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An Abelian Quotient of the Mapping Class Group $\mathscr{I}_g$

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1. Introduction and Preliminaries

Let $M = M_{g,1}$ be a compact oriented surface with one boundary component and let $\mathscr{M} = \mathscr{M}_{g,1}$ be its mapping class group (that is, orientation preserving homeomorphisms of M which are 1 on the boundary mod homeomorphisms which are isotopic to 1 by an isotopy which is pointwise fixed on the boundary), and let $\mathscr{I} = \mathscr{I}_{g,1}$ be the subgroup of maps in $\mathscr{M}$ which induce the identity map on the homology group $H_1(M, Z)$. The group $\mathscr{I}$ is of interest both for its possible topological applications and for the group-theoretic questions which it supports. The former has been championed mainly in the work of Birman. In the latter category we have, for example, the open question: is $\mathscr{I}$ finitely generated? Another problem of interest (which might have bearing on the first and which also has specific topological applications) is the computation of the abelianization $\mathscr{I}/\mathscr{I}'$. Abelian quotients of $\mathscr{I}$ were produced originally by Birman and Craggs in [BC] using the Rochlin invariant of homology 3-spheres. These quotients are finite in number and defined by means of homomorphisms: $\mathscr{I} \to Z_2$. In this paper we construct a different abelian quotient of $\mathscr{I}$, this time free of rank $\binom{2g}{3}$, by examining its action on a certain nilpotent quotient of $\pi_1(M)$. The new abelian representation is then applied to the two problems:

a) (Birman): Does the subgroup $\mathscr{T}$ of $\mathscr{I}$ which is generated by twists on bounding simple closed curves have finite index in $\mathscr{I}$?

b) (Chillingworth): If $f \in \mathscr{I}$ and f "preserves winding numbers" of all curves on M, is $f \in \mathscr{T}$? (See [C 1] for relevant definitions.)

The results are then extended to the case of a closed surface, and in the final section we look at the connection between the Birman-Craggs quotients and the quotient defined here.

Throughout the paper, all surfaces are compact, orientable and *oriented*. M_g will denote a closed surface of genus g and $M_{g,1}$ a surface with one boundary

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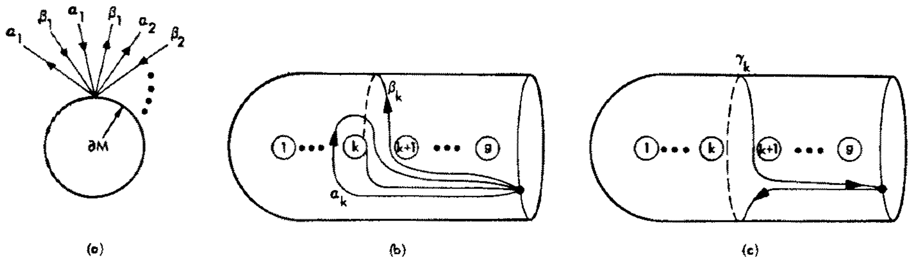

Fig. 1a–c

component; the latter will be commonly referred to as "an open surface". We also use the following notations and abbreviations:

SCC means *simple closed curve*; BSCC means *bounding* SCC, that is, an SCC which bounds in M. If the surface is closed, a BSCC γ separates M into two surfaces each of which it is the boundary. For an open surface, γ is the boundary of only one of these subsurfaces: the other contains ∂M. In this case then, we define the *genus of* γ to be the genus of the unique subsurface it bounds.

All homology groups use Z coefficients. We abbreviate $H_1(M,Z)$ and $\pi_1(M)$ to H and π respectively. For an open surface the base point of π will always be chosen on ∂M. The product $\alpha\beta$ in π indicates that we traverse α first, then β, and $[\alpha,\beta]$ means $\alpha\beta\alpha^{-1}\beta^{-1}$. We will commonly confuse curves with their classes in π, using the same notation for both.

H has the standard bilinear intersection form, which is *symplectic* (that is, $x\cdot x=0$ and the form establishes a self-duality on H, i.e., an isomorphism $H\to H^*$). A basis of H, which is free abelian of rank $2g$, is *symplectic* if it is of the form a_i, b_i ($i=1,\ldots,g$) and $a_i\cdot a_j=b_i\cdot b_j=0$, $a_i\cdot b_j=\delta_{ij}$ (Kronecker delta). An automorphism of H is *symplectic* if it preserves the form; these form a group which we denote simply by Sp.

