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Intertwining Operators and Polynomials Associated with the Symmetric Group

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Abstract. There is an algebra of commutative differential-difference operators which is very useful in studying analytic structures invariant under permutation of coordinates. This algebra is generated by the Dunkl operators $T_i := \frac{\partial}{\partial x_i} + k \sum_{j\neq i} \frac{1-(ij)}{x_i - x_j}$, $(i = 1, \ldots, N$, where (ij) denotes the transposition of the variables $x_i\, x_j$ and k is a fixed parameter). We introduce a family of functions $\{p_\alpha\}$, indexed by m-tuples of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_m)$ for $m \leqslant N$, which allow a workable treatment of important constructions such as the intertwining operator V. This is a linear map on polynomials, preserving the degree of homogeneity, for which $T_i V = V \frac{\partial}{\partial x_i}$, $i = 1, \ldots, N$, normalized by $V1 = 1$ (see Dunkl, Canadian J. Math. **43** (1991), 1213–1227). We show that $T_i p_\alpha = 0$ for $i > m$, and

$$V(x_1^{\alpha_1} \cdots x_m^{\alpha_m}) = \frac{\lambda_1!\lambda_2! \cdots \lambda_m!}{(Nk+1)_{\lambda_1}(Nk-k+1)_{\lambda_2} \cdots (Nk-(m-1)k+1)_{\lambda_m}} p_\alpha + \sum_\beta A_{\beta\alpha} p_\beta,$$

where $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ is the partition whose parts are the entries of α (That is, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \lambda_m \geqslant 0$), $\beta = (\beta_1, \ldots, \beta_m)$, $\sum_{i=1}^m \beta_i = \sum_{i=1}^m \alpha_i$ and the sorting of β is a partition strictly larger than λ in the dominance order. This triangular matrix representation of V allows a detailed study. There is an inner product structure on span$\{p_\alpha\}$ and a convenient set of self-adjoint operators, namely $T_i \rho_i$, where $\rho_i p_\alpha := p_{(\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_m)}$. This structure has a bi-orthogonal relationship with the Jack polynomials in m variables. Values of k for which V fails to exist are called singular values and were studied by De Jeu, Opdam, and Dunkl in Trans. Amer. Math. Soc. **346** (1994), 237–256. As a partial verification of a conjecture made in that paper, we construct, for any $a = 1, 2, 3, \ldots$ such that $\gcd(N - m + 1, a) < (N - m + 1)/m$ and $m \leqslant N/2$, a space of polynomials annihilated by each T_i for $k = -a/(N - m + 1)$ and on which the symmetric group S_N acts according to the representation $(N - m, m)$.

When spaces of functions in several variables have an analytic structure which is invariant under permutation of coordinates, there is often a connection to a commutative algebra of differential-difference operators. Examples of such structures are orthogonal decompositions with respect to the measure

$$\prod_{1 \leqslant i < j \leqslant N} |x_i - x_j|^{2k} e^{-|x|^2/2} dx \text{ on } \mathbf{R}^N,$$

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or eigenfunctions of the differential operator

$$\Delta + 2k \sum_{1\leqslant i<j\leqslant N} \frac{\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}}{x_i - x_j}.$$

The algebra of operators is generated by the Dunkl operators [3],

$$T_i := \frac{\partial}{\partial x_i} + k \sum_{j\neq i} \frac{1-(ij)}{x_i - x_j}.$$

($i = 1, \ldots, N$, where (ij) denotes the transposition of the variables x_i, x_j and k is a fixed parameter, often positive). On the one hand, these operators make it easy to construct a complete set of commuting invariant (symmetric) differential operators, but on the other hand, it is difficult to work with the effect of the operators on actual polynomials. In this paper we introduce a family of functions which allow a workable treatment of important constructions such as the intertwining operator V. This is a linear map on polynomials, preserving the degree of homogeneity, for which $T_iV = V\frac{\partial}{\partial x_i}$, $i = 1, \ldots, N$, normalized by $V1 = 1$. This operator was introduced for arbitrary finite reflection groups by the author in [4]; later the "singular" values of k, those for which V fails to exist, were studied by DE JEU, OPDAM, and the author [8]. These values have an interesting representation-theoretic interpretation. When V can be realized as an integral transform, it is a fractional integral for several variables. This was done by the author [6] for the case $N = 3$, the smallest nonabelian group, by analysis on the unitary group in complex 3-space.

In this paper a more algebraic approach is taken. We construct a family of functions, $\{p_\alpha\}$, indexed by m-tuples of nonnegative integers, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ for $m \leqslant N$. We show that $T_i p_\alpha = 0$ for $i > m$, and

$$V(x_1^{\alpha_1} \cdots x_m^{\alpha_m}) = \frac{\alpha_1!\alpha_2!\cdots\alpha_m!}{(Nk+1)_{\lambda_1}(Nk-k+1)_{\lambda_2}\cdots(Nk-(m-1)k+1)_{\lambda_m}} p_\alpha + \sum_\beta A(\beta,\alpha)p_\beta,$$

where $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ is the partition whose parts are the entries of α, (that is, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m \geqslant 0$), $\beta = (\beta_1, \ldots, \beta_m)$, $\sum_{i=1}^m \beta_i = \sum_{i=1}^m \alpha_i$ and the sorting of β is a partition strictly larger than λ in the dominance order; also $(t)_j = t(t+1)\cdots(t+j-1)$, the Pochhammer symbol. This triangular matrix representation of V allows a detailed study. A key device is the operator $T_i\rho_i$ where $\rho_i p_{(\alpha_1,\ldots,\alpha_m)} = p_{(\alpha_1,\ldots,\alpha_i+1,\ldots,\alpha_m)}$. The introduction of the formal map $\xi : p_\alpha \mapsto x_1^{\alpha_1} \cdots x_m^{\alpha_m}/(\alpha_i! \cdots \alpha_m!)$ leads to a complete eigenfunction decomposition of $V\xi$. The eigenfunctions have several interesting properties: the coefficients do not depend on N, the operators $\{T_i\rho^i\}$ commute with $V\xi$ and there is an inner product structure for which they are self-adjoint, and there is a bi-orthogonal relation to complex analytic polynomials in variables $z_1, \ldots, z_m$ with the inner product

$$(f,g)\mapsto$$
$$\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} f(e^{i\theta_1},\ldots,e^{i\theta_m})g(e^{-i\theta_1},\ldots,e^{-i\theta_m}) \prod_{1\leqslant j<l\leqslant m} \left|e^{i\theta_j}-e^{i\theta_l}\right|^{2k} d\theta_1\cdots d\theta_m.$$

The Jack polynomials form an orthogonal basis for the symmetric polynomials (see Beerends and Opdam [2]); this leads to useful evaluation results for the aforementioned eigenfunctions. Sahi [13] has found the structure constants of an inner product defined in terms of $\{p_\alpha\}$ and their generating function for the non-symmetric Jack polynomials. Results of the present paper link these polynomials to the generalized Hermite polynomials studied by Baker and Forrester [1].

In section 1, the polynomials p_α are defined, the action of the Dunkl operators is worked out in detail, with the aid of product formulas. In section 2, the eigenfunction decomposition of $V\xi$ is produced; this requires some very involved induction procedures, and heavy reliance on the triangular nature of $\{T_i\rho_i\}$ with respect to associated partial orderings.

In section 3, the results on eigenfunctions are used for a new proof of a formula of Dunkl and Hanlon [7] for the evaluation of $g(T_1,\ldots,T_N)g(x_1,\ldots,x_N)$ where g is the Garnir polynomial associated to a partition of N (essentially, a product of alternating polynomials, one for each column of an associated tableau). Also an irreducibility property is proven for the space of eigenfunctions of $V\xi$ with given eigenvalue and this leads to an orthogonality result in the next section.

In section 4, the orthogonality structure associated with Jack polynomials is introduced and is used to define an inner product on the span of $\{p_\alpha\}$. When the parameter k is positive, the inner products are positive-definite, and there is a boundedness property for V, namely, if f is a homogeneous polynomial on $\mathbf{R}^N$, then

$$\sup_{|x|=1}|Vf(x)|\leqslant \sup_{|x|=1}|f(x)| \quad (\text{where } |x|=(\Sigma_i x_i^2)^{\frac{1}{2}})$$

(Dunkl [5]). Also, there is a description of the inner product used by Sahi, and its relation to $V\xi$ and to the Hermite-type polynomials.

In section 5, a part of the conjecture made in [8] about singular values is verified: if $1\leqslant m\leqslant N/2$ and $a=1,2,3,\ldots$ such that $\gcd(a,N-m+1)<$ $<(N-m+1)/m$, then there is a space of polynomials (defined in terms of $p_{(a,a\ldots,a)}$) realizing the representation $(N-m,m)$ of the symmetric group, which are annihilated by each T_i, when $k=-a/(N-m+1)$.

The eigenfunctions of $V\xi$ have coefficients in $\mathbf{Q}(k)$; conjectures are made about the poles of these coefficients in terms of hook-length products of tableaux. There are indicators to subsequent research on complete orthogonal decompositions and norm formulas. This will provide orthogonal polynomials whose symmetrizations are the Jack polynomials.

Notation used throughout:

- $\mathbf{Z}_+=\{0,1,2,3,\ldots\}$, $\mathcal{N}_m=\{\alpha=(\alpha_1,\ldots,\alpha_m):\alpha_i\in\mathbf{Z}_+\}$, $m=1,2,3,\ldots$;
- for $\alpha\in\mathcal{N}_m$, $|\alpha|:=\sum_{i=1}^m \alpha_i$, $\alpha!:=\prod_{i=1}^m \alpha_i!$,

$\mathcal{N}_{m,n} = \{\alpha \in \mathcal{N}_m : |\alpha| = n\}$,
$\mathcal{N}^P_m = \{\lambda \in \mathcal{N}_m : \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m \geqslant 0\}$ (partitions with m or fewer parts);
$\mathcal{N}^P_{m,n} = \{\lambda \in \mathcal{N}^P_m : |\lambda| = n\}$;

- for $\alpha \in \mathcal{N}_m$, α^s denotes the sorting of α into a partition, that is, $\alpha^s_1 \geqslant \alpha^s_2 \cdots \geqslant \alpha^s_m$ and $(\alpha^s_1, \alpha^s_2, \ldots, \alpha^s_m)$ is a permutation of $(\alpha_1, \alpha_2, \ldots, \alpha_m)$;
- for λ, $\mu \in \mathcal{N}^P_m$, the dominance ordering is defined by $\lambda \succeq \mu$ if and only if $\sum_{i=1}^j \lambda_i \geqslant \sum_{i=1}^j \mu_i$, $1 \leqslant j \leqslant m$;
- also $\lambda \succ \mu$ means $\lambda \succeq \mu$ and $\lambda \neq \mu$;
- for $\alpha \in \mathcal{N}_m$ and $\alpha_j \geqslant 1$, let $\delta_j \alpha = (\alpha_1, \ldots, \alpha_j - 1, \ldots, \alpha_m) \in \mathcal{N}_m$;
- for $x = (x_1, \ldots, x_m) \in \mathbf{R}^m$, and $\alpha \in \mathcal{N}_m$, the monomial (basis element for polynomials) $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}$;
- for a polynomial $p(x)$, the transposition $(ij)p(x) := p(\ldots, \overset{i}{x_j}, \ldots, \overset{j}{x_i}, \ldots)$ (acts on the variables);
- for a polynomial $p(x)$ expressed in some basis $\{g_\alpha\}$ as $p(x) = \sum_\alpha c_\alpha g_\alpha(x)$, let $\operatorname{cof}(p, g_\beta) := c_\beta$ denote the coefficient;
- for $n = 0, 1, 2, \ldots$, the Pochhammer symbol (shifted factorial) is defined by $(a)_0 = 1$, $(a)_{n+1} := (a)_n (a + n)$; for $\lambda \in \mathcal{N}^p_m$ and a parameter k, the generalized shifted factorial is

$$(a)_{\lambda;k} := \prod_{i=1}^m (a - (i-1)k)_{\lambda_i};$$

- vector spaces are over the field $\mathbf{Q}(k)$; "generic values" specifically exclude negative rational numbers;
- the cardinality of a set Ω is denoted by $\#\Omega$, and for $a, b \in \mathbf{Z}$, $a \wedge b := \min(a, b)$.

1. The Fundamental Polynomials

The symmetric group S_N acts on $\mathbf{R}^N$ by permutation of coordinates. For a parameter k we define the first-order differential-difference ("Dunkl") operators

$$T_i := \frac{\partial}{\partial x_i} + k \sum_{j \neq i} \frac{1 - (ij)}{x_i - x_j}, \quad 1 \leqslant i \leqslant N.$$

The commutativity of $\{T_i : 1 \leqslant i \leqslant N\}$ was proved in (DUNKL [3]). The polynomials underlying the analysis of $\{T_i\}$ and V are defined by a generating function.

1.1 Definition. For $i = 1, \ldots, N$ and $n = 0, 1, 2, \ldots$, the polynomial $p_n(x_i, x)$ is defined by

$$\mathcal{F}_i(x, r) := (1 - x_i r)^{-1} \prod_{j=1}^N (1 - x_j r)^{-k} = \sum_{n=0}^\infty p_n(x_i, x) r^n.$$

If $k = -m$ for some $m = 1, 2, 3, \ldots$, then $\mathcal{F}_i$ is a polynomial of degree $Nm - 1$ and $p_n(x_i, x) = (-Nm + 1)_n / n! \neq 0$ for $n = 0, 1, \ldots, Nm - 1$ at $x \underset{i}{=} (1, 1, \ldots, 1)$. If $k + 1 \notin -\mathbf{Z}_+$, then $p_n(x_i, x) = (k+1)_n / n! \neq 0$ at $x = (0, 0, \ldots, \underset{i}{1}, \ldots, 0)$. Also $p_n(x_i, x)$ is symmetric in the variables $\{x_j : j \neq i\}$.

1.1 Proposition. $T_j p_n(x_i, x) = \delta_{ij}(Nk + n)p_{n-1}(x_i, x)$, for $1 \leqslant i, j \leqslant N$, $n \in \mathbf{Z}_+$.

