^4 - 3X^3 - 2X^2 - 3$. It is a Perron number which is neither Pisot nor Salem. We have $d(1) = 3203$. We show that $E(\theta, [\theta]) = (3202)^*32030^\omega \cup (3202)^\omega$, which is a recognizable set. Let $s = (s_n)_{n\geq 1}$ be an infinite word of $\{0, \ldots, 3\}^{\mathbb{N}}$ such that $\pi(s) = \sum_{n\geq 1} s_n\theta^{-n} = 1$. If $s_1s_2s_3s_4 >_{\text{lex}} 3203$, $\pi(s) > 1$, because the θ-development 3203 is greater in the lexicographical ordering than any θ-representation of 1. Next if $s_1s_2s_3s_4 <_{\text{lex}} 3202$, then $\pi(s) < 1$. The conclusion follows.

Example 3.3. Let θ be the dominant root of the polynomial $X^4 - 2X^3 - 2X^2 - 2X + 1$. θ is a Salem number and $d(1) = 2(211)^\omega$. Then $E(\theta, [\theta]) = 2(211)^\omega$: let $s = (s_n)_{n\geq 1}$ be an infinite word of $\{0, 1, 2\}^{\mathbb{N}}$ such that $\pi(s) = \sum_{n\geq 1} s_n\theta^{-n} = 1$. If $s_1s_2s_3s_4 >_{\text{lex}} 2211$, then $\pi(s) > 1$. If $s_1s_2s_3s_4 <_{\text{lex}} 2211$, $\pi(s) < 1$. Then there remains only a finite number of cases to consider.

3.2.3. Application to Normalization in Basis θ. From the above results we deduce that if $Z(\theta, c)$ is recognizable, then S_θ is sofic, and so the set of θ-developments D_θ is recognizable. Using the same tools as in Proposition 2.3 we are able to show

Proposition 3.5. *The normalization ν_C: $C^{\mathbb{N}} \to A^{\mathbb{N}}$ is rational if and only if $Z(\theta, c)$ is recognizable.*

First we show

Lemma 3.4. *If the set $Z(\theta, c)$ is recognizable, then ν is rational.*

Proof. The graph of ν is $\hat{\nu} = \{(s, t) \in C^{\mathbb{N}} \times A^{\mathbb{N}} | t \in D_\theta, s - t \in Z(\theta, c)\}$ where $s - t$ denotes the infinite word $(s_n - t_n)_{n \geq 0}$, with $s = (s_n)_{n \geq 0}$ and $t = (t_n)_{n \geq 0}$. Let $\mathscr{A} = (A, Q_1, I_1, T_1)$ be the automaton recognizing D_θ and $\mathscr{B} = (\tilde{C}, Q_2, I_2, T_2)$ be the automaton recognizing $Z(\theta, c)$. From $\mathscr{B}$ an automaton $\mathscr{S}$ on $C \times A$ which recognizes words $(s, t) \in C^{\mathbb{N}} \times A^{\mathbb{N}}$ such that $s - t \in Z(\theta, c)$ is constructed as follows: for every edge $p \xrightarrow{z} q$ of $\mathscr{B}$, $z \in \tilde{C}$, a finite set of edges $p \xrightarrow{(a,b)} q$ for every a in C and b in A such that $a - b = z$ is defined in $\mathscr{S}$. The states of $\mathscr{S}$ are kept for $\mathscr{B}$: $\mathscr{S} = (C \times A, Q_2, I_2, T_2)$. Then the infinite behaviour of $\mathscr{S}$ is $L_{\text{inf}}(\mathscr{S}) = \{(s, t) \in C^{\mathbb{N}} \times A^{\mathbb{N}} | s - t \in Z(\theta, c)\}$.

To recognize $\hat{\nu}$, only the couples (s, t) of $L_{\text{inf}}(\mathscr{S})$ such that the second component t belongs to D_θ, are retained. Let $\mathscr{C}$ be the following automaton on $C \times A$:

$$\mathscr{C} = (C \times A, Q_1 \times Q_2, I_1 \times I_2, T_1 \times T_2),$$

where the edges are: $(p_1, p_2) \xrightarrow{(a,b)} (q_1, q_2)$ is an edge if and only if $p_1 \xrightarrow{(a,b)} q_1$ in $\mathscr{S}$ and $p_2 \xrightarrow{b} q_2$ in $\mathscr{A}$. Then

$$L_{\text{inf}}(\mathscr{C}) = \{(s, t) \in C^{\mathbb{N}} \times A^{\mathbb{N}} | (s, t) \in L_{\text{inf}}(\mathscr{S}) \text{ and } t \in L_{\text{inf}}(\mathscr{A})\} = \hat{\nu}$$

and thus ν is rational. □

As in the case of integers we need a precise characterization of normalization.