For an open surface M, π is free and has a set of free generators represented by SCC's α_i, β_i ($i=1,\ldots,g$), where the α_i and β_i's are disjoint except at the basepoint and are arranged there as in Fig. 1a. (Such a set of curves is known as a "canonical basis", and cutting the surface along these curves reduces it to a disc.) Figure 1b shows the curves α_k, β_k on the form of the surface we will use in this paper. The situation is essentially the same for a closed surface, except that π is no longer free: we have the well known relation $\prod_{i=1}^{g}[\alpha_i,\beta_i]=1$. If we compute the partial product $\prod_{i=1}^{k}[\alpha_i,\beta_i]$ in the *open* surface, we find it to be represented by the curve γ_k of Fig. 1c. In particular, $\prod_{i=1}^{g}[\alpha_i,\beta_i]$ is represented by ∂M itself, with orientation *opposite* to that acquired in the natural way from M (the natural orientation of ∂M puts the interior of M on its left).

$\mathcal{M}_g$ (resp. $\mathcal{M}_{g,1}$) is the mapping class group of M_g (resp. $M_{g,1}$). The homology functor gives homomorphisms from these groups to Sp, and it is well known that these homomorphisms are surjective; the kernels are $\mathcal{I}_g$ (resp. $\mathcal{I}_{g,1}$). We shall also systematically confuse a homeomorphism with its isotopy class in $\mathscr{M}$. For the most part we will work with an open surface, deriving the corresponding results for a closed surface from those for an open surface. The subscripts g, 1, etc., will be suppressed when clear.

2. The Nilpotent Group $\pi/[\pi, \pi']$

Let $M=M_{g,1}$; it will be implicitly assumed that all homeomorphisms of M are the identity on ∂M, and that a fixed basepoint bp is given on ∂M. Then any homeomorphism f of M gives a well defined automorphism f^* of π. Furthermore, any isotopy which is trivial on ∂M induces the identity map on π, and hence f^* depends only on its mapping class. This is consistent with our stated policy of confusing f with its mapping class, and we obtain a homomorphism $*:\mathscr{M}\to\mathrm{Aut}(\pi)$. In the case of a closed surface we do not have such a homomorphism: the homeomorphisms and isotopies of M_g do not fix a natural basepoint. Another simplifying feature of the open surface case is, as we have seen, that π is free. These facts illustrate the general principle we have found that, in most situations, $\mathscr{M}_{g,1}$ is easier to work with than $\mathscr{M}_g$; the desired results for $\mathscr{M}_g$ can usually be derived from those for $\mathscr{M}_{g,1}$.

We are going to exploit the action of $\mathscr{M}$ on π to get an abelian quotient of $\mathscr{I}$. Our main tool will be its action on a certain quotient group of π.

Definition. a) $E=\pi/[\pi,\pi']$.
b) $N=\pi'/[\pi,\pi']$.

Note that E is just π with centralized commutator subgroup N. We have thus a central extension

$$0\to N\to E\xrightarrow{p} H\to 0.$$

Although the group operation in E (and N) will be written multiplicatively, we maintain the additive notation in H, so that $p(e_1e_2)=p(e_1)+p(e_2)$.

Recall that Λ^2H is the quotient of $H\otimes H$ by the subgroup generated by $x\otimes x$; more generally, Λ^kH is the quotient of the k^{th} tensor power H^k of H by the subgroup generated by $x_1\otimes x_2\otimes\ldots\otimes x_k$ in which two of the factors x_i are equal. The image of $y_1\otimes\ldots\otimes y_k$ in Λ^kH is denoted by $y_1\wedge y_2\wedge\ldots\wedge y_k$. An alternate definition which we will find more akin to our purposes is the following: let S_k be the symmetric group, acting on H^k by permuting the order of the factors, and define the map $\lambda: H^k\to H^k$ by $\lambda(x)=\sum_{\pi\in S_k}\mathrm{sign}(\pi)\cdot\pi(x)$ for all $x\in H^k$. It is well known that $\mathrm{Ker}\,\lambda$ is precisely the subgroup generated by all $x_1\otimes\ldots\otimes x_k$ having two equal factors, and we may then *define* Λ^kH to be $\mathrm{Im}\,\lambda$, writing

$$\lambda(x_1\otimes\ldots\otimes x_k)=x_1\wedge\ldots\wedge x_k=\sum_{\pi\in S_k}\mathrm{sign}(\pi)\cdot x_{\pi(1)}\otimes\ldots\otimes x_{\pi(k)}.$$

The fact that H is free abelian of rank $2g$ implies that Λ^kH is free abelian of rank $\binom{2g}{k}$.