Proof. For $j \neq i$,

$$T_j \mathscr{F}_i(x, r) = \frac{kr}{1 - x_j r}\mathscr{F}_i + \frac{k}{x_j - x_i}\left(\frac{1}{1 - x_i r} - \frac{1}{1 - x_j r}\right)\prod_{l=1}^{N}(1 - x_l r)^{-k}$$
$$= \left(\frac{kr}{1 - x_j r} - \frac{kr}{1 - x_j r}\right)\mathscr{F}_i = 0.$$

Also,

$$T_i \mathscr{F}_i - r^2 \frac{\partial}{\partial r}\mathscr{F}_i = \frac{(k+1)r}{1 - x_i r}\mathscr{F}_i + k\sum_{j \neq i}\frac{1}{x_i - x_j}\left(\frac{1}{1 - r x_i} - \frac{1}{1 - r x_j}\right)\prod_{l=1}^{N}(1 - r x_l)^{-k}$$
$$- r^2\left(\frac{(k+1)x_i}{1 - r x_i} + k\sum_{j \neq i}\frac{x_j}{1 - r x_j}\right)\mathscr{F}_i$$
$$= \mathscr{F}_i\left((k+1)r + k\sum_{j \neq i} r\right) = (Nk + 1)r\mathscr{F}_i.$$

$T_i p_n(x_i, x)$ is the coefficient of r^n in

$$r\left(Nk + 1 + r\frac{\partial}{\partial r}\right)\mathscr{F}_i = \sum_{n=0}^{\infty} p_n(x_i, x)(Nk + 1 + n)r^{n+1}. \qquad \square$$

We establish a product rule for T_i with emphasis on certain invariance relations.

1.2 Lemma. *For any two polynomials f, g on $\mathbf{R}^N$ and $1 \leqslant i \leqslant N$,*

$$T_i(fg) = f(T_i g) + (T_i f)g - k\sum_{j \neq i}(f - (ij)f)\frac{g - (ij)g}{x_i - x_j}.$$

1.3 Lemma. *Suppose $f_1, f_2, \ldots, f_m$ are polynomials satisfying $(ij)f_l = f_l$ if $l \neq i$ and $l \neq j$, then*

$$T_i(f_1 f_2 \cdots f_m) = \sum_{l=1}^{m}(T_i f_l)\prod_{s \neq l} f_s - k\sum_{l \neq i}\frac{(f_i - (il)f_i)(f_l - (il)f_l)}{x_i - x_l}\prod_{s \neq l, i} f_s.$$

Proof. For labeling convenience, assume that $i = 1$, and then induct on m. Suppose the formula holds for $1 \leqslant m \leqslant n$; by the product rule

$$T_1(f_1 \cdots f_n f_{n+1}) = (T_1(f_1 \cdots f_n))f_{n+1} + (f_1 \cdots f_n)T_1 f_{n+1}$$
$$- k\sum_{j \neq 1}\frac{1}{x_1 - x_j}((f_1 \cdots f_n) - (1j)(f_1 \cdots f_n))(f_{n+1} - (1j)f_{n+1}).$$

In the sum, only the $j = n + 1$ term can be nonzero by the invariance property of f_{n+1}; and this term is exactly

$$-k(f_1 - (1, n+1)f_1)(f_{n+1} - (1, n+1)f_{n+1}) \cdot (f_2 \cdots f_n)/(x_1 - x_{n+1}). \qquad \square$$

1.4 Corollary. *For a subset* $J = \{j_1, j_2, \ldots, j_m\} \subset \{1, 2, \ldots, N\}$ *and* $l \notin J$,

$$T_l\left(\prod_{i=1}^{m} p_{\alpha_i}(x_{j_i}, x)\right) = 0, \quad \textit{for every } \alpha \in \mathcal{N}_m.$$

Proof. Apply the lemma with $f_{j_i} = p_{\alpha_i}(x_{j_i}, x), f_s = 1$ if $s \notin J$; each term in the formula 1.3 evaluates to zero (indeed, $T_l f_l = T_l 1 = 0$ and $(f_l - (lj)f_l) = 0$ for all $j \neq l$). □

1.5 Lemma. *Let* $n_1, n_2 \in \mathbf{Z}_+$, $1 \leqslant i < j \leqslant N$, *then*

$$\begin{aligned} &-(p_{n_1}(x_i, x) - (ij)p_{n_1}(x_i, x))(p_{n_2}(x_j, x) - (ij)p_{n_2}(x_j, x))/(x_i - x_j) \\ &= \sum_{s=0}^{n_2-1} [p_{n_1+n_2-1-s}(x_i, x)p_s(x_j, x) - p_s(x_i, x)p_{n_1+n_2-1-s}(x_j, x)]. \end{aligned}$$

The right-hand side is symmetric in (n_1, n_2) *and the summation can be taken over the range* $0 \leqslant s \leqslant (n_1 - 1) \wedge (n_2 - 1)$.

Proof. Denote the expression to be evaluated by

$$Q(n_1, n_2) = (p_{n_1}(x_i) - p_{n_1}(x_j))(p_{n_2}(x_i) - p_{n_2}(x_j))/(x_i - x_j)$$

(suppressing the argument "x"). Denote

$$Q'(m_1, m_2) := p_{m_1}(x_i)p_{m_2}(x_j) - p_{m_2}(x_i)p_{m_1}(x_j)$$

for $m_1, m_2 \in \mathbf{Z}_+$. We will show that $Q(n_1, n_2) - Q(n_1+1, n_2 - 1) = Q'(n_1, n_2 - 1)$ for the case $n_1 \geqslant n_2$. Express

$$\mathscr{F}_i(x, r) = \sum_{n=0}^{\infty} p_n(x_i, x)r^n = (1 - rx_i)^{-1} \sum_{l=0}^{\infty} \pi_l^{(k)}(x)r^l$$

with each $\pi_l^{(k)}$ being a symmetric polynomial in $(x_1, \ldots, x_N)$. Then

$$p_n(x_i) = \sum_{s=0}^{n} x_i^{n-s}\pi_s^{(k)} = x_i p_{n-1}(x_i) + \pi_n^{(k)}.$$

Thus

$$p_n(x_i) - p_n(x_j) = x_i p_{n-1}(x_i) - x_j p_{n-1}(x_j).$$

Substituting this formula for $n = n_2$ and $n = n_1 + 1$, we obtain

$$\begin{aligned} &Q(n_1, n_2) - Q(n_1 + 1, n_2 - 1) \\ &= \left(\frac{1}{x_i - x_j}\right)\Big[(p_{n_1}(x_i) - p_{n_1}(x_j))(x_i p_{n_2-1}(x_i) - x_j p_{n_2-1}(x_j)) \\ &\quad - (x_i p_{n_1}(x_i) - x_j p_{n_1}(x_j))(p_{n_2-1}(x_i) - p_{n_2-1}(x_j))\Big] \\ &= Q'(n_1, n_2 - 1). \end{aligned}$$

Since $Q(n_1+n_2,0)=0$, we have

$$\begin{aligned}Q(n_1,n_2)&=\sum_{s=0}^{n_2-1}(Q(n_1+s,n_2-s)-Q(n_1+s+1,n_2-s-1))\\&=\sum_{s=0}^{n_2-1}Q'(n_1+s,n_2-s-1)\\&=\sum_{t=0}^{n_2-1}Q'(n_1+n_2-1-t,t)\end{aligned}$$

(setting $t=n_2-1-s$). If $n_1<n_2$ the same argument shows the validity of the formula with the upper summation limit replaced by n_1-1. In this case consider

$$\sum_{s=n_1}^{n_2-1}Q'(n_1+n_2-1-s,s)=\sum_{t=n_1}^{n_2-1}Q'(t,n_1+n_2-1-t),$$

(where $t=n_1+n_2-1-s$), but $Q'(m_1,m_2)=-Q'(m_2,m_1)$ so this sum equals zero. □

Observe that the term $p_{n_1-1}(x_i)p_{n_2}(x_j)$ appears in $Q(n_1,n_2)$ exactly when $n_1<n_2$.

1.2 Definition. For $\alpha\in\mathcal{N}_m$, with $m\leqslant N$, let $p_\alpha:=p_{\alpha_1}(x_1,x)p_{\alpha_2}(x_2,x)\cdots p_{\alpha_m}(x_m,x)$, and let $\mathcal{V}_m:=\operatorname{span}\{p_\alpha:\alpha\in\mathcal{N}_m\}$ (as in other vector spaces, the span is over $\mathbf{Q}(k)$). also $\mathcal{V}_{m,n}:=\operatorname{span}\{p_\alpha:\alpha\in\mathcal{N}_{m,n}\}$. Sometimes $p(\alpha_1,\alpha_2,\ldots,\alpha_m)$ is a synonym for p_α.

1.6 Proposition. *For $\alpha\in\mathcal{N}_m$ and $\alpha_i\geqslant 1$,*

$$\begin{aligned}T_ip_\alpha&=(Nk-k\#\{l:\alpha_l\geqslant\alpha_i,l\neq i\}+\alpha_i)p(\alpha_1,\ldots,\alpha_i-1,\ldots)\\&\quad+k\sum_{s=1,s\neq i}^{m}\left(\sum_{l=0}^{(\alpha_i-1)\wedge(\alpha_s-1)}p(\ldots,\overset{i}{\alpha_i+\alpha_s-1-l},\ldots,\overset{s}{l},\ldots)\right.\\&\quad\left.-\sum_{l=0}^{(\alpha_i-2)\wedge(\alpha_s-1)}p(\ldots,\overset{i}{l},\ldots,\overset{s}{\alpha_i+\alpha-1-l},\ldots)\right);\end{aligned}$$

when $\alpha_i=0$ then $T_ip_\alpha=0$.

Proof. This a direct consequence of Lemmas 1.3, 1.4, and 1.5. □

Example. For $m=3$,

$$\begin{aligned}T_1p_{321}&=(Nk+3)p_{221}+k(p_{401}-p_{041}+p_{311}-p_{131}+p_{320}-p_{023});\\T_2p_{321}&=(Nk-k+2)p_{311}+k(p_{041}-p_{401}+p_{131}+p_{320}-p_{302});\\T_3p_{321}&=(Nk-2k+1)p_{320}+k(p_{023}-p_{302}),\end{aligned}$$

As customary in analysis of several variables, the triangularity of linear operators with respect to some partial ordering is very useful. Here we will define a family of orderings, one for each T_i.

Define the raising operator $\rho_i p_\alpha = p(\alpha_1, \ldots, \alpha_i \overset{i}{+} 1, \ldots, \alpha_m)$ for $\alpha \in \mathcal{N}_m$, then

$$T_i \rho_i p_\alpha = (Nk - k\#\{l : \alpha_l \geqslant \alpha_i + 1\} + \alpha_i + 1) p_\alpha + k \sum_{s=1, s\neq i}^{m} \left(\sum_{l=0}^{\alpha_i \wedge (\alpha_s - 1)} p(\ldots, \overset{i}{\alpha_i + \alpha_s - l}, \ldots, \overset{s}{l}, \ldots) - \sum_{l=0}^{(\alpha_i - 1)\wedge(\alpha_s - 1)} p(\ldots, \overset{i}{l}, \ldots, \overset{s}{\alpha_i + \alpha_s - l}, \ldots) \right). \tag{1.1}$$

For each $i = 1, \ldots, m$ define the binary relation R_i on $\mathcal{N}_m$ by $\alpha R_i \beta$ if and only if $\alpha \neq \beta$ and cof $(T_i \rho_i p_\beta, p_\alpha) \neq 0$. Then let $\geqslant_{(i)}$ denote the transitive closure of R_i.

1.7 Proposition. *The transitive closure $\geqslant_{(i)}$ of R_i is a partial ordering. If $\alpha, \beta \in \mathcal{N}_{m,n}$ and $\alpha \geqslant_{(i)} \beta$, then one of the following conditions hold:* (i) $\alpha = \beta$, (ii) $\alpha^s = \beta^s$ *and* $\alpha_i > \beta_i$, (iii) $\alpha^s \succ \beta^s$.

Proof. From the formula (1.1), we see that $\alpha R_i \beta$ when there is an s, $1 \leqslant s \leqslant m$ such that $\alpha_j = \beta_j$ for $j \neq s$ and one of the following cases occurs:

(i) $\beta_s > \beta_i$ and $\{\alpha_i, \alpha_s\} = \{\beta_i - l, \beta_s + l\}$ for some $l = 1, \ldots, \beta_i$;
(ii) $\beta_i = \alpha_s < \beta_s = \alpha_i$
(iii) $\beta_s \leqslant \beta_i$ and $\{\alpha_i, \alpha_s\} = \{\beta_s - l, \beta_i + l\}$ for some $l = 1, \ldots, \beta_s$.

Induction on l shows that if α satisfies (i) or (iii), then $\alpha^s \succ \beta^s$ (recall β^s denotes the partition whose parts are the entries of β); in case (ii) $\alpha^s = \beta^s$. Thus if $\alpha R_i \beta$, then $\alpha^s = \beta^s$ and $\alpha_s = \beta_i < \beta_s = \alpha_i$ for some s and $\alpha_j = \beta_j$ for $j \notin \{i, s\}$, or $\alpha^s \succ \beta^s$. This shows that the transitive closure of R_i is a partial ordering (there can be no loops). Suppose $\{\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(t)}\} \subset \mathcal{N}_{m,n}$ satisfy $\alpha^{(j)} R_i \alpha^{(j+1)}$ for $1 \leqslant j \leqslant t-1$, then $\alpha^{(1)} \neq \alpha^{(t)}$. Indeed $(\alpha^{(1)})^s \succeq (\alpha^{(2)})^s \cdots \succeq (\alpha^{(t)})^s$ implies either $(\alpha^{(1)}) \succ (\alpha^{(t)})$ or "=" holds in each step, but then case (ii) shows $\alpha_i^{(1)} > \alpha_i^{(t)}$. □

By construction, $T_i \rho_i$ is triangular for $\geqslant_{(i)}$.