Definition 3.2. A rational relation of $E^{\mathbb{N}} \times F^{\mathbb{N}}$ has a *bounded delay* if it is the infinite behavior of a transducer $\mathscr{T} = (E \times F, Q, \alpha, T)$ with edges labeled by elements of $E \times F$, equipped with an initial partial function α: $Q \to (E^{\leq k} \times \varepsilon) \cup (\varepsilon \times F^{\leq k})$.

A function of infinite words has a *bounded delay* if its graph has a bounded delay.

Proposition 3.6. *If the normalization ν in basis θ is rational, then it has a bounded delay.*

Proof. Let $R = \widehat{\nu^{-1} \circ \nu}$; it is a rational relation of $C^{\mathbb{N}} \times C^{\mathbb{N}}$. If R has an unbounded delay, every transducer which recognizes R has a loop, the label of which is not length-preserving [12]. Thus there exists a couple $(ufs, vgt) \in R$ with $u, f, v, g \in C^*$, $s, t \in C^{\mathbb{N}}$ such that $|f| \neq |g|$, and $(uf^n s, vg^n t) \in R$ for every $n \geq 0$.

So $\pi(uf^n s) = \pi(vg^n t)$. Suppose that $|f| = j$, $|g| = p$, $p > j$. A straightforward computation gives

$$\pi(t) = \theta^{np}(\pi(u) - \pi(v) + \pi(f)\theta^{-|u|}(1 + \cdots + \theta^{-j(n-1)}) + \pi(s)\theta^{-jn} - \pi(g)\theta^{-|v|}(1 + \cdots + \theta^{-p(n-1)}))$$

so $\lim_{n\to\infty} \pi(t) = \infty$, which is impossible since $\pi(t) \leq c/(\theta - 1)$. □

Lemma 3.5. *If the normalization ν: $C^{\mathbb{N}} \to A^{\mathbb{N}}$ is rational, then $Z(\theta, c)$ is recognizable.*

Proof. $Z(\theta, c) = \{s - t \mid (s, t) \in R = \widehat{\nu^{-1} \circ \nu}\}$. Since R is a rational relation of bounded delay, it is the infinite behavior of a transducer of the previous type. From this transducer we construct a finite automaton $\mathcal{S}$ whose infinite behavior is equal to $Z(\theta, c)$, as in Lemma 2.3. □

The previous results can be put together into the following statement.

Theorem 3.3. *The normalization ν_C in basis θ is rational on any alphabet C if and only if the minimal polynomial of θ has no root of modulus 1 and if $|s_n| \leq c$, $\sum_{n\geq 0} s_n\theta^{-n} = 0$ imply $\sum_{n\geq 0} s_n\alpha^{-n} = 0$ for every conjugate α of modulus >1.*

Corollary 3.6. *Let θ be a Pisot number. For every alphabet C, the normalization ν_C in basis θ is rational (and in particular the addition also).*

Corollary 3.7. *Let θ be a Salem number. There exists an integer c_0 such that for every integer $c \geq c_0$ the normalization ν_C in basis θ is not rational.*

Example 3.4. Let $\theta = (1 + \sqrt{5})/2$. Then θ is a Pisot number, the θ-shift is of finite type since $d(1) = 11$. Normalization is rational on any alphabet.

Example 3.5. Let $\theta = (3 + \sqrt{5})/2$. The minimal polynomial of θ is $X^2 - 3X + 1$ and θ is a Pisot number. Since $d(1) = 21^\omega$, the θ-shift is a sofic system and normalization is rational on any alphabet.

Example 3.6. Let θ be the dominant root of the polynomial $X^4 - 2X^3 - 2X^2 - 2X + 1$. θ is a Salem number and $d(1) = 2(211)^\omega$. There exists c_0 such that for every $c \geq c_0$ normalization on C is not rational.