Suppose now that x, $y \in H$. Lift them to $\tilde{x}$, $\tilde{y}$ in E and form $[\tilde{x}, \tilde{y}]$. Using the centrality of N, standard arguments (see, e.g., [M], as on p. 63) show that $[\tilde{x}, \tilde{y}] \in N$ and does not depend on the liftings; we denote this element by $\{x, y\}$. Also following Milnor's exposition we have:

Lemma 1A. *The map* $\{-,-\}: H \times H \to N$ *is bilinear and antisymmetric, that is,* $\{x, y+z\} = \{x, y\} \cdot \{x, z\}$ *and* $\{y, x\} = \{x, y\}^{-1}$, *and hence defines a surjective homomorphism* $j: \Lambda^2 H \to N$ *given by* $j(x \wedge y) = \{x, y\}$.

Proof. If $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ are liftings of x, y, z, then $\tilde{y}\tilde{z}$ lifts $y+z$, and hence $\{x, y+z\} = [\tilde{x}, \tilde{y}\tilde{z}]$. By a standard commutator identity the latter is

$$[\tilde{x}, \tilde{y}] \cdot \tilde{y}[\tilde{x}, \tilde{z}]\tilde{y}^{-1} = [\tilde{x}, \tilde{y}] \cdot [\tilde{x}, \tilde{z}] = \{x, y\} \cdot \{x, z\},$$

since N is central. That $\{y, x\} = \{x, y\}^{-1}$ is obvious. Thus we do indeed get a well defined homomorphism from $\Lambda^2 H$ to N; its surjectivity follows from the fact that N is generated by all $[\tilde{x}, \tilde{y}]$ for $\tilde{x}$, $\tilde{y} \in E$.

Lemma 1B. *j is an isomorphism.*

Proof. N is free abelian of rank $\binom{2g}{2}$ (see [MKS], Theorems 5.11 and 5.12). But $\Lambda^2 H$ is also free abelian of rank $\binom{2g}{2}$, and a surjective homomorphism between f.g. free abelian groups of the same rank must be an isomorphism, QED. We shall henceforth identify N and $\Lambda^2 H$ via this isomorphism. It is natural in the following sense. Any automorphism f of π induces an automorphism of E taking N to N, also denoted by f. The action of f on H also induces a natural automorphism f on $\Lambda^2 H$ (sometimes written $\Lambda^2 f$) by $f(x \wedge y) = f(x) \wedge f(y)$. The naturality of $j: \Lambda^2 H \to N$ is then expressed by:

Lemma 1C. $fj(x \wedge y) = jf(x \wedge y)$ *for* x, $y \in H$

Proof. Let $\tilde{x}$, $\tilde{y}$ lift x, y to E; then $fj(x \wedge y) = f\{x, y\} = f[\tilde{x}, \tilde{y}] = [f\tilde{x}, f\tilde{y}]$. Note that $f(\tilde{x})$, $f(\tilde{y})$ are liftings of $f(x)$, $f(y)$, so we get $fj(x \wedge y) = \{f(x), f(y)\} = j(f(x) \wedge f(y))$ $= jf(x \wedge y)$. QED.

Corollary. $\mathcal{I}$ acts trivially on N, and thus N is an $\mathcal{M}/\mathcal{I} = \mathrm{Sp}$-module; j is an Sp-module isomorphism of $\Lambda^2 H$ with N.

Proof. The first statement follows from the corresponding one for $\Lambda^2 H$; the second is then obvious.

3. The Action of $\mathcal{I}$ on E

Let $f \in \mathcal{I}$; to avoid messy notation, we shall abbreviate the induced maps f^* on π, H, and E to simply f. Let now $e \in E$ and put $p(e) = x \in H$. Since $f(x) = x$, we get $f(e) = ke$, where $k \in N$. If $p(e') = x$ also, we have $e' = en$ for some $n \in N$, and $f(e') = f(e) \cdot f(n) = f(e) \cdot n$ since (by Lemmas 1) f acts trivially on N. Hence $f(e') \cdot e'^{-1} = f(e)n \cdot (en)^{-1} = f(e) \cdot e^{-1} = k$, and we have:

Lemma 2A. *For* $f \in \mathcal{I}$, $x \in H$ *and e a lifting of x, the element* $f(e) \cdot e^{-1}$ *is in N and independent of the lifting.*

We denote this element of N by $\delta f(x)$.