1.8 Proposition. *For each $i = 1, \ldots, m$ the linear operator $T_i \rho_i$ has the eigenvalues $Nk - k\#\{l : \alpha_l \geqslant \alpha_i + 1\} + \alpha_i + 1$ for $\alpha \in \mathcal{N}_{m,n}$ and is invertible. For any $\beta \in \mathcal{N}_{m,n}$ the subspace* span$\{p_\alpha : \alpha \in \mathcal{N}_{m,n}$ *and* $\alpha \geqslant_{(i)} \beta\}$ *is invariant under $T_i \rho_i$.*

1.9 Theorem. *For $m = 1, 2, \ldots, N, n \in \mathbf{Z}_+$, generic k,*

$$\dim \mathcal{V}_{mn} = \binom{m+n-1}{n},$$

and

$$\{f \in \mathcal{V}_{mn} : T_m f = T_{m-1} f = \cdots = T_{m-j+1} f = 0\} = \mathcal{V}_{m-j,n}, 1 \leqslant j \leqslant m-1.$$

Proof. We use double induction. First $\mathcal{V}_{1,n} = \text{span}\{p_n(x_1, x)\}$ and $p_n(1, (1, 0, \ldots, 0)) = (k+1)_n / n! \neq 0$, thus dim $\mathcal{V}_{1,n} = 1$. Also dim $\mathcal{V}_{m,0} = 1$. Assume dim $\mathcal{V}_{l,s} = \binom{l+s-1}{s}$ for $l = 1, 2, \ldots, m-1$ and $s \in \mathbf{Z}_+$ and also for $l = m$

and $s = 0, 1, 2, , n-1$. Now

$$\begin{aligned}\mathscr{V}_{m,n} &= \rho_m \mathscr{V}_{m,n-1} \oplus \mathscr{V}_{m-1,n} \\ &= \operatorname{span}\{p_\alpha : \alpha_m \geqslant 1, |\alpha| = n\} + \operatorname{span}\{p_\alpha : \alpha_m = 0, |\alpha| = n\}.\end{aligned}$$

By Proposition 1.8 ρ_m is one-to-one on $\mathscr{V}_{m,n-1}$, and so

$$\begin{aligned}\dim \mathscr{V}_{m,n} &= \dim \mathscr{V}_{m,n-1} + \dim \mathscr{V}_{m-1,n} = \\ &= \binom{m+n-2}{n-1} + \binom{m+n-2}{n} = \binom{m+n-1}{n}.\end{aligned}$$

This completes the induction. Essentially the same argument shows that $(\ker T_N) \cap \cap \mathscr{V}_{N,n} = \mathscr{V}_{N-1,n}$; and by induction that $(\ker T_N) \cap (\ker T_{N-1}) \cdots \cap (\ker T_{N-j+1}) \cap \cap \mathscr{V}_{N,n} = \mathscr{V}_{N-j,n}$. □

1.10 Corollary. *For fixed m and arbitrary $N \geqslant m$, the projection of $\mathscr{V}_{m,n}$ considered as a space of polynomials in $(x_1, x_2, \ldots, x_N)$ onto $\mathscr{V}_{m,n}$ in the variables $(x_1, \ldots, x_m)$ induced by the specialization $x_{m+1} = x_{m+1} = \cdots = x_N = 0$ is a linear isomorphism. As a consequence, in the action of T_i, the value of N can be taken as a generic parameter.*

2. The Intertwining Operator V

The operator V is a linear map on polynomials in $(x_1, x_2, \ldots, x_N)$ preserving degree of homogeneity, such that $V1 = 1$ and $T_i V p(x) = V\left(\frac{\partial}{\partial x_i} p(x)\right)$ for $i = 1, \ldots, N$ and any polynomial p. It was shown in (Dunkl, De Jeu, Opdam [8]) that V exists and is an isomorphism for any $k \notin \{-j/m - n : 1 \leqslant j < m \leqslant N$ and $n \in \mathbf{Z}_+\}$. The inverse exists for all k; it has the simple definition

$$V^{-1} p(y) = \exp\left(\sum_{i=1}^{N} y_i T_i\right) p(x)\Big|_{x=0}.$$

Since p is a polynomial the formal exponential is actually a terminating series (see Torossian [15] for another application).

2.1 Proposition. *Let $\alpha \in \mathscr{N}_{m,n}$, then $Vx^\alpha \in \mathscr{V}_{m,n}$.*

Proof. By Theorem 1.9 $\mathscr{V}_{N,n}$ is the space of homogeneous polynomials of degree n in $(x_1, \ldots, x_N)$. By the existence theorem, $Vx^\alpha \in \mathscr{V}_{N,n}$. For $i > m$, $T_i V x^\alpha = V\left(\frac{\partial}{\partial x_i} x^\alpha\right) = 0$, because $x^\alpha = x_1^{\alpha 1} \cdots x_m^{\alpha_m}$. Thus $Vx^\alpha \in \mathscr{V}_{m,n}$. □

Because of Corollary 1.10 and the formula for V^{-1} we can consider the action of V on span $\{x^\alpha : \alpha \in \mathscr{N}_{m,n}\}$ and on $\mathscr{V}_{m,n}$ in terms of a generic parameter N.

Example. $Vx_1^n = (n!/(Nk+1)_n) p_n(x_1, x)$. By Proposition 2.1, $Vx_1^n = c_n p_n(x_1, x)$ for some $c_n \in \mathbf{Q}(k)$; also

$$T_1(Vx_1^n) = (Nk+n)c_n p_{n-1}(x_1,x) = V\left(\frac{\partial}{\partial x_1}x_1^n\right)$$
$$= nVx_1^{n-1} = nc_{n-1}p_{n-1}(x_1,x),$$

for each $n \geqslant 1$

2.2 Theorem. *For* $\alpha \in \mathcal{N}_{m,n}$,

$$Vx^\alpha = \frac{\alpha!}{(Nk+1)_{\alpha^s;k}} p_\alpha + \sum_{\beta \in \mathcal{N}_{m,n}} A(\beta,\alpha) p_\beta,$$

where $A(\beta,\alpha) \in \mathbf{Q}(k)$ *and* $A(\beta,\alpha) = 0$ *unless* $\beta^s \succ \alpha^s$ *(or* $\beta = \alpha$*).*

Proof. The claim holds for $n = 1$ (indeed, $Vx_j = (1/Nk+1))p_1(x_j,x)$, $1 \leqslant j \leqslant m$). Assume the claim is valid for all α with $|\alpha| \leqslant n-1$. By symmetry, $A(w\beta, w\alpha) = A(\beta,\alpha)$ for any $w \in S_m$, we may restrict our attention to Vx^λ with $\lambda \in \mathcal{N}_{m,n}^P$ (that is, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$). Also assume $\lambda_m > 0$ (changing m, if necessary). Write $Vx^\lambda = \sum_{\beta \in \mathcal{N}_{m,n}} A(\beta,\lambda)p_\beta$, and for each j, let $Vx^\lambda = g_j + g_{j,0}$ where $g_j := \sum\{A(\beta,\lambda)p_\beta : \beta_j \geqslant 1\}$ and $T_j g_{j,0} = 0$. By the defining property of V,

$$T_j Vx^\lambda = \lambda_j V\left(x_1^{\lambda_1} \cdots x_j^{\lambda_j - 1} \cdots x_m^{\lambda_m}\right) =: \lambda_j h_j \text{ (defining } h_j).$$

Then $g_j = \lambda_j \rho_j (T_j \rho_j)^{-1} h_j$, and we can use the triangular properties of $T_j\rho_j$ and the inductive hypothesis on h_j to get the desired results for $A(\beta,\lambda)$. Let $\delta_j\lambda = (\lambda_1, \ldots, \lambda_j - 1, \ldots, \lambda_m) \in \mathcal{N}_{m,n-1}$ (not necessarily a partition). By the inductive hypothesis and Proposition 1.8,

$$(T_j\rho_j)^{-1} h_j \in \operatorname{span}\{p_\alpha : \alpha \in \mathcal{N}_{m,n-1}, \alpha^s \succ (\delta_j\lambda)^s \text{ or } \alpha \geqslant_{(j)} \delta_j\lambda\}.$$

The argument starts at the smallest part of λ.

Part 1. We show if $A(\beta,\lambda) \neq 0$ and $\beta_m \geqslant 1$, then $\beta = \lambda$ or $\beta^s \succ \lambda$. Let $\mu = (\delta_m\beta)^s$. By Proposition 1.7 we have (i) $\delta_m\beta = \delta_m\lambda$, or (ii) $\mu = \delta_m\lambda$ and $(\delta_m\beta)_m > (\delta_m\lambda)_m$, or (iii) $\mu \succ \delta_m\lambda$. In case (i) $\beta = \lambda$. In case (ii) $\mu = \delta_m\lambda$, and β is obtained from $\delta_m\lambda$ by adding 1 to a part of λ larger than λ_m, so that $\beta^s = (\lambda_1, \ldots, \lambda_s + 1, \ldots, \lambda_m - 1)$ for some s which implies $\beta^s \succ \lambda$. In case (iii) $\sum_{i=1}^j \mu_i \geqslant \sum_{i=1}^j \lambda_i$ for $j = 1, \ldots, m-1$, and at least one of these inequalities is strict. Then β^s is obtained by adding 1 to some part of μ, which implies $\beta^s \succ \lambda$.

Part 2. We show if $A(\beta,\lambda) \neq 0$, $\beta_{j+1} = \beta_{j+2} = \cdots = \beta_m = 0$, and $\beta_j \geqslant 1$ for some $j < m$, then $\beta^s \succ \lambda$. Consider $g_j = \lambda_j\rho_j(T_j\rho_j)^{-1}h_j$, let $\delta_j\beta = (\beta_1, \ldots, \beta_j - 1, \ldots, \beta_m)$, and let $\mu = (\delta_j\beta)^s \in \mathcal{N}_{m,n-1}^P$. As before, (ii) $\mu = (\delta_j\lambda)^s$ and $(\delta_j\beta)_j > (\delta_j\lambda)_j$ or (iii) $\mu \succ (\delta_j\lambda)^s$ must hold (the case (i) $\delta_j\beta = \delta_j\lambda$ is ruled out because $(\delta_j\beta)_m = 0 < (\delta_j\lambda)_m$). In case (iii),

$$\sum_{i=1}^l \mu_i \geqslant \sum_{i=1}^l \lambda_i \quad \text{for } 1 \leqslant l \leqslant j-1,$$

and

$$n - 1 = \sum_{i=1}^j \mu_i \geqslant \sum_{i=1}^{j-1} \lambda_i + \tilde{\lambda}_j,$$

where $\tilde{\lambda}_j = \lambda_j - 1$ if $\lambda_j > \lambda_{j+1}$ or $\tilde{\lambda}_j = \lambda_j$ if $\lambda_j = \lambda_{j+1}$. If $n-1 = \sum_{i=1}^{j} \mu_i > \sum_{i=1}^{j-1} \lambda_i + \tilde{\lambda}_j$, then adding 1 to any part μ_i $(i \leqslant j)$ produces $\beta \in \mathcal{N}_{m,n}$ with $\beta^s \succ \lambda$ (recall $\beta_{j+1} = \beta_{j+2} = \cdots = \beta_m = 0$, thus μ has j or fewer parts). If there is equality $n-1 = \sum_{i=1}^{j} \mu_i = \sum_{i=1}^{j-1} \lambda_i + \tilde{\lambda}_j$, then $j = m-1$ and $\lambda_m = \lambda_{m-1} = 1$. Again β^s is obtained by adding 1 to some part of μ and $\beta^s \succ \lambda$ (because $n = \sum_{i=1}^{m-1} \beta_i > \sum_{i=1}^{m-1} \lambda_i$). In case (ii), $\mu = (\delta_j \lambda)^s$ implies $\lambda_{m-1} = \lambda_m = 1$ and $j = m-1$; also $(\delta_j \beta)_j = \beta_j - 1 > (\delta_j \lambda)_j = \lambda_j - 1 = 0$ implies that β^s is obtained by adding 1 to some part of λ larger than $\lambda_m = \lambda_{m-1} = 1$; hence $\beta^s \succ \lambda$.

Finally, $A(\lambda, \lambda)$ is the coefficient of p_λ in $\lambda_m \rho_m (T_m \rho_m)^{-1} h_m$ where

$$h_m = V\left(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_m^{\lambda_m - 1}\right).$$

By the inductive hypothesis,

$$\operatorname{cof}(h_m, p_{\delta_m \lambda}) = \frac{(\delta_m \lambda)!}{(Nk+1)_{\delta_m \lambda; k}}.$$

By the triangular property of $T_m \rho_m$ and the inductive hypothesis

$$\operatorname{cof}\Big((T_m \rho_m)^{-1} h_m, p_{\delta_m \lambda}\Big) = (Nk + (m-1)k + \lambda_m)^{-1} \operatorname{cof}(h_m, p_{\delta_m \lambda}).$$

Thus

$$\begin{aligned} A(\lambda, \lambda) &= \operatorname{cof}(\lambda_m \rho_m (T_m \rho_m)^{-1} h_m, p_\lambda) \\ &= \frac{(\lambda_m / (Nk + (m-1)k + \lambda_m))(\lambda_1! \lambda_2! \cdots (\lambda_m - 1)!)}{(Nk+1)_{\delta_m \lambda; k}} \\ &= \frac{\lambda!}{(Nk+1)_{\lambda; k}}. \end{aligned}$$

□

By setting up a simple formal correspondence between x^α and p_α we can study V as an endomorphism on $\mathcal{V}_m$. Define $\xi p_\alpha = \frac{1}{\alpha!} x^\alpha$, for $\alpha \in \mathcal{N}_m$.

2.3 Proposition. *The linear map $V\xi$ is a diagonalizable automorphism of $\mathcal{V}_{m,n}$ (each $n \in \mathbf{Z}_+$) with eigenvalues $1/(Nk+1)_{\lambda;k}$ for $\lambda \in \mathcal{N}^P_{m,n}$.*

Proof. For any $\lambda \in \mathcal{N}^P_{m,n}$ and $\alpha \in \mathcal{N}_{m,n}$ with $\alpha^s = \lambda$ (that is, α is a permutation of λ),

$$V\xi p_\alpha = \frac{1}{\lambda!}\left(\frac{\lambda!}{(Nk+1)_{\lambda;k}} p_\alpha + \sum_{\substack{\beta \in \mathcal{N}_{m,n} \\ \beta^s \succ \lambda}} A(\beta, \alpha) p_\beta\right).$$

Thus $V\xi$ is triangular for the partial order on $\mathcal{N}_{n,m}$ induced by $\beta^s \succ \alpha^s$. The formula for $V\xi$ shows that its matrix representation in the $\{p_\alpha : \alpha \in \mathcal{N}_{m,n}\}$ basis has a (upper) triangular block structure with respect to the subspaces span $\{p_\alpha : \alpha^s = \lambda\}$ for $\lambda \in \mathcal{N}^P_{m,n}$. The on-diagonal blocks are multiples of the identity matrix, and the eigenvalues $1/(Nk+1)_{\lambda;k}$ are distinct for different blocks. Hence $V\xi$ is diagonalizable. □

In this argument we tacitly used the generic dependence on N (see Corollary 1.10). Also the formula for V^{-1} (involving $T_i, 1 \leqslant i \leqslant m$) shows that N is just a parameter. In fact for specializations there may well be "accidental" equalities: example for $\lambda = (6,3,3,2,1,1)$ and $\mu = (5,3,3,3,2)$, $(Nk+1)_{\lambda;k} = (Nk+1)_{\mu;k}$ when $N = 6$.

2.1 Definition. For $\lambda \in \mathcal{N}^P_{m,n}$, let

$$E_\lambda = \left\{ f \in \mathcal{V}_{m,n} : V\xi f = \frac{1}{(Nk+1)_{\lambda;k}} f \right\}.$$

For $\alpha \in \mathcal{N}_{m,n}$ with $\alpha^s = \lambda$, let ω_α be the element of E_λ with $\mathrm{cof}(\omega_\alpha, p_\alpha) = 1$ and $\mathrm{cof}(\omega_\alpha, p_\beta) = 0$ for any $\beta \neq \alpha$ with $\beta^s = \lambda$.

By Proposition 2.3, $\mathcal{V}_{m,n}$ is the direct sum of E_λ, $\lambda \in \mathcal{N}^P_{m,n}$. The canonical basis for E_λ is $\{\omega_\alpha : \alpha^s = \lambda\}$. If $f \in E_\lambda$, then $f = \sum_\alpha \mathrm{cof}(f, p_\alpha)\omega_\alpha$ (with $\alpha^s = \lambda$).