4. Back to the Integers

To a number θ, such that the θ-shift is sofic, is associated a linear recurrence. This defines a linear numeration system for the integers, the set of normal forms of which is strongly related to the set of θ-developments of real numbers.

Let θ be an algebraic integer > 1, such that S_θ is a sofic system, and let $d(1) = t_0 \cdots t_N(t_{N+1} \cdots t_{N+p})^\omega$ be the eventually periodic θ-development of 1. From

$$\theta = t_0 + \frac{t_1}{\theta} + \cdots + \frac{t_N}{\theta^N} + \left(\frac{t_{N+1}}{\theta^{N+1}} + \cdots + \frac{t_{N+p}}{\theta^{N+p}}\right)\left(1 + \frac{1}{\theta^p} + \frac{1}{\theta^{2p}} + \cdots\right)$$

we deduce that θ is a root of the polynomial P of $\mathbf{Z}[X]$:

$$P(X) = X^{N+p+1} - t_0 X^{N+p} - \cdots - t_{p-2}X^{N+2} - (t_{p-1} + 1)X^{N+1} - (t_p - t_0)X^N - \cdots - (t_{N+p} - t_N).$$

If P is irreducible over $\mathbf{Z}[X]$, then it is the minimal polynomial of θ and so θ is the dominant root of P, because it is a Perron number. If P is reducible, then it could happen that θ is not the dominant root of P. In the following we omit this case.

To P is associated a linear recurrence on $\mathbf{Z}$, of order $N + p + 1$, with P as characteristic polynomial: for $k \geq 0$,

$$u_{k+N+p+1} = t_0 u_{k+N+p} + \cdots + t_{p-2}u_{k+N+2} + (t_{p-1} + 1)u_{k+N+1} + (t_p - t_0)u_{k+N} + \cdots + (t_{N+p} - t_N)u_k$$

with $u_0 = 1$ and u_i for $1 \leq i \leq N + p$ such that the sequence $U = (u_n)_{n\geq 0}$ is strictly increasing. U is said to be *associated* to θ, and defines a linear numeration system *associated* to the θ-shift.

Remark 4.1. If the θ-development of 1 is finite $d(1) = t_0 \cdots t_N$, we get

$$u_{k+N+1} = t_0 u_{k+N} + \cdots + t_N u_k$$

with the convention $p = 0$.

Proposition 4.1. *Let θ be an algebraic integer such that*

$$d(1) = t_0 \cdots t_N(t_{N+1} \cdots t_{N+p})^\omega.$$

The sequence defined by

$$u_{k+N+p+1} = t_0 u_{k+N+p} + \cdots + t_{p-2}u_{k+N+2} + (t_{p-1} + 1)u_{k+N+1} + (t_p - t_0)u_{k+N} + \cdots + (t_{N+p} - t_N)u_k$$

for $k \geq 0$, with $u_0 = 1$ and initial conditions such that the sequence U is strictly increasing, induces a linear numeration system such that the set $L(U)$ of normal forms of the integers is rational. More precisely, there exists $b \geq 0$ such that

$$L(U) = L(D_\theta)B,$$

where B is the set of normal forms of length $\leq b$.

Proof. (1) In the proof of Proposition 2.2 it is established that $u_n \sim \lambda\theta^n$ when $n \to \infty$. So there exists a sequence $(\varepsilon_n)_{n\geq 0}$ of positive integers, decreasing toward 0 and a positive number $b \geq 0$ such that, for every $n \geq b$, $u_n = \lambda\theta^n(1 + \varepsilon_n)$. Let us show that $L(U) \subseteq L(D_\theta)B$. Let $f \in L(U)$. If $|f| \leq b$, by definition $f \in L(D_\theta)B$. If $|f| > b$, $f = f_0 \cdots f_n$, $n \geq b$, with $f_0 \neq 0$. Let $K = \pi(f)$. Since f is in normal form $u_n \leq K < u_{n+1}$ and $K - f_0 u_n$ is normally represented by $f_1 \cdots f_n$. By induction $f_1 \cdots f_n \in L(D_\theta)B$. Denote $\theta = (t_i)_{i\geq 0}$ with $t_{N+jp+k} = t_{N+k}$ for $1 \leq k \leq p$, $j \geq 0$. If f is not in $L(D_\theta)B$, then there exists a prefix of f equal to $t_0 \cdots t_{i-1}(t_i + h)$, $h \geq 1$, $0 \leq i \leq n - b$. We would then get