Lemma 2B. *$\delta f: H \to N$ is a homomorphism.*

Proof. Let $x_1, x_2 \in H$ and e_1, e_2 be the liftings; then

$$\delta f(x_1+x_2) = f(e_1 e_2)\cdot(e_1 e_2)^{-1} = f(e_1)\cdot(f(e_2)\cdot e_2^{-1})\cdot e_1^{-1}$$
$$= f(e_1)e_1^{-1}\cdot f(e_2)e_2^{-1} = \delta f(x_1)\cdot\delta f(x_2), \quad \text{QED.}$$

Making now our standard identification of N with $\Lambda^2 H$, we have then $\delta f \in \mathrm{Hom}(H, \Lambda^2, H)$, and the latter is an additive abelian group.

Lemma 2C. *The function $\delta: \mathscr{I} \to \mathrm{Hom}(H, \Lambda^2 H)$ is a homomorphism.*

Proof. We must show that $\delta(fg) = \delta f + \delta g$, that is $\delta(fg)(x) = \delta f(x) + \delta g(x)$ for $x \in H$. Let e lift x; we get $\delta(fg)x = fg(e)\cdot e^{-1} = f[g(e)\cdot e^{-1}]\ f(e)\cdot e^{-1} = f(\delta g(x))\cdot\delta f(x)$. But $\delta g(x) \in N$ and f acts trivially on N, so the latter is just $\delta g(x)\cdot\delta f(x)$ in N, that is, $\delta f(x) + \delta g(x)$ in $\Lambda^2 H$, QED.

Note that $\mathrm{Ker}\,\delta$ is precisely all maps $f \in \mathscr{I}$ such that $f(e) = e$, all $e \in E$, i.e., the set of maps acting trivially on E.

The group $\mathrm{Hom}(H, \Lambda^2 H)$ has a natural Sp-module structure, namely, for $h \in \mathrm{Sp}$ and $\alpha: H \to \Lambda^2 H$, we put

$$h(\alpha) = (H \xrightarrow{h^{-1}} H \xrightarrow{\alpha} \Lambda^2 H \xrightarrow{h} \Lambda^2 H) = h\alpha h^{-1}.$$

This action also then defines an action of $\mathscr{M}$ on $\mathrm{Hom}(H, \Lambda^2 H)$ via the map $\mathscr{M} \to \mathrm{Sp}$.

Lemma 2D. *If $h \in \mathscr{M}$, $f \in \mathscr{I}$, then $\delta(hfh^{-1}) = h(\delta f)$.*

Proof.

$$\delta(hfh^{-1})(x) = hfh^{-1}(e)\cdot e^{-1} = hfh^{-1}(e)\cdot hh^{-1}(e^{-1})$$
$$= h[f(h^{-1}e)\cdot(h^{-1}(e))^{-1}] = h[\delta f(h^{-1}(x))] = [h(\delta f)](x), \quad \text{QED.}$$

In summary, Lemmas 2 tells us that $\delta: \mathscr{I} \to \mathrm{Hom}(H, \Lambda^2 H)$ is a homomorphism of groups commuting with the action of $\mathscr{M}$ on these groups, where $\mathscr{M}$ acts on $\mathscr{I}$ by conjugation. Note that $\mathrm{Hom}(H, \Lambda^2 H)$ is free abelian of rank $2g\cdot\binom{2g}{2}$, so $\mathrm{Im}\,\delta$ is free abelian of rank no greater than this. Our first main result will be to determine this image precisely.

We first shift our viewpoint slightly through the use of the self-duality on H. We have $\mathrm{Hom}(H, \Lambda^2 H)$ naturally isomorphic to $\Lambda^2 H \otimes H^*$, and the latter is isomorphic via the symplectic duality to $\Lambda^2 H \otimes H$. These isomorphisms commute with the natural actions of Sp on the groups in question[1] and are thus Sp-module isomorphisms. The isomorphism from $\Lambda^2 H \otimes H$ to $\mathrm{Hom}(H, \Lambda^2 H)$ is given specifically by defining the value of $\theta \otimes x$ on y to be $(y\cdot x)\theta$ for $\theta \in \Lambda^2 H$, x, $y \in H$. To