The spaces E_λ have a number of interesting properties which will be proved mostly by induction. For $n = 0, 1, 2, \ldots$, let $\mathcal{S}_n$ denote the following statements applying to any E_λ with $\lambda \in \mathcal{N}^P_{m,n}$:

(i) $\omega_\lambda = p_\lambda + \sum\{A_{\beta\lambda} p_\beta : \beta \in \mathcal{N}_{m,n}, \beta^s \succ \lambda\}$ and the coefficients $A_{\beta\lambda}$ are independent of N;

(ii) $T_i \rho_i E_\lambda = E_\lambda$ for $i = 1, \ldots, m$;

(iii) if $f \in E_\lambda$, $i = 1, \ldots, m$, then $T_i f \in \mathrm{span}\{E_{\delta_j\lambda} : \lambda_j > \lambda_{j+1}\}$, and the projection of $T_i f$ on $E_{\delta_j\lambda}$ is an eigenvector of $T_i\rho_i$ with eigenvalue $(Nk - k(j-1) + \lambda_j)$.

An equivalent formulation to (iii) is

(iii)$'$ any $f \in E_\lambda$ has the expansion $f = \rho_i \sum_j \{f_{ij} : \lambda_j > \lambda_{j+1}\} + f_{i,0}$ where $f_{ij} \in E_{\delta_j\lambda}$, $T_i\rho_i f_{ij} = (Nk - (j-1)k + \lambda_j) f_{ij}$ (for j with $\lambda_j > \lambda_{j+1}$, so that $\delta_j\lambda$ is a partition), $T_i f_{i,0} = 0$, and $T_i f = \sum_j (Nk - k(j-1) + \lambda_j) f_{ij}$.

Under the hypothesis $(\mathcal{S}_n)$ one can determine the specific form of the eigenvectors of $T_i\rho_i$.

2.4 Theorem. *If $T_i\rho_i E_\lambda \subset E_\lambda$, for some $\lambda \in \mathcal{N}^P_{m,n}$, then for each $\alpha \in \mathcal{N}_{m,n}$ with $\alpha^s = \lambda$ there is a unique eigenvector $\phi_{\alpha,i} = \omega_\alpha + \sum\{B(\beta,\alpha)\omega_\beta : \beta^s = \lambda, \beta_i > \alpha_i\}$ with eigenvalue $(Nk - k\#\{l : \lambda_l > \alpha_i\} + \alpha_i + 1)$. The coefficients $B(\beta, \alpha)$ depend on k,* but not on N. Any eigenvector f of $T_i\rho_i$ in E_λ with the same eigenvalue has the expansion

$$f = \sum \{\mathrm{cof}\,(f, p_\beta)\phi_{\beta,i} : \beta^s = \lambda \quad \textit{and} \quad \beta_i = \alpha_i\}.$$

Proof. The expansion of $T_i\rho_i\omega_\beta$ in terms of ω_γ, $\gamma^s = \lambda$ is determined by the coeffcients of p_γ. By formula (1.1),

$$\begin{aligned} T_i\rho_i\omega_\beta = {} & (Nk - k\#\{l : \lambda_l > \beta_i\} + \beta_i + 1)\omega_\beta \\ & + k\sum\{(ij)\omega_\beta : \beta_j > \beta_i\}. \end{aligned}$$

First we characterize the possible β for which $B(\beta,\alpha)\neq 0$. If $\alpha_i=\lambda_1$, then $\phi_{\alpha,i}=\omega_\alpha$ is an eigenvector for $Nk+\lambda_1+1$. Henceforth assume $\mu_i<\lambda_1$. There is a permutation $\pi\in S_m$ such that $\alpha_j=\lambda_{\pi(j)}$ for all j and $\lambda_{\pi(i)-1}>\lambda_{\pi(i)}$. Let $\pi_0=\pi(i)$. By the triangular property of $T_i\rho_i$ the eigenvalue of $\phi_{\alpha,i}$ must be $(Nk-(\pi_0-1)k+\lambda_{\pi_0}+1)$. Say $(j_1,j_2,\dots,j_s)$ is a permissible string if $1\leqslant j_1<j_2\cdots<j_s<\pi_0$ and $\lambda_{j_1}>\lambda_{j_2}>\cdots>\lambda_{j_s}>\lambda_{\pi_0}$. For such a string, let $\tilde{j}_l=\#\{t:\lambda_t>\lambda_{j_l}\}$, $l=1,\dots,s$. Then $B(\beta,\alpha)=(-k)^s/\prod_{l=1}^s(k(\pi_0-1-\tilde{j}_l)+$ $+\lambda_{j_l}-\lambda_{\pi_0})$ for $\omega_\beta=(i,\pi^{-1}(j_1))(i,\pi^{-1}(j_2))\cdots(i,\pi^{-1}(j_s))\omega_\alpha$. The construction of β implies that β and α differ in exactly $s+1$ entries, namely those indexed by elements of $\{i\}\cup\{\pi^{-1}(j_t):t=1,\dots,s\}$. Also $\beta_i=\alpha_{\pi^{-1}(j_1)}$, $\beta_{\pi^{-1}(j_t)}=$ $=\alpha_{\pi^{-1}(j_{t+1})}=\lambda_{j_{t+1}}$ for $t=1,\dots,s-1$ and $\beta_{\pi^{-1}(j_s)}=\alpha_i$. Further $\beta_i>\beta_{\pi^{-1}(j_1)}>\cdots>\beta_{\pi^{-1}(j_s)}>\alpha_i$ (these values coincide with $\lambda_{j_1}>\lambda_{j_2}\cdots>\lambda_{\pi_0}$). This shows that there is a one-to-one correspondence between the permissible string and β (so that $B(\beta,\alpha)$ is uniquely defined). Set $B(\gamma,\alpha)=0$ for $\gamma\in\mathcal{N}_{m,n}$, $\gamma^s=\lambda$ if γ does not arise in this way.

To show $\phi_{\alpha,i}$ is an eigenvector consider $\operatorname{cof}(T_i\rho_i\phi_{\alpha,i},\omega_\beta)$, the sum of the contributions from $T_i\rho_i\omega_\beta$ and $T_i\rho_i\omega_\gamma$ where $\omega_\gamma:=(i,\pi^{-1}(j_2))\cdots(i,\pi^{-1}(j_s))\omega_\alpha$. Indeed,

$$\begin{aligned}\operatorname{cof}(T_i\rho_i\phi_{\alpha,i}\omega_\beta)&=\frac{(-k)^{s-1}}{\prod_{l=2}^{s}(k(\pi_0-1-\tilde{j}_l)+\lambda_{j_l}-\lambda_{\pi_0})}\\&\quad\cdot\left(\frac{(Nk+1-k\tilde{j}_1+\lambda_{j_1})(-k)}{k(\pi_0-1-\tilde{j}_1)+\lambda_{j_1}-\lambda_{\pi_0}}+k\right)\\&=(Nk-k(\pi_0-1)+\lambda_{\pi_0}+1)\operatorname{cof}(\phi_{\alpha,i,}\omega_\beta).\end{aligned}$$ □

For two linear operators U_1, U_2 the commutator is $[U_1,U_2]:=U_1U_2-U_2U_1$. Define a "variable-changing" operator on $\mathcal{V}_m$ for each $i,j=1,\dots,m$ with $i\neq j$ by

$$\zeta_{i,j}p_\alpha:=p\left(\alpha_1,\dots\overset{i}{\alpha_i+\alpha_j},\dots,\overset{j}{0},\dots\right),\quad\alpha\in\mathcal{N}_m.$$

2.5 Lemma. *For $i,j=1,\dots,m$ with $i\neq j$:*

(i) $T_i\rho_i^2=\rho_iT_i\rho_i+\left(1+k\sum_{l\neq i}(il)\right)\rho_i-k\sum_{l\neq i}\zeta_{l,i}\rho_i$;

(ii) $T_j\rho_i\rho_j=\rho_i(T_j\rho_j-k(ij))+k\zeta_{j,i}\rho_i$;

(iii) $[T_j\rho_j,\ T_i\rho_i-k(ij)]=0$.

Proof. For part (i), let $\alpha\in\mathcal{N}_m$ and evaluate

$$
\begin{aligned}
&T_i\rho_i^2 p_\alpha - \rho_i T_i \rho_i p_\alpha \\
&\quad = (Nk+\alpha_i+2)\rho_i p_\alpha + k\sum_{l\neq i}\sum_{s=0}^{\alpha_i+1}\Big(p\Big(\dots,\overset{i}{\alpha_i+\alpha_l+1-s},\dots\overset{l}{s},\dots\Big) \\
&\qquad -p\Big(\dots,\overset{i}{s},\dots,\overset{l}{\alpha_i+\alpha_l+1-s},\dots\Big)\Big) \\
&\qquad -(NK+\alpha_i+1)\rho_i p_\alpha - k\sum_{l\neq i}\Big(\sum_{s=0}^{\alpha_i} p\Big(\dots,\overset{i}{\alpha_i+\alpha_l+1-s},\dots,\overset{l}{s},\dots\Big) \\
&\qquad -p\Big(\dots,\overset{i}{s+1},\dots,\overset{l}{\alpha_i+\alpha_l-s},\dots\Big)\Big) \\
&\quad = \rho_i p_\alpha + k\sum_{l\neq i}\Big(p\Big(\dots,\overset{i}{\alpha_l}\dots,\overset{l}{\alpha_i+1},\dots\Big) - p\Big(\dots,\overset{i}{0},\dots,\overset{l}{\alpha_i+\alpha_l+1},\dots\Big)\Big) \\
&\quad = \Big(1+k\sum_{l\neq i}(il)\Big)\rho_i p_\alpha - k\sum_{l\neq i}\zeta_{li}\rho_i p_\alpha .
\end{aligned}
$$

Similarly, for part (ii),

$$
\begin{aligned}
&T_j\rho_i\rho_j p_\alpha - \rho_i T_j \rho_j p_\alpha \\
&\quad = (Nk+\alpha_j+1)\rho_i p_\alpha + k\sum_{l=0}^{\alpha_j}\Big(p\Big(\dots,\overset{i}{l},\dots,\overset{j}{\alpha_i+\alpha_j+1-l},\dots\Big) \\
&\qquad -p\Big(\dots,\overset{i}{\alpha_i+\alpha_j+1-l},\dots,\overset{j}{l},\dots\Big)\Big) - (Nk+\alpha_j+1)\rho_i p_\alpha \\
&\qquad -k\sum_{l=0}^{\alpha_j} p\Big(\dots,\overset{i}{l+1},\dots,\overset{j}{\alpha_i+\alpha_j-l},\dots\Big) - p\Big(\dots,\overset{i}{\alpha_i+\alpha_j+1-l},\dots,\overset{j}{l},\dots\Big) \\
&\quad = k\Big(p\Big(\dots,\overset{i}{0},\dots,\overset{j}{\alpha_i+\alpha_j+1},\dots\Big) - p\Big(\dots,\overset{i}{\alpha_j+1},\dots,\overset{j}{\alpha_i},\dots\Big)\Big) \\
&\quad = k\zeta_{j,i}\rho_i p_\alpha - k\rho_i(ij)p_\alpha .
\end{aligned}
$$

For part (iii),

$$
\begin{aligned}
[T_j\rho_j, T_i\rho_i] &= T_j(T_i\rho_j - \rho_j T_i))\rho_i - T_i(T_j\rho_i - (T_j\rho_i - \rho_i T_j))\rho_j \\
&= T_i(k\zeta_{j,i}\rho_i - k\rho_i(ij)) - T_j(k\zeta_{i,j}\rho_j - k\rho_j(ij)) \\
&= k(T_j\rho_j - T_i\rho_i)(ij)
\end{aligned}
$$

(because $[T_i, T_j] = 0$, Theorem 1.9 [3] and $T_j\zeta_{i,j} = 0$). From this relation $[T_j\rho_j, T_i\rho_i - k(ij)] = 0$ follows easily. □

2.6 Lemma. *If $\mathscr{S}_{n-1}$ holds and $\lambda \in \mathcal{N}^P_{m,n}$, then $f \in E_\lambda$ if and only if for each $i = 1, \dots, m$*

$$f = \rho_i \sum\{f_{ij} : j = 1, \dots, m \text{ and } \lambda_j > \lambda_{j+1}\} + f_{i,0},$$

with $f_{ij} \in E_{\delta_j\lambda}$, $T_i\rho_i f_{ij} = (Nk - k(j-1) + \lambda_j)f_{ij}$, and $T_i f_{i,0} = 0$.

Proof. Let $f \in E_\lambda$, $i = 1, \ldots, m$. Break up f into two parts:

$$f'_i := \sum\{\mathrm{cof}(f, p_\alpha)p_\alpha : \alpha_i \geqslant 1\},$$

and

$$f_{i,0} := \sum\{\mathrm{cof}(f, p_\alpha)p_\alpha : \alpha_i = 0\}.$$

By the direct sum decomposition of $\mathscr{V}_{m,n-1}$ expand

$$f'_i = \rho_i \sum\{f'_\mu :\in \mathscr{N}^P_{m,n-1}, f'_\mu \in E_\mu, f'_\mu \neq 0\}.$$

By the definition of V and E_λ we have

$$T_i V \xi f = (1/(Nk+1)_{\lambda;k}) T_i f = V \frac{\partial}{\partial x_i}(\xi f).$$

But

$$\frac{\partial}{\partial x_i}\xi p_\alpha = \frac{\partial}{\partial x_i}(x_1^{\alpha_1} \cdots x_m^{\alpha_m}/(\alpha_1!\alpha_2! \cdots \alpha_m!))$$

$$= \begin{cases} 0 & \alpha_i = 0, \\ \xi p(\alpha_1, \ldots, \alpha_i - 1, \ldots) & \alpha_i \geqslant 1, \end{cases}$$

thus

$$\left(\frac{1}{(Nk+1)_{\lambda;k}}\right) T_i f = V \Sigma_\mu f'_\mu \frac{1}{(Nk+1)_{\mu;k}} f'_\mu.$$

The property $(\mathscr{S}_{n-1}ii)$ implies that $\dfrac{(Nk+1)_{\lambda;k}}{(Nk+1)_{\mu;k}}$ is an eigenvalue of $T_i\rho_i$ on E_μ because $T_i f = \Sigma_\mu T_i \rho_i f'_\mu = \Sigma_\mu \dfrac{(NK+1)_{\lambda;k}}{(Nk+1)_{\mu;k}} f'_\mu$. Since N is generic, and the eigenvalues of $T_i\rho_i$ on E_μ are known by $(\mathscr{S}_{n-1}ii)$ and Theorem 2.4, we see that $(Nk+1)_{\mu;k}$ must divide $(Nk+1)_{\lambda;k}$ (when $f'_\mu \neq 0$). This implies $\mu_s = \lambda_s$ except $\mu_j = \lambda_j - 1$ for some j with $\lambda_j > \lambda_{j+1}$; the quotient $(Nk - k(j-1) + \mu_j + 1) =$ $= (Nk - k(j-1) + \lambda_j)$ is indeed an eigenvalue of $T_i\rho_i$ on E_μ (Theorem 2.4).