$$K = \pi(f) \geq t_0 u_n + \cdots + t_{i-1}u_{n-i+1} + (t_i + 1)u_{n-i}.$$

But

$$\begin{aligned} t_0 u_n &+ \cdots + t_{i-1}u_{n-i+1} + (t_i + 1)u_{n-i} \\ &= \lambda(t_0\theta^n(1 + \varepsilon_n) + \cdots + t_{i-1}\theta^{n-i+1}(1 + \varepsilon_{n-i+1}) + (t_i + 1)\theta^{n-i}(1 + \varepsilon_{n-i})) \\ &\geq \lambda(1 + \varepsilon_{n+1})(t_0\theta^n + \cdots + t_{i-1}\theta^{n-i+1} + (t_i + 1)\theta^{n-i}) \\ &> \lambda(1 + \varepsilon_{n+1})\theta^{n+1} \end{aligned}$$

since $t_0 t_1 \cdots >_{\text{lex}} t_i t_{i+1} \cdots$. So $K > u_{n+1}$, which is impossible.

(2) As above, for every $n \geq b$, $u_n = \lambda\theta^n(1 - \varepsilon_n)$, where $(\varepsilon_n)_{n\geq 0}$ is a decreasing sequence of positive numbers of limit 0. Let us suppose that $f \in L(D_\theta)B$. If $|f| \leq b$, then $f \in B$ and $f \in L(U)$. If $|f| > b$, $f = f_0 \cdots f_n$ with $n \geq b$ and $f_0 \neq 0$. Suppose that f is not in normal form; let g be the normal form of f. If $|f| = |g|$, then $g >_{\text{lex}} f$. Let $g = g_0 \cdots g_n$. There exists i, $0 \leq i \leq n$, such that $g_0 \cdots g_{i-1} = f_0 \cdots f_{i-1}$ and $g_i > f_i$. Let $K = \pi(f_i \cdots f_n) = \pi(g_i \cdots g_n)$. As g is in normal form, $K = g_i u_{n-i} + r$, with $r < u_{n-i}$. Since $f_{i+1} \cdots f_n \in L(D_\theta)B$, by induction $f_{i+1} \cdots f_n \in L(U)$, and so $K' = \pi(f_{i+1} \cdots f_n) < u_{n-i}$. We thus get $K = f_i u_{n-i} + K'$, which is in contradiction with $K = g_i u_{n-i} + r$, $r < u_{n-i}$.

If $|g| > |f|$, then $\pi(g) = \pi(f) \geq u_{n+1}$. Since $f \in L(D_\theta)B$, we have $f = t_0 \cdots t_{j-1}(t_j - h)f'$, $h \geq 1$, $0 \leq j \leq n - b$, with $f' \in L(D_\theta)B$. By induction $\pi(f') < u_{n-j+1}$. Thus

$$\begin{aligned} \pi(f) &\leq t_0 u_n + \cdots + t_{j-1}u_{n-j+1} + t_j u_{n-j} \\ &= \lambda(t_0\theta^n(1 - \varepsilon_n) + \cdots + t_{j-1}\theta^{n-j+1}(1 - \varepsilon_{n-j+1}) + t_j\theta^{n-j}(1 - \varepsilon^{n-j})) \\ &\leq \lambda(1 - \varepsilon_{n+1})(t_0\theta^n + \cdots + t_{j-1}\theta^{n-j+1} + t_j\theta^{n-j}) \\ &< \lambda(1 - \varepsilon_{n+1})\theta^{n+1} \end{aligned}$$

since $t_0 t_1 \cdots >_{\text{lex}} t_i t_{i+1} \cdots$. So $K < u_{n+1}$, which is impossible. □

From Proposition 2.1 we derive

Proposition 4.2. *If the θ-shift is a sofic system, then normalization in basis U, where U is associated to θ, is rational if and only if the set of words equal to 0 in basis U is recognizable.*

In [10] we have proved by combinatorial methods that normalization in basis

U and in basis θ is rational on any alphabet for a special type of linear recurrence:

$$u_{n+m} = au_{n+m-1} + \cdots + au_{n+1} + bu_n,$$
$$a \geq b \geq 1, \quad u_i = (a+1)^i, \quad 0 \leq i \leq m-1.$$

In that case, the dominant root θ of the characteristic polynomial happens to be a Pisot number [6].

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