1 The natural action of Sp on $\Lambda^2 H \otimes H$ is $g(x \wedge y \otimes z) = gx \wedge gy \otimes gz$

describe the inverse isomorphism, choose a symplectic basis a_i, b_i of H; then for $\delta \in \operatorname{Hom}(H, \Lambda^2 H)$ the corresponding element of $\Lambda^2 H \otimes H$ is:

$$\sum_{i=1}^{g} [\delta(a_i) \otimes b_i - \delta(b_i) \otimes a_i].$$

For $f \in \mathscr{I}$ we denote the image of δf in $\Lambda^2 H \otimes H$ via this isomorphism by τ_f, and have the following form of Lemma 2:

Lemma 3. *$\tau: \mathscr{I} \to \Lambda^2 H \otimes H$ is a homomorphism commuting with the action of $\mathscr{M}$ on these groups. Its image is thus an* Sp-*submodule of $\Lambda^2 H \otimes H$.*

We will find this form of the homomorphism δ more convenient for our purposes.

4. The Image of τ

Classical matter on the representations of Gl tells us that $\Lambda^2 H \otimes H$ is not irreducible over Gl but (at least over a field) splits into two Gl-submodules, and this suggests that we examine these submodules as possibilities for $\operatorname{Im} \tau$. One of these submodules is $\Lambda^3 H$; in fact, using the alternate definition of $\Lambda^k H$ found in Sect. 2, we have:

a) $\Lambda^2 H =$ the subgroup of H^2 generated by $x \wedge y = x \otimes y - y \otimes x$. Hence $\Lambda^2 H \otimes H =$ the subgroup of H^3 generated by $(x \wedge y) \otimes z = x \otimes y \otimes z - y \otimes x \otimes z$.

b) $\Lambda^3 H =$ the subgroup of H^3 generated by

$$\begin{aligned} x \wedge y \wedge z &= x \otimes y \otimes z - y \otimes x \otimes z + y \otimes z \otimes x - z \otimes y \otimes x \\ &\quad + z \otimes x \otimes y - x \otimes z \otimes y = (x \wedge y) \otimes z + (y \wedge z) \otimes x + (z \wedge x) \otimes y. \end{aligned}$$

Thus $\Lambda^3 H \subset \Lambda^2 H \otimes H$. We shall show that $\operatorname{Im} \tau$ is precisely $\Lambda^3 H$ by examining τ on generators of $\mathscr{I}$.

Powell (see [P]) has shown that $\mathscr{I}$ is generated by two types of maps. First, let γ be a bounding SCC (BSCC); then the twist $T_\gamma \in \mathscr{I}$. Second, let (γ, δ) be a *bounding pair* (BP), that is, a pair of disjoint homologous SCC's which are *not* homologically trivial; then $T_\gamma T_\delta^{-1} \in \mathscr{I}$. Powell showed that (for a closed surface) these maps, taken over all BSCC's which bound a subsurface of genus 1 or 2 in M, and all BP's which bound a surface of genus 1 in M, generate $\mathscr{I}$.[2] The author has shown in [J 1] that the latter maps suffice, for an open surface as well as closed. The next two lemmas calculate τ_f when f is a Powell generator.

Lemma 4A. *Let γ be a* BSCC *and put $f = T_\gamma$; then $\tau_f = 0$.*

Proof. Let γ be as pictured in Fig. 2, bounding S of genus k. We choose an arc ε in $M - S$ connecting bp to γ and let U be a thin band neighborhood of ε in $M - S$, also as shown in the figure. A canonical basis α_i, β_i may then be chosen for π, as shown in Fig. 1b, so that α_i, $\beta_i \subset M - S$ for $i > k$, and for $i \leqq k$, (α_i or β_i)$\cap M - S \subset U$. Let

2 For an open surface $M_{g,1}$, the genus of a BP is defined as was done for a BSCC, namely, the genus of the unique subsurface of M which it bounds