For the converse, suppose $g \in \mathscr{V}_{m,n}$ and for each $i = 1, \ldots, m$, $g =$ $= \rho_i \Sigma_j\{g_{ij} : \lambda_j > \lambda_{j+1}\} + g_{i,0}$ such that $g_{ij} \in E_{\delta_j\lambda}$, $T_i\rho_i g_{ij} = (Nk - k(j-1) + \lambda_j)g_{ij}$ and $T_i g_{i,0} = 0$. We must show $g \in E_\lambda$. Expand $g = \Sigma\{g_\nu : \nu \in \mathscr{N}^P_{m,n}\}$ with $g_\nu \in E_\nu$. By the first part of this lemma we can assume $g_\lambda = 0$ (by subtraction). For some fixed $\nu \neq \lambda$ consider the contribution of g_ν to $g_{ij} \in E_{\delta_j\lambda}$, an eigenvector of $T_i\rho_i$ with eigenvalue $(Nk - k(\#\{l : \nu_l \geqslant \nu_s\} - 1) + \nu_s)$ where $\nu_t = \lambda_t$ for all t except $\nu_j = \lambda_j - 1$ and $\nu_s = \lambda_s + 1$, some s (subject to $\nu_s \leqslant \nu_{s-1}$ or $s = 1$). This eigenvalue differs from the hypothetical $(Nk - k(j-1) + \lambda_j)$ because $\lambda_j = \nu_s$ is impossible; this requires $s = j+1$ and $\lambda_{j+1} = \lambda_j - 1$, then $\nu_{j+1} > \nu_j$. Thus $g_{ij} = 0$ for each j (with $\lambda_j > \lambda_{j+1}$). This leaves $T_i g = 0$ for each i, which implies $g = 0$. Recall the desired component of g in E_λ was subtracted off. □

2.7 Theorem. *$\mathscr{S}_n$ holds for all $n \in \mathbf{Z}_+$.*

Proof. Assume $\mathscr{S}_{n-1}$ holds. Lemma 2.6 shows that $(\mathscr{S}_n iii)$ is valid. Let $\lambda \in \mathscr{N}^P_{m,n}$. We will start by showing that $T_i\rho_i E_\lambda \subset E_\lambda$, for $i = 1, \ldots, m$. Let $f \in E_\lambda$ and by the lemma expand $f = \rho_j \Sigma\{f_{j,l} : \lambda_l > \lambda_{l+1}\} + f_{j,0}$ (for any $j = 1, \ldots, m$;

and $f_{j,l} \in E_{\delta_l\lambda}$, $T_j\rho_j f_{j,l} = (Nk - k(l-1) + \lambda_l)f_{j,l}$ and $T_j f_{j,0} = 0$. For any $j \neq i$, by Lemma 2.5 (ii),

$$\begin{aligned} T_i\rho_i f &= T_i\rho_i\rho_j\Sigma_l f_{j,l} + T_i\rho_i f_{j,0} \\ &= \rho_j(T_i\rho_i - k(ij))\Sigma_l f_{j,l} + k\zeta_{ij}(\rho_j\Sigma_l f_{j,l}) + T_i\rho_i f_{j,0}. \end{aligned}$$

Rewrite this as $T_i\rho_i f = \rho_j\Sigma_l g_{j,l} + g_{j,0}$ where $g_{j,l} := (T_i\rho_i - k(ij))f_{j,l}$ and $g_{j,0} := k\zeta_{i,j}(\rho_j\Sigma_l f_{j,l}) + T_i\rho_i f_{j,0}$. By definition of ζ_{ij} and properties of $T_i\rho_i$, $T_j g_{j,0} = 0$. Because $[T_j\rho_j, T_i\rho_i - k(ij)] = 0$ (Lemma 2.5(iii)) and by property $\mathscr{S}_{n-1}(iii)$, $g_{j,l} \in E_{\delta_l\lambda}$ and $T_j\rho_j g_{j,l} = (Nk - k(l-1) + \lambda_l)g_{j,l}$.

To apply Lemma 2.6 to show $T_i\rho_i f \in E_\lambda$ we also have to do the case $j = i$. By Lemma 2.5 (i),

$$\begin{aligned} T_i\rho_i f &= T_i\rho_i^2\Sigma_l f_{i,l} + T_i\rho_i f_{i,0} \\ &\quad \text{(the range of summation is again}\{l : \lambda_l > \lambda_{l+1}\}) \\ &= \left(\rho_i T_i\rho_i + \left(1 + k\sum_{j\neq i}(ij)\right)\rho_i\right)\Sigma_l f_{i,l} + T_i\rho_i f_{i,0} - k\sum_{j\neq i}\zeta_{j,i}(\rho_i\Sigma_l f_{i,l}) \\ &= \rho_i T_i\rho_i\Sigma_l f_{i,l} + \left(1 + k\sum_{j\neq i}(ij)\right)(f - f_{i,0}) \\ &\quad + (Nk + 1 - (m-1)k)f_{i,0} + k\sum_{j\neq i}(ij)f_{i,0} - k\sum_{j\neq i}\zeta_{ji}(\rho_i\Sigma_l f_{i,l}) \\ &= \rho_i\Sigma_l(Nk - k(l-1) + \lambda_l)f_{i,l} + \left(1 + k\sum_{j\neq i}(ij)\right)f \\ &\quad + (Nk - (m-1)k)f_{i,0} - k\sum_{j\neq i}\zeta_{ji}(\rho_i\Sigma_l f_{i,l}) \end{aligned}$$

(because $T_i\rho_i p_\alpha = (Nk + 1 - (m-1)k)p_\alpha + k\sum_{s\neq i}(is)p_\alpha$ when $\alpha \in \mathscr{N}_{m,n}$ and $\alpha_i = 0$). Further $T_i\zeta_{s,i} = 0$, and $(ij)f \in E_\lambda$; thus $T_i\rho_i f$ satisfies the hypotheses of Lemma 2.6 and is an element of E_λ.

Now assume $\lambda_m > 0$ (simply note that if $\lambda_m = 0$, then $E_\lambda = (1 + (1,m))E_{(\lambda_1,\lambda_2,\ldots,\lambda_{m-1})}$; or argue that E_λ is the span of the S_m-orbit of ω_μ where μ is obtained from λ by dropping the zero parts). Let $\{t_1, t_2, \ldots, t_r\} = \{j : \lambda_j > \lambda_{j+1}\}$ and $1 \leqslant t_1 < t_2 \cdots < t_r = m$. From Theorem 2.2, $\omega_\lambda = p_\lambda + \Sigma\{A_{\alpha\lambda}p_\alpha : \alpha \in \mathscr{N}_{m,n}$ and $\alpha^s \succ \lambda\}$. As in the previous part of the proof for each $j = 1, \ldots, m$ expand $\omega_\lambda = \rho_j\sum_{s=1}^r f_{j,s} + f_{j,0}$ with $f_{j,s} \in E_{\delta_{t_s}\lambda}$, $f_{j,s}$ is an eigenvector of $T_j\rho_j$ with eigenvalue $(Nk - k(t_s - 1) + \lambda_{t_s})$, and $T_j f_{j,0} = 0$. We claim that $f_{j,s} = 0$ if $j > t_s$. By property $\mathscr{S}_{n-1}(i)$, $f_{j,s}$ is uniquely determined by $\mathrm{cof}(f_{j,s}, p_\beta)$ for $\beta^s = \delta_{t_s}\lambda$. Since $s > i$ implies $\delta_{t_i}\lambda \succ \delta_{t_s}\lambda$ we have

$$\mathrm{cof}(f_{j,s}, p_\beta) = \mathrm{cof}(\omega_\lambda, p_{\tilde\beta}) - \sum_{l=1}^{s-2}\mathrm{cof}(f_{j,l}, p_\beta),$$

where $\beta \in \mathscr{N}_{m,n-1}$, $\beta^s = (\delta_j\lambda)^s$ (note $\delta_j\lambda$ is not necessarily a partition), and $\tilde\beta := (\beta_1, \ldots, \beta_j + 1, \ldots)$. If $f_{j,s} \neq 0$ for some s, then it is an eigenvector of $T_j\rho_j$

with eigenvalue $(Nk - k(t_s - 1) + \lambda_{t_s})$ which requires $\operatorname{cof}(f_{j,s}, p_\beta) \neq 0$ for some β with $\beta_j = \lambda_{t_s} - 1$ and $\operatorname{cof}(f_{j,s}, p_\gamma) = 0$ for all γ with $\gamma^s = \delta_{t_s}\lambda$ and $\gamma_j < \lambda_{t_s} - 1$ (Theorem 2.4). Suppose $t_1 < t_2 \cdots < t_s < j$. If $f_{j,1} \neq 0$, then $0 \neq \operatorname{cof}(f_{j,1}, p_\beta) = = \operatorname{cof}(\omega_\lambda, p_{\tilde{\beta}})$ with $\beta_j = \lambda_{t_1} - 1$, but this implies $\tilde{\beta}^s = \lambda$ and $\tilde{\beta} \neq \lambda$ (since $\tilde{\beta}_j \neq \lambda_j$) so $\operatorname{cof}(\omega_\lambda, p_{\tilde{\beta}}) = 0$, a contradiction. The same argument inductively applied shows $f_{j,l} = 0$ for $l \leqslant s$.

The last step is to show that the coefficients $A_{\alpha\lambda}$ are independent of N. Specialize the above conclusion to $j = m$, to obtain $\omega_\lambda = \rho_m f_m + f_0$ (changing notation) with $f_m \in E_{\delta_m \lambda}$ and $T_m f_0 = 0$.

Since f_m is an eigenvector of $T_m \rho_m$ with eigenvalue $(Nk - (m-1)k + \lambda_m)$, Theorem 2.4 shows that $f_m = \omega_{\delta_m \lambda} + \Sigma\{B(\mu)\omega_\mu : \mu^s = \delta_m \lambda \text{ and } \mu_m > \lambda_m - 1\}$ for certain coefficients $B(\mu)$ which are independent of N (and in $\mathbf{Q}(k)$). By property $\mathcal{S}_{n-1}(i)$ the coefficients of $\rho_m f_m$ (in the $\{p_\alpha\}$ basis) are independent of N.

We find an explicit formula for f_0 in terms of f_m, again independent of N. From the first part of the proof $(\mathcal{S}_n ii)$ we see that ω_λ is an eigenvector of $(T_m\rho_m - k\sum_{i<m}(im))$ with eigenvalue $(Nk - k(m-1) + \lambda_m + 1)$ (similar to the argument in Theorem 2.4). Write this as an equation for f_0: indeed,

$$\left(\left(T_m\rho_m - k\sum_{i<m}(im)\right) - (Nk - k(m-1) + \lambda_m + 1)\right)\rho_m f_m$$
$$= \left((Nk - k(m-1) + \lambda_m + 1) - \left(T_m\rho_m - k\sum_{i<m}(im)\right)\right)f_0.$$

But for $\alpha \in \mathcal{N}_{m,n}$ with $\alpha_m = 0$, $(T_m\rho_m - k\sum_{i<m}(im))p_\alpha = (Nk - (m-1)k + 1)p_\alpha$ and this applies to each term of the right-hand side of the equation. The commutation relation 2.5i shows that the left-hand side equals

$$\rho_m T_m \rho_m f_m + \left(1 + k\sum_{i<m}(im)\right)\rho_m f_m - k\sum_{i<m}\zeta_{im}\rho_m f_m$$
$$- (Nk - k(m-1) + \lambda_m + 1)\rho_m f_m - k\sum_{i<m}(im)\rho_m f_m = -k\sum_{i<m}\zeta_{im}\rho_m f_m,$$

while the right-hand side reduces to $\lambda_m f_0$. That is $f_0 = -(k/\lambda_m)\sum_{i<m}\zeta_{im}\rho_m f_m$, and the $\{p_\alpha\}$-expansion coefficients of f_0 are independent of N. □

Observe that the proof actually provided an algorithm for ω_λ in terms of $E_{\delta_m\lambda}$. The fact that $V\xi$ acts as a multiple of the identity on E_λ and $T_i\rho_i E_\lambda \subset E_\lambda$ for each $\lambda \in \mathcal{N}^P_{m,n}$ shows that $T_i\rho_i$ commutes with $V\xi$. There does not seem to be a direct (non-inductive) way of proving this.

We state the $m = 2$ results for illustration and leave the proofs as exercises. Let $\lambda = (\lambda_1, \lambda_2)$ (with $\lambda_1 \geqslant \lambda_2$), then

$$\omega_\lambda = p_\lambda + \sum_{j=1}^{\lambda_2}\frac{(-k)_j}{(k + \lambda_1 - \lambda_2 + 1)_j j!}((\lambda_1 - \lambda_2 + 1)_j p(\lambda_1 + j, \lambda_2 - j)$$
$$+ j(\lambda_1 - \lambda_2 + 1)_{j-1}p(\lambda_2 - j, \lambda_1 + j)).$$

As Theorem 2.4 asserts, $T_1\rho_1\omega_\lambda=(Nk+\lambda_1+1)\omega_\lambda$ and $T_2\rho_2\omega_\lambda=(Nk-k+$ $+\lambda_2+1)\omega_\lambda+k\omega_{(\lambda_2,\lambda_1)}$ (for $\lambda_1>\lambda_2$).

The transition from E_λ to $E_{(\lambda_1-1,\lambda_2-1,\ldots,\lambda_m-1)}$ (for $\lambda_m\geqslant 1$) is remarkably easy. Define the linear map $\varepsilon_m:\mathscr{V}_{m,n}\to\mathscr{V}_{m;n-m}$ by

$$\varepsilon_m p_\alpha=\begin{cases}p(\alpha_1-1,\alpha_2-1,\ldots,\alpha_m-1) & \text{if each } \alpha_i\geqslant 1;\\ 0 & \text{if any } \alpha_i=0.\end{cases}$$

We will show $\varepsilon_m\omega_\lambda=\omega_{(\lambda_1-1,\ldots,\lambda_m-1)}$.

2.8 Lemma. $\varepsilon_m T_i\rho_i-T_i\rho_i\varepsilon_m=\varepsilon_m$, $i=1,\ldots,m$.

Proof. For $\alpha\in\mathscr{N}_{m,n}$,

$$\begin{aligned}(\varepsilon_m T_i\rho_i-T_i\rho_i\varepsilon_m)p_\alpha &= [\varepsilon_m(Nk+\alpha_i+1)-(Nk+\alpha_i)\varepsilon_m]p_\alpha\\ &\quad+k\Bigg\{\sum_{s\neq i}\Bigg(\sum_{l=0}^{\alpha_i}\varepsilon_m p\Big(\ldots,\alpha_i+\overset{i}{\alpha_s}-l,\ldots,\overset{s}{l},\ldots\Big)\\ &\quad-\sum_{l=0}^{\alpha_i-1}p\Big(\alpha_1-1,\ldots,\alpha_i+\alpha_s\overset{i}{-}2-l,\ldots,\overset{s}{l},\ldots\Big)\Bigg)\\ &\quad-\sum_{s\neq i}\Bigg(\sum_{l=0}^{\alpha_i}\varepsilon_m p\Big(\ldots,\overset{i}{l},\ldots,\alpha_i+\overset{s}{\alpha_s}-l,\ldots,\Big)\\ &\quad-\sum_{l=0}^{\alpha_i-1}p\Big(\alpha_1-1,\ldots,\overset{i}{l},\ldots,\alpha_i+\alpha_s\overset{s}{-}2-l,\ldots,\Big)\Bigg)\Bigg\}.\end{aligned}$$

any term with an entry of "-1" vanishes; then replace l to $l+1$ in the $0\leqslant l\leqslant\alpha_i$ summations, and everything in { } cancels. □

2.9 Theorem. *Let* $\lambda\in\mathscr{N}^P_{m,n}$ *with* $\lambda_m\geqslant 1$, *and let* $\mu=(\lambda_1-1,\lambda_2-1,\ldots,$ $\lambda_m-1)\in\mathscr{N}_{m,n-m}$, *then* $\varepsilon_m E_\lambda=E_\mu$, $\varepsilon_m\omega_\lambda=\omega_\mu$ *and*

$$T_1T_2\cdots T_m\omega_\lambda=(Nk+\lambda_1)(Nk-k+\lambda_2)\cdots(Nk-(m-1)k+\lambda_m)\omega_\mu.$$

Proof. Suppose that $\varepsilon_m E_\lambda=E_\mu$ and $\varepsilon_m\omega_\lambda=\omega_\mu$ is true for all $\lambda\in\mathscr{N}^P_{m,s}$ with $m\leqslant s\leqslant n-1$. The induction starts at $\varepsilon_m\omega_{(1,1,\ldots,1)}=1$; since $\omega_{(1,\ldots,1)}=$ $=p_{(1,\ldots,1)}+\Sigma\{A_{\alpha,(1,\ldots,1)}p_\alpha:\alpha\in\mathscr{N}_{m,m}$ and $\alpha^s\succ(1,1,\ldots,1)\}$ and $\alpha^s\succ(1,\ldots,1)$ implies at least one $\alpha_i=0$ so that $\varepsilon_m p_\alpha=0$. Let $\lambda\in\mathscr{N}^P_{m,n}$ with $\lambda_m\geqslant 1$, and let $f\in E_\lambda$. As before, let $\{t_1,\ldots,t_r\}=\{j:\lambda_j>\lambda_{j+1}\}$ and $1\leqslant t_1\leqslant t_2\cdots<t_r=m$. For each $i=1,\ldots,m$, expand $f=\rho_i\sum_{s=1}^r f_{i,s}+f_{i,0}$ with $f_{i,s}\in E_{\delta_{t_s}}\lambda$ and $T_i\rho_i f_{i,s}=(Nk-k(t_s-1)+\lambda_{t_s})f_{i,s}$ and $T_i f_{i,0}=0$ (by Lemma 2.6). Then $\varepsilon_m f=\varepsilon_m\rho_i\Sigma_s\varepsilon_m f_{i,s}=\rho_i\Sigma_s\varepsilon_m f_{i,s}+\varepsilon_m\Sigma_s\rho_i f'_{i,s}$, where

$$f'_{i,s}:=\Sigma\{\operatorname{cof}(f_{i,s},p_\alpha)p_\alpha:\alpha_i=0,\alpha\in\mathscr{N}_{m,n-1}\}.$$

Also, $T_i(\varepsilon_m\Sigma_s\rho_i f'_{i,s})=0$ (any p_α appearing in the expression has $\alpha_i=0$). By the inductive hypothesis $\varepsilon_m f_{i,s}\in E_{\delta_{t_s}\mu}$ with the appropriate eigenvalue for $T_i\rho_i$, or $\varepsilon_m f_{i,s}=0$ (necessary when $\mu_{t_s}=0=\lambda_{t_s}-1$). By Lemma 2.6, $\varepsilon_m f\in E_\mu$. Because

$\operatorname{cof}(\varepsilon_m f, p_\alpha) = \operatorname{cof}(\omega_\lambda, p(\alpha_1+1,\ldots,\alpha_m+1))$ for any $\alpha \in \mathcal{N}_{m,n-m}$ with $\alpha^s = \lambda$, $\varepsilon_m\omega_\lambda = \omega_\mu$. By the formula for $(V\xi)^{-1}$, we have

$$(V\xi)^{-1}\omega_\lambda = \sum_{\alpha\in\mathcal{N}_{m,n}} p_\alpha T_1^{\alpha_1}\cdots T_m^{\alpha_m}\omega_\lambda = (Nk+1)_{\lambda;k}\omega_\lambda.$$

Thus

$$\begin{aligned}(Nk+1)_{\lambda;k}\varepsilon_m\omega_\lambda &= \Sigma\{p(\alpha_1-1,\alpha_2-1,\ldots)T_1^{\alpha_1-1}\cdots T_m^{\alpha_m-1}(T_1T_2\cdots T_m\omega_\lambda) :\\ &\qquad\qquad \alpha\in\mathcal{N}_{m,n},\ \text{each } \alpha_i \geqslant 1\}\\ &= (V\xi)^{-1}(T_1T_2\cdots T_m\omega_\lambda),\end{aligned}$$

and so

$$\begin{aligned}(T_1T_2\cdots T_m)\omega_\lambda &= (Nk+1)_{\lambda;k}(V\xi)\omega_\mu\\ &= \frac{(Nk+1)_{\lambda;k}}{(Nk+1)_{\mu;k}}\omega_\mu\\ &= (Nk+\lambda_1)(Nk-k+\lambda_2)\cdots(Nk-(m-1)k+\lambda_m)\omega_\mu.\end{aligned}$$

Of course, the same factor applies to any ω_α with $\alpha^s = \lambda$. □

3. Some Further Developments

As an application we give a new proof of a formula of DUNKL and HANLON [7] for Garnir polynomials. Fix a partition μ of $N_1 \leqslant N$ and form the product of alternating polynomials corresponding to each column of the tableau for μ (enter the numbers $1,2,\ldots,N_1$ in the tableau in order and filling up the columns one by one). That is, partition the set $\{1,2,\ldots,N_1\}$ as $\{1,2,\ldots,\mu_1'\}\cup$ $\cup\{\mu_1'+1,\ldots,\mu_1'+\mu_2'\}\cup\cdots$, where $\mu'=(\mu_1',\mu_2',\ldots)$ is the transposed (conjugate) partition for μ. Let G_0 be the group of permutations of $\{1,\ldots,N_1\}$ which leave each part of the set partition invariant as sets, that is, $G_0 \simeq S_{\mu_1'}\times S_{\mu_2'}\times\cdots$. Let ε denote the sign character of G_0, and let

$$\alpha = \left(\mu_1'-1,\mu_1'-2,\ldots,1,\overset{\mu_1'}{0},\mu_2'-1,\mu_2'-2,\ldots,\overset{\mu_1'+\mu_2'}{0},\ldots\right)$$

(note $|\alpha| = \Sigma_i\binom{\mu_i'}{2}$ and α has $N_1-\mu_1$ nonzero entries). Then $g_\mu(x) := \sum_{w\in G_0}\varepsilon(w)x^{w\alpha}$ is the Garnir polynomial for μ.

3.1 Theorem. **([7])** *Let* $\lambda = \alpha^s$, *then* $g_\mu(T)g_\mu(x) = \prod_i(\mu_i')!(Nk+1)_{\lambda;k}$.

Proof. Let $m = N_1$, so that $\lambda = \alpha^s \in \mathcal{N}_m$. Let $f_\mu := \sum_{w\in G_0}\varepsilon(w)\omega_{w\alpha} \in E_\lambda$. The degree of f_μ (as a polynomial in x) is the same as the degree of g_μ, and f_μ has the same alternating properties for G_0 as g_μ hence $f_\mu = c(k)g_\mu$ with $c(k)\in\mathbf{Q}(k)$. In fact, $c(k)$ is the coefficient of x^α in f_μ, which is independent of $N(\geqslant N_1)$, by property $(\mathcal{S}i)$. The formula for $(V\xi)^{-1}$ shows that $\operatorname{cof}((V\xi)^{-1}f_\mu, p_\alpha) = T^\alpha f_\mu = (Nk+1)_{\lambda;k}$ (because $(V\xi)^{-1}f_\mu = (Nk+1)_{\lambda;k}f_\mu$ and $\operatorname{cof}(f_\mu,p_\alpha)=1$). By the G_0-alternating property, $g_\mu(T)f_\mu = \#(G_0)T^\alpha f_\mu = \#(G_0)(Nk+1)_{\lambda;k}$. Thus

$g_\mu(T)g_\mu(x) = \#(G_0)(Nk+1)_{\lambda;k}/c(k)$. But $g_\mu(T)g_\mu(x)$ is a polynomial in k of degree $|\lambda|$ (constant in x) and so $c(k)$ must be constant in k, because the degree of $(Nk+1)_{\lambda;k}$ is $|\lambda|$ and $c(k)$ is independent of N. It is straightforward to compute (for $k=0$)

$$g_\mu\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots\right) g_\mu(x) = \#(G_0)\lambda!,$$

thus $c(k) = 1$. □

We intend to construct an inner product for $\mathscr{V}_m$ (which is positive-definite when $k \geqslant 0$) in which the spaces E_λ are pairwise orthogonal. The following irreducibility result will be instrumental.

3.2 Proposition. *Suppose $\lambda \in \mathscr{N}_m^P$, $k \neq 0$, and $C : E_\lambda \to E_\lambda$ is a linear transformation which commutes with each $T_i\rho_i$, $i = 1, \dots, m$, then $C = cI$ (multiple of the identity), some $c \in \mathbf{Q}(k)$.*

Proof. We may assume $\lambda_1 \neq \lambda_m$, or else E_λ is one-dimensional. Suppose C has the matrix representation $C\omega_\alpha = \sum_\beta C(\beta,\alpha)\omega_\beta$ (with $\alpha^s = \lambda = \beta^s$). Then $[T_i\rho_i, C] = 0$ is equivalent to

$$C(\beta,\alpha)(h_i(\beta) - h_i(\alpha)) + k\left\{\sum_{\beta_l<\beta_i} C((il)\beta,\alpha) - \sum_{\alpha_l>\alpha_i} C(\beta,(il)\alpha)\right\}$$
$$= 0, \text{ all } \alpha,\beta \in \mathscr{N}_m,$$

with $\alpha^s = \lambda = \beta^s$, where the eigenvalue of $T_i\rho_i$ associated to ω_α by Theorem 2.4 is denoted

$$h_i(\alpha) := Nk - k\#\{l : \lambda_l > \alpha_i\} + \alpha_i + 1.$$

For fixed i, if $\beta_i = \lambda_m$ and $\alpha_i = \lambda_1 (> \lambda_m)$, then $C(\beta,\alpha) = 0$ since the sums in (3.2) are vacuous and $h_i(\alpha) \neq h_i(\beta)$. Doubly inducting, suppose that $\beta_i < \alpha_i$ and $C(\tilde\beta,\tilde\alpha) = 0$ for all permutations $\tilde\alpha$, $\tilde\beta$ of λ such that (i) $\tilde\beta_i < \beta_i$ and $\alpha_i \leqslant \tilde\alpha_i$, or (ii) $\tilde\beta_i \leqslant \beta_i$ and $\alpha_i < \tilde\alpha_i$, then $C(\beta,\alpha) = 0$ (again, each of the sums equals zero). For any permutations α,β of λ if $\alpha \neq \beta$ there exists i such that $\beta_i < \alpha_i$, and the argument shows $C(\beta,\alpha) = 0$.

Suppose β differs by a tranposition from α, that is, $\beta = (ij)\alpha$, labeled so that $\alpha_i < \alpha_j$. Using (3.2) and $C(\beta,\alpha) = 0$ we have

$$k\left(\sum_{\beta_l<\beta_i} C((il)\beta,\alpha) - \sum_{\alpha_l<\alpha_i} C(\beta,(il)\alpha)\right) = 0.$$

But $(il)\beta = \alpha$ only when $l = j$, and this shows $C(\alpha,\alpha) = C(\beta,\beta)$. The transpositions generate all permutations of λ, hence C is scalar. □

4. The Inner Product and Jack Polynomials

There is a bi-orthogonal relation of $\{E_\lambda : \lambda \in \mathscr{N}_m^P\}$ to Jack polynomials in m variables. These are introduced (MACDONALD [11], STANLEY [14], BEERENDS and

Opdam [2], Lapointe and Vinet [9, 10]) as orthogonal basis for symmetric polynomials in $(z_1, z_2, \dots, z_m)$ with respect to the measure

$$\prod_{1\leqslant i<j\leqslant m} |(z_i - z_j)(z_i^{-1} - z_j^{-1})|^k du_m(z),$$

where du_m is supported by the m-torus,

$$\mathbf{T}^m := \{(z_1, \dots, z_m) : z_j = e^{i\theta_j}, -\pi < \theta_j \leqslant \pi, j = 1, \dots, m\},$$

and $du_m = d\theta_1 d\theta_2 \cdots d\theta_m$. We will go beyond the symmetric polynomials to all the polynomials in z. Also we will evaluate ω_λ at the point $x^{(L)} := (1, \dots, \overset{L}{1}, 0, \dots, \overset{N}{0})$ for $L \geqslant m$.

For polynomials in $z = (z_1, \dots, z_m)$ with coefficients in $\mathbf{Q}(k)$, and for $k \geqslant 0$ define the inner product

$$\langle f, g\rangle_k = \frac{c_k}{(2\pi)^m} \int_{\mathbf{T}^m} f(z) g^\vee(z) \left| \prod_{j<l} (z_j - z_l)(z_j^{-1} - z_l^{-1}) \right|^k du_m(z),$$

where $g^\vee(z) := g(z_1^{-1}, z_2^{-1}, \dots, z_m^{-1})$. Equivalently, for $k = 1, 2, 3, \dots$, $\langle f, g\rangle_k$ is the constant term in the Laurent polynomial $c_k f(z) g^\vee(z) \left(\prod_{j<l} (z_j - z_l)(z_j^{-1} - z_l^{-1}) \right)^k$, and $c_k = \Gamma(k+1)^m / \Gamma(km + 1)$ the normalizing constant chosen so that $\langle 1, 1\rangle_k = 1$ (a Selberg-type integral). Let $\mathscr{H}_n := \operatorname{span}\{z^\alpha : \alpha \in \mathscr{N}_{m,n}\}$.

For $\alpha, \beta \in \mathscr{N}_m$, let $H_{\alpha\beta} := \langle z^\alpha, z^\beta\rangle_k$, a real symmetric matrix, positive-definite for $k \geqslant 0$; since $H_{\alpha\beta} = 0$ whenever $|\alpha| \neq |\beta|$, the infinite matrix H is a direct sum of finite matrices $((\dim \mathscr{H}_n)$-square). As an illustration for $m = 2$,

$$\langle z_1^\alpha z_2^{n-\alpha}, z_1^\beta z_2^{n-\beta}\rangle_k = (-k)_l / (k+1)_l \quad \text{for} \quad l = |\alpha - \beta|.$$

Define an inner product on $\mathscr{V}_m$ by $\langle p_\alpha, p_\beta\rangle_k^\wedge := H_{\alpha\beta}^{-1}$, extended by linearity. When $k \geqslant 0$, H^{-1} is positive-definite. The caret in the notation is to suggest duality. Define a bilinear map $\mathscr{V}_{m,n} \times \mathscr{H}_n \to \mathbf{Q}(k)$ by $[p_\alpha, z^\beta] := \delta_{\alpha\beta}$ (Kronecker delta), extended by linearity. We find the adjoint of $T_i \rho_i$ with respect to this pairing. Let τ_i denote the Dunkl operator for the S_m-action on $(z_n, \dots, z_m)$, that is,

$$\tau_i := \frac{\partial}{\partial z_i} + k \sum_{j \neq i} \frac{1 - (ij)}{z_i - z_j}, \quad i = 1, \dots, m.$$

4.1 Proposition. *For $f \in \mathscr{V}_{m,n}$ and $g \in \mathscr{H}_n$,*

$$[T_i \rho_i f, g] = [f, (\tau_i z_i + (N - m + 1)k) g], \quad i = 1, \dots, m.$$

Proof. Denote the generating function for $\{p_\alpha\}$ by

$$F(x, z) = \sum_{n=0}^{\infty} \sum_{\alpha \in \mathscr{N}_{m,n}} p_\alpha z^\alpha = \prod_{j=1}^{m} \left((1 - x_j z_j)^{-1} \prod_{l=1}^{N} (1 - x_l z_j)^{-k} \right).$$

We first show

$$T_i F(x, z) - z_i \tau_i z_i F(x, z) = k(N - m + 1) z_i F(x, z), \quad \text{for} \quad i = 1, \dots, m.$$

The partial product

$$F_0(x,z)=\prod_{j=1}^{m}\prod_{l=1}^{N}(1-x_lz_j)^{-k}$$

is symmetric in both x and z, facilitating the following calculations:

$$\begin{aligned}T_iF(x,z)&=\left(\frac{z_i}{1-x_iz_i}+k\sum_{j=1}^{m}\frac{z_j}{1-x_iz_j}\right)F(x,z)\\&\quad+\left\{\sum_{j=1,j\neq i}^{m}\frac{1}{x_i-x_j}\left(\frac{1}{(1-z_ix_i)(1-z_jx_j)}\right.\right.\\&\quad\left.-\frac{1}{(1-z_ix_j)(1-z_jx_i)}\right)\prod_{s\neq i,j}(1-z_sx_s)^{-1}\\&\quad\left.+\sum_{j=m+1}^{N}\frac{1}{x_i-x_j}\left(\frac{1}{1-z_ix_i}-\frac{1}{1-z_ix_j}\right)\prod_{s=1,s\neq i}^{m}(1-z_sx_s)^{-1}\right\}F_0(x,z)\\&=F(x,z)\left(\frac{(k+1)z_i}{1-z_ix_i}+k\sum_{j=1,j\neq i}^{m}\left\{\frac{z_j}{1-z_jx_i}+\frac{z_i-z_j}{(1-z_jx_i)(1-z_ix_j)}\right\}\right.\\&\quad\left.+k\sum_{j=m+1}^{N}\frac{z_i}{1-z_ix_j}\right).\end{aligned}$$

Also

$$\begin{aligned}z_i\tau_iz_iF(x,z)&=z_i\left(1+\frac{x_iz_i}{1-z_ix_i}+k\sum_{j=1}^{N}\frac{x_jz_i}{1-z_ix_j}\right)F(x,z)\\&\quad+k\left\{\sum_{j=1,j\neq i}^{m}\frac{z_i}{z_i-z_j}\left(\frac{z_i}{(1-z_ix_i)(1-z_jx_j)}-\frac{z_j}{(1-z_jx_i)(1-z_ix_j)}\right)\times\right.\\&\quad\left.\times\prod_{s=1,s\neq i,j}^{m}(1-z_sx_s)^{-1}\right\}F_0(x,z)\\&=z_iF(x,z)\left[\frac{(k+1)z_ix_i}{1-z_ix_i}+k\sum_{j=1,j\neq i}^{N}\frac{z_ix_j}{1-z_ix_j}+1\right.\\&\quad\left.+k\sum_{j=1,j\neq i}^{m}\frac{1-(z_i+z_j)x_j+z_iz_jx_ix_j}{(1-z_jx_i)(1-z_ix_j)}\right].\end{aligned}$$

Thus $T_iF(x,z)-z_i\tau_iz_iF(x,z)=k(N-m+1)z_iF(x,z)$ (here $\dfrac{(k+1)z_i}{1-z_ix_i}-\dfrac{(k+1)z_i^2x_i}{1-z_ix_i}=(k+1)z_i$, the terms for $\sum_{j=m+1}^{N}$ contributed $(N-m)kz_1$ and the terms with denominator $(1-z_jx_i)(1-z_ix_j)$ cancel out). This exhibits the adjoint of T_i. Also $\rho_i^* z^\alpha = z^\alpha / z_i$ if $\alpha_i \geqslant 1$, 0 if $\alpha_i = 0$. Then $(T_i \rho_i)^* = \rho_i^* T_i^* = \tau_i z_i + k(N - m + 1)$. □

Lapointe and Vinet [9, 10] made intensive use of the operator $z_i \tau_i$, which provided some stimulation for the present section. Sahi [13] computed the expansion of the generating function $F(x, z)$ (when $m = N$) in terms of the orthogonal basis of non-symmetric Jack polynomials.

4.2 Proposition. *The operators $z_i \tau_i$, $T_i \rho_i$ are self-adjoint on $\mathscr{H}_n$, $\mathscr{V}_{m,n}$ respectively, for the inner products $\langle \cdot, \cdot \rangle_k$, $\langle \cdot, \cdot \rangle_k^\wedge$, $n \in \mathbf{Z}_+$, $i = 1, \ldots, m$.*

Proof. Assume that $k = 1, 2, 3, \ldots$ (the entries of H, namely $\langle z^\alpha, z^\beta \rangle_k$, are known to be in $\mathbf{Q}(k)$). Integration by parts on the torus shows that

$$\int_{\mathbf{T}} z \frac{\partial f(z)}{\partial z} g(z) du_1(z) = -\int_{\mathbf{T}} f(z) z \frac{\partial g(z)}{\partial z} du_1(z)$$

(Since $\frac{\partial}{\partial \theta} f(e^{i\theta}) = ie^{i\theta} \frac{\partial f}{\partial z}(e^{i\theta})$). Let

$$h(z) := \left(\sum_{l<j} (z_l - z_j)(z_l^{-1} - z_j^{-1}) \right)^k.$$

Then

$$\begin{aligned}
\int_{\mathbf{T}^m} z_j \tau_j f(z) g^\vee(z) h(z) du_m(z) &= -\int_{\mathbf{T}^m} f(z) \left(z_j \frac{\partial}{\partial z_j} g^\vee(z) \right) h(z) du_m(z) \\
&\quad - \int_{\mathbf{T}^m} f(z) g^\vee(z) k h(z) \sum_{l \neq j} \frac{z_j + z_l}{z_j - z_l} du_m(z) \\
&\quad + k \sum_{l \neq j} \int_{\mathbf{T}^m} z_j \frac{f(z) - (jl) f(z)}{z_j - z_l} g^\vee(z) h(z) du_m(z) \\
&= \int_{\mathbf{T}^m} f(z) \left(\frac{1}{z_j} \left(\frac{\partial}{\partial z_j} g \right)^\vee (z) \right) h(z) du_m(z) \\
&\quad + k \sum_{l \neq j} \int_{\mathbf{T}^m} f(z) \left(\frac{z_j^{-1} (g^\vee(z) - (jl) g^\vee(z))}{z_j^{-1} - z_l^{-1}} \right) h(z) du_m(z) \\
&= \int_{\mathbf{T}^m} f(z) (z_j \tau_j g)^\vee (z) h(z) du_m(z).
\end{aligned}$$

In the calculation, we transformed

$$\int_{\mathbf{T}^m} z_j \frac{(jl) f(z)}{z_j - z_l} g^\vee(z) h(z) du_m(z)$$

to

$$\int_{\mathbf{T}^m} z_l f(z) \frac{(jl) g^\vee(z)}{z_l - z_j} h(z) du_m(z);$$

valid since $k \geqslant 1$ and $h(z)$ is invariant under (jl). Now $\tau_i z_i - z_i \tau_i = I + k \sum_{l \neq i}(li)$, thus

$$(T_i \rho_i)^* = z_i \tau_i + (1 + k(N - m + 1))I + k \sum_{l \neq i}(il),$$

which is also self-adjoint for $\langle \cdot, \cdot \rangle_k$. Let M be the matrix representation of $(T_i \rho_i)^*$ in the z-basis $((T_i \rho_i)^* z^\alpha = \sum_\beta M_{\beta\alpha} z^\beta,\ \alpha, \beta \in \mathcal{N}_{m,n})$, then the transpose M^T is the matrix for $T_i \rho_i$ in the $\{p_\alpha\}$ basis, and $T_i \rho_i$ is self-adjoint for $\langle \cdot, \cdot \rangle_k^\wedge$ if and only if $MH^{-1} = H^{-1}M^T \Leftrightarrow HM^{-1} = (M^T)^{-1}H \Leftrightarrow M^T H = HM$, the condition that $(T_i \rho_i)^*$ be self-adjoint (recall $T_i \rho_i$ is invertible for generic k, in particular for $k > 0$). □

4.3 Theorem. *If $\lambda, \mu \in \mathcal{N}_m^P$ and $\lambda \neq \mu$, then $E_\lambda \perp E_\mu$ in the $\langle \cdot, \cdot \rangle_k^\wedge$ pairing.*

Proof. In this pairing, homogeneous polynomials of different degrees are always orthogonal; assume $|\lambda| = |\mu| = n \geqslant 2$. The eigenvalues of $T_i \rho_i$ on E_λ, E_μ are $(Nk - k\#\{l : \lambda_l > \lambda_j\} + \lambda_j + 1)$ and $(Nk - k\#\{l : \mu_l > \mu_j\} + \mu_j + 1), j = 1, \ldots, m$, respectively. There must be at least one of these values not equal to any of the other set, say that it is $c_0 \in \mathbf{Q}(k)$ and occurs on E_λ (changing lables, if necessary). The linear space $Y = E_\lambda \cap E_\mu^\perp$ is invariant under $T_i \rho_i$ for each i. The orthogonal projection of E_λ on Y (which certainly exists for $k > 0$) commutes with each $T_i \rho_i$, hence either $Y = \{0\}$ or $Y = E_\lambda$ (Proposition 3.2). Suppose $Y = \{0\}$ and let $f \in E_\lambda$ be an eigenvector of $T_1 \rho_1$ for the eigenvalue c_0; since $f \notin E_\mu^\perp$ there exists $g \in E_\mu$ with $\langle f, g \rangle_k^\wedge \neq 0$. Expand g as a sum of eigenvectors of $T_1 \rho_1$; all of the occurring eigenvalues differ from c_0 hence $\langle f, g \rangle_k^\wedge = 0$, a contradiction. Thus $E_\lambda \subset E_\mu^\perp$. □

By inverting the expansions of ω_λ in $\{p_\alpha\}$ we develop the link to Jack polynomials.

As in property $(\mathcal{S}i)$ in Section 2, define the connection matrix $A_{\alpha\beta}(\alpha, \beta \in \mathcal{N}_{m,n})$ by $\omega_\alpha = \sum_\beta A_{\beta\alpha} p_\beta$; Theorem 2.7 shows that $A_{\alpha\alpha=1}$ and $A_{\beta\alpha} = 0$ unless $\alpha = \beta$ or $\beta^s \succ \alpha^s$. Let B denote the inverse matrix, so that $p_\alpha = \sum_\beta B_{\beta\alpha} \omega_\beta$ (and again, $B_{\alpha\alpha} = 1$ and $B_{\beta\alpha} = 0$ unless $\alpha = \beta$ or $\beta^s \succ \alpha^s$). For $\alpha \in \mathcal{N}_{m,n}$ define an element of $\mathcal{H}_n, g_\alpha(z) := \sum_\beta B_{\alpha\beta} z^\beta$ (note the transpose!). Then $\{g_\alpha : \alpha \in \mathcal{N}_{m,n}\}$ is a basis for $\mathcal{H}_n$ and $\langle g_\alpha, g_\beta \rangle_k = 0$ if $\alpha^s \neq \beta^s$. Further $[\omega_\alpha, g_\beta] = \delta_{\alpha\beta}$. The proofs are trivial: let $\Omega_{\alpha\beta} := \langle \omega_\alpha, \omega_\beta \rangle_k^\wedge$, a matrix, then $\Omega = A^T H^{-1} A$; the Gram matrix

$$\Psi_{\alpha\beta} := \langle g_\alpha, g_\beta \rangle_k = \sum_{\gamma_1} \sum_{\gamma_2} B_{\alpha,\gamma_1} B_{\beta,\gamma_2} \langle z^{\gamma_1}, z^{\gamma_2} \rangle_k$$
$$= (BHB^T)_{\alpha\beta} = (\Omega^{-1})_{\alpha\beta}.$$

The block structure of Ω with respect to the spaces $\{E_\lambda : \lambda \in \mathcal{N}_{m,n}^P\}$ implies that Ω^{-1} has the same block structure; and this shows $\langle g_\alpha, g_\beta \rangle_k = 0$ when $\alpha^s \neq \beta^s$.

The pairing $[\omega_\alpha, g_\beta] = [\sum_{\gamma_1} A_{\gamma_1,\alpha} p_{\gamma_1}, \sum_{\gamma_2} B_{\beta,\gamma_2} z^{\gamma_2}] = \sum_{\gamma_1} A_{\gamma_1,\alpha} B_{\beta,\gamma_1} = (BA)_{\beta\alpha} = \delta_{\beta\alpha}$.

Recall the generating function from Proposition 4.1; the bi-orthogonality implies

$$\begin{aligned} F(x,z) &= \sum_{n=0}^{\infty} \sum_{\alpha \in \mathcal{N}_{m,n}} p_\alpha z^\alpha \\ &= \prod_{i=1}^{m} \left((1 - z_i x_i)^{-1} \prod_{j=1}^{N} (1 - z_i x_j)^{-k} \right) \\ &= \sum_{n=0}^{\infty} \sum_{\alpha \in \mathcal{N}_{m,n}} \omega_\alpha g_\alpha(z) \end{aligned} \tag{4.1}$$

(absolute convergence if $|x_j| < 1$ all j, and $|z_i| = 1$, all i).

Fix $\lambda \in \mathcal{N}^P_{m,n}$, let $\tilde{J}_\lambda(z) = \sum\{g_\alpha(z) : \alpha^s = \lambda\}$ (summing over distinct permutations of λ). Then $\tilde{J}_\lambda(z)$ is symmetric in $(z_i, z_2, \ldots, z_m)$, for λ, $\mu \in \mathcal{N}^P_{m,n}$ with $\lambda \neq \mu$, $\langle \tilde{J}_\lambda, \tilde{J}_\mu \rangle_k = 0$ and $\tilde{J}_\lambda(z) = m_\lambda(z) + \sum_{\nu \prec \lambda} B'_{\lambda\nu} m_\nu(z)$, where m_ν denotes an element of the monomial basis, that is, $m_\nu(z) = \Sigma\{z^\alpha : \alpha^s = \lambda\}$ and

$$B'_{\lambda\nu} = \Sigma\{B_{\alpha\nu} : \alpha^s = \lambda\}.$$

By the defining properties of Jack polynomials $\tilde{J}_\lambda(z)$ is a scalar $(\mathbf{Q}(k))$-multiple of $J_\lambda(z; 1/k)$ (Stanley [14], Beerends and Opdam [12]). Recall that the Jack functions form a basis for symmetric functions in infinitely many variables, while Jack polynomials are the specializations to m variables and partitions with m or fewer parts.

For the tableau corresponding to the partition λ (the set of lattice points (i,j), $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant \lambda_i$) there are two hook-length products, the upper

$$h^*(\lambda) := \prod_{(i,j) \in \lambda} (\lambda'_j - i + k^{-1}(\lambda_i - j + 1)),$$

and the lower

$$h_*(\lambda) := \prod_{(i,j) \in \lambda} (\lambda'_j - i + 1 + k^{-1}(\lambda_i - j))$$

(where λ' denotes the conjugate partition). Stanley showed that $\operatorname{cof}(J_\lambda(z; 1/k), m_\lambda) = h_*(\lambda)$ ([14] Theorem 5.6), and also that the "symmetric function" squared norm $j_\lambda := h_*(\lambda) h^*(\lambda)$. This allows us to evaluate w_α at the point $x^{(L)} := (\overbrace{1, \ldots, 1}^{L}, \overbrace{0, \ldots, 0}^{N})$ for $L \geqslant m$. By Beerends and Opdam (Cor. 3.6 in [12]),

$$\langle J_\lambda, J_\lambda \rangle_k = \frac{(km)_{\lambda;k}}{(k(m-1)+1)_{\lambda;k}} j_\lambda.$$

4.4 Proposition. *For* $\alpha \in \mathcal{N}_{m,n}, m \leqslant L \leqslant N$,

$$\omega_\alpha(x^{(L)}) = (Lk+1)_{\lambda;k} \left(\prod_{(i,j) \in \lambda} (\lambda_i - j + 1 + k(\lambda'_j - i)) \right)^{-1},$$

where $\lambda = \alpha^s \in \mathcal{N}^P_{m,n}$.

Proof. Substitute $x_i = 1$ for $i \leq L$, $x_i = 0$ for $i > L$ in the identity (4.1) obtaining

$$\sum_{n=0}^{\infty} \sum_{\alpha \in \mathcal{N}_{m,n}} \omega_\alpha(x^{(L)}) g_\alpha(z) = \prod_{i=1}^{m} (1 - z_i)^{-(Lk+1)}.$$

Since $\omega_\alpha(x^{(L)})$ depends only on α^s, the identity can be restated as

$$\sum_{n=0}^{\infty} \sum_{\lambda \in \mathcal{N}_{m,n}^P} \omega_\alpha(x^{(L)}) \tilde{J}_\alpha(z) = \prod_{i=1}^{m} (1 - z_i)^{-(Lk+1)}.$$

The ${}_1F_0$-hypergeometric series for Jack polynomials is

$$\prod_{i=1}^{m} (1 - z_i)^{-(Lk+1)} = \sum_{n=0}^{\infty} \sum_{\lambda \in \mathcal{N}_{m,n}^P} \frac{(Lk+1)_{\lambda;k}}{k^n} \frac{J_\lambda(z;1/k)}{j_\lambda}$$

(Z. YAN [16], also BEERENDS and OPDAM [2]). The coefficient of $(Lk+1)_{\lambda;k}$ is $\dfrac{h_*(\lambda)\tilde{J}_\lambda(z)}{k^n h_*(\lambda) h^*(\lambda)} = \dfrac{\tilde{J}_\lambda(z)}{k^n h^*(\lambda)}$, the required result.

It is not hard to show that the symmetrized $\tilde{\omega}_\lambda := \sum\{\omega_\alpha : \alpha^s = \lambda\}$ has the squared norm

$$\langle \tilde{\omega}_\lambda, \tilde{\omega}_\lambda \rangle_k^\wedge = (\#\{\alpha : \alpha \in \mathcal{N}_{mn}, \alpha^s = \lambda\})^2 \frac{(k(m-1)+1)_{\lambda;k} h_*(\lambda)}{(km)_{\lambda;k} h^*(\lambda)}.$$

We leave the computation of $\langle \omega_\alpha, \omega_\beta \rangle$ for $\alpha^s = \lambda = \beta^s$, $\lambda \in \mathcal{N}_{m,n}^P$ for subsequent work; this computation probably can be done using the self-adjointness of each $T_i\rho_i$ acting on E_λ. OPDAM [12] briefly mentioned bases for arbitrary (not just symmetric) polynomials associated to root systems. It should be possible to use the commuting set of self-adjoint operators $\{T_1\rho_1, T_2\rho_2 - k(12), \ldots, T_i\rho_i - k\sum_{j<i}(ij), \ldots,\}$ to produce a complete orthogonal decomposition of E_λ (the duals of these operators on $\mathcal{H}_n$ were introduced by LAPOINTE and VINET [10]).

The inner product on polynomials in $x_1, \ldots, x_N$ used by SAHI [13] is defined as follows: the generating function $F(x,z) = \prod_{i=1}^N ((1 - x_i z_i)^{-1} \prod_{j=1}^N (1 - x_i z_j)^{-k}) = \sum_\alpha z^\alpha p_\alpha(x) = \sum_{\alpha,\beta} C_{\alpha\beta} z^\alpha x^\beta$ defines the matrix C, for polynomials $f = \sum_\alpha a_\alpha x^\alpha$ and $g = \sum_\beta b_\beta x^\beta$ the inner product $\langle f, g \rangle_p = \sum_{\alpha,\beta} (C^{-1})_{\alpha\beta} a_\alpha b_\beta$. Restricted to each E_λ this inner product is proportional to the torus-type inner product $\langle f, g \rangle_k$; this follows from the irreducibility shown in Proposition 3.2. Another pairing for polynomials was used in Section 3; it is $[f, g] = f(T)g(x)|_{x=0}$. It was shown in [5] that this is related to the Hermite-type inner product by the formula

$$[f,g] = c_k \int_{\mathbf{R}^N} \exp(-\Delta_k/2) f(x) \exp(-\Delta_k/2) g(x) \prod_{i<j} |x_i - x_j|^{2k} \exp(-|x|^2/2) dx$$

where c_k is the normalizing constant and $\Delta_k = \sum_{i=1}^N T_i^2$. Now suppose $f \in E_\lambda$ and $g \in E_\mu$ with $\lambda, \mu \in \mathcal{N}_N^P$ and let $f = \sum_\alpha a_\alpha x^\alpha$ and $g = \sum_\beta b_\beta x^\beta$, then $f(T)g(x)|_{x=0} = \sum_\alpha a_\alpha T^\alpha g(x)|_{x=0}$; but $T^\alpha g = (Nk+1)_{\mu;k} \mathrm{cof}(g, p_\alpha)$ by the properties of E_μ. Now $g = \sum_{\alpha,\beta} b_\beta (C^{-1})_{\alpha\beta} p_\alpha$. Thus $[f, g] = (Nk+1)_{\mu;k} \sum_{\alpha,\beta} a_\alpha (C^{-1})_{\alpha,\beta} b_\beta = (Nk+1)_{\mu;k} \langle f, g\rangle_p$. The inner product is zero unless $\lambda = \mu$. This exhibits the proportionality constant for the two inner products on each E_λ. The mapping $\exp(-\Delta_k/2)$ transfers orthogonality relations and norms from E_λ to the Hermite-type inner product, studied by Baker and Forrester [1].

The inverse matrix B can be used in a formula for $Vx^\alpha (\alpha \in \mathcal{N}^P_{m,n})$. Indeed,

$$x^\alpha = \alpha! \xi p_\alpha = \alpha! \xi \Sigma_\beta B_{\beta\alpha} \omega_\beta$$

(and $B_{\alpha\alpha} = 1$, $B_{\beta\alpha} = 0$ unless $\beta = \alpha$ or $\beta^s \succ \alpha^s$). For example, when $m = 2$, $\lambda_1 \geqslant \lambda_2$,

$$\begin{aligned} p_{\lambda_1,\lambda_2} = \omega_{\lambda_1,\lambda_2} + \sum_{j=1}^{\lambda_2} \frac{(k)_j}{(k+\lambda_1-\lambda_2+j)_j j!} \big((\lambda_1 - \lambda_2 + j)_j \omega_{(\lambda_1+j,\lambda_2-j)} \\ + j(\lambda_1 - \lambda_2 + j + 1)_{j-1} \omega_{(\lambda_2-j,\lambda_1+j)} \big). \end{aligned}$$

(Proof left as exercise.)

5. Singular Polynomials

In Dunkl, De Jeu, Opdam [8] the singular polynomials were studied, for general finite reflection groups. For each value of k for which V^{-1} has nontrivial kernel (V does not exist), there is a space of homogeneous polynomials in $x_1, \ldots, x_N$ annihilated by each T_i, $i = 1, \ldots, N$. A conjecture was made regarding in which irreducible S_N-modules one could find these singular polynomials. The present work confirms part of the conjecture: for each $a = 1, 2, 3, \ldots$ such that $\gcd(N - m + 1, a) < (N - m + 1)/m$, there is a space of polynomials annihilated by each T_i for $k = -a/(N - m + 1)$ and on which S_N acts according to the representation $(N - m, m)$ $(m \leqslant N/2)$.

We will show that the S_N-orbit of $\omega_{(a,a,\ldots,a)}$ (rescaled to have $\mathbf{Q}[k]$-coefficients) provides this space. Property $\mathscr{S}iii'$ of Theorem 2.7 shows that

$$T_i \omega_{(a,a,\ldots,a)} = (Nk - (m-1)k + a) f_i$$

for some $f_i \in E_{(a,a,\ldots,a,a-1)}$ for $i = 1, \ldots, m$ and, of course, $T_i \omega_{(a,\ldots,a)} = 0$ for all $i > m$. This indicates the desired property for $k = -a/(N - m + 1)$, but how do we know that $\omega_{(a,\ldots,a)}$ is defined for this special value? Let $\lambda = (a, \ldots, a) \in \mathcal{N}^P_m$; the hook-length product used in Proposition 4.3 has the value

$$\begin{aligned} k^{ma} h^*(\lambda) &= a!(k+1)_a (2k+1)_a \cdots ((m-1)k+1)_a \\ &= ((m-1)k+1)_{\lambda;k}. \end{aligned}$$

Now Proposition 4.4 with $L = m$ shows that

$$\omega_\lambda(x^{(m)}) = \frac{(mk+1)_{\lambda;k}}{((m-1)k+1)_{\lambda;k}} = \frac{(mk+1)_a}{a!}.$$

If $f_a := ((m-1)k+1)_{\lambda;k} \omega_\lambda$ has coefficients in $\mathbf{Q}[k]$ (polynomials in k), then the polynomial identity $f_a(x^{(m)}) = (mk+1)_{\lambda;k}$ shows explicitly that $f_a \neq 0$ when

$k=-a/(N-m+1)$ and $\gcd(a,N-m+1)<(N-m+1)/m$. We show that this is indeed the case.

5.1 Lemma. *Let* $0\leqslant b\leqslant a$ *and* $\mu=(a,\dots,b)\in\mathcal{N}_m^P$, *then* $k^{(m-1)a+b}h^*(\mu)\omega_\mu$ *is in the* $\mathbf{Q}[k]$*-span of* $\{p_\alpha\}$.

Proof. Let $\nu=(a,\dots,a,b+1)\in\mathcal{N}_m^P$ for $0\leqslant b\leqslant a-1$. Then

$$k^{(m-1)a+b}h^*(\mu)=b!(a-b)!\left(\prod_{i=1}^{m-2}(1+ik)_a\right)(a-b+1+(m-1)k)_b,$$

and

$$\frac{k^{(m-1)a+b+1}h^*(\nu)}{k^{(m-1)a+b}h^*(\mu)}=\frac{(b+1)}{(a-b)}(a-b+(m-1)k).$$

Assume the statement of the lemma is true for μ. By the construction used in Theorem 2.7, $\omega_\nu=\rho_m f_m+f_0$ where $f_m\in E_\mu$ and $T_m\rho_m f_m=(Nk-(m-1)k+$ $+b+1)f_m$. By the formula in Theorem 2.4,

$$f_m=\omega_\mu-\frac{k}{(m-1)k+a-b}\sum_{j<m}(jm)\omega_\mu.$$

By the inductive hypothesis $k^{(m-1)a+b+1}h^*(\nu)f_m$ has all of its coefficients ($\{p_\alpha\}$) in $\mathbf{Q}[k]$.

The method in Theorem 2.7 constructs f_0 from f_m with no divisions (other than rational numbers).

Repetition of this procedure reduces the problem to $k^{(m-1)a}h^*\times$ $\times((a,a,\dots,a,0))\omega_{(a,\dots,a,0)}$; now reduce m by 1, and so forth. The induction begins at $\omega_a=p_a(x_1,x)$ (just one part).

The singular polynomials not associated to two-part partitions of N appear to be considerably more complicated. For example, for $N=5$, the conjecture calls for the representation (3,1,1) to give rise to singular polynomials for $k=-\frac{1}{2},-\frac{3}{2},\dots$

We conjecture that $k^{|\lambda|}h^*(\lambda)\omega_\lambda$ has coefficients in $\mathbf{Q}[k]$ for arbitrary $\lambda\in\mathcal{N}_m^P$. We also expect that more detailed information can be found about the denominators in $\omega_\lambda=\Sigma_\beta A_{\beta\lambda}p_\beta$ in terms of β.

This has been a rather algebraic approach to the intertwining operator. The problem of constructing an integral transform which implements the operator for $k>0$ remains open beyond $N=3$. It would also be interesting to find an analytic definition of the inner product $\langle\cdot,\cdot\rangle_k^\wedge$, introduced for $\mathcal{V}_m$.

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