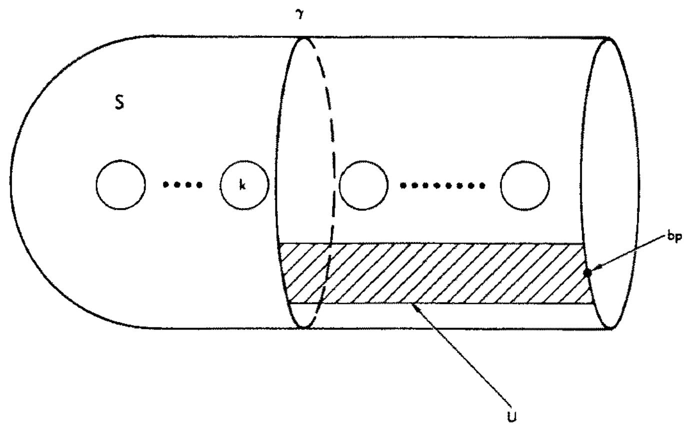

Fig. 2

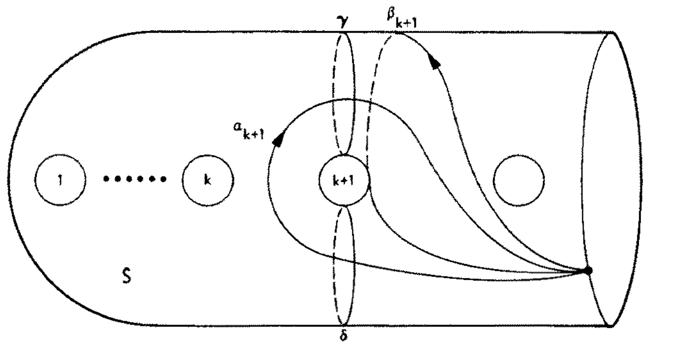

Fig. 3

now c be that element of π given by traveling out along ε, turning right at γ and running around it once, and returning to bp along ε. We find that the induced map of f on π leaves α_i, β_i fixed for $i>k$, and for $i \leqq k$ and $x=\alpha_i$ or β_i we have $f(x) = cxc^{-1} = [c,x]\cdot x$. But $c \in \pi'$ since it is homologically trivial, so $[c,x] \in [\pi,\pi']$. This shows that $f(x) \equiv x \operatorname{mod}[\pi,\pi']$ for *all* basis elements x, that is, $\delta f(y)=0$ for the corresponding basis elements y of H, i.e., $\delta f=0$; τ_f is then also zero.

We turn now to the case of a BP (γ,δ) bounding a surface S of genus k, as pictured in Fig. 3.

Again we choose a canonical basis of π, suitably adapted to the splitting of M into S and $M-S$ (see Figs. 3 and also 1b). To begin with, α_i, $\beta_i \subset M-S$ for $i>k+1$, and $\beta_{k+1} \subset M-S$ also. α_{k+1} travels along an arc ε to δ (that is, ε is the initial arc of $\alpha_{k+1} \cap M-S$), crosses it into S, and exits S at a crossing of γ, as shown in Fig. 3. Let $\eta = \alpha_{k+1} \cap S$ and define d to be the path traveling out from ε to δ, turning *left*, once around δ and then back along ε; also define c by running out along ε and then η to γ, left once around γ, then back along η and ε to the bp. These curves c and d are clearly homologous, and in fact homologous to β_{k+1}. Let now $f=T_\gamma T_\delta^{-1}$. Its

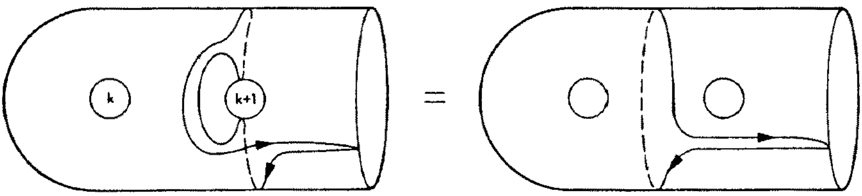

Fig. 4

action on π leaves α_i, β_i fixed for $i>k+1$ as well as β_{k+1}, and for $x=\alpha_i$ or β_i with $i\leqq k$ we get $f(x)=dxd^{-1}=[d,x]\cdot x$. Finally, $f(\alpha_{k+1})=dc^{-1}\alpha_{k+1}$. Now dc^{-1} is represented by the curve of Fig. 4. Referring to Fig. 1c, Sect. 1, we see that this is precisely the curve γ_k, and hence by the results given there, we have $dc^{-1} = \prod_{i=1}^{k} [\alpha_i, \beta_i]$. If a_i, b_i are the homology classes of α_i, β_i respectively (so that the class of d is b_{k+1}), then in $N=\Lambda^2 H$ we have dc^{-1} is just $\sum_{i=1}^{k} a_i \wedge b_i$. Summarizing then, we get:

$$\left.\begin{aligned}\delta f(a_i)&=b_{k+1}\wedge a_i\\ \delta f(b_i)&=b_{k+1}\wedge b_i\end{aligned}\right\} i\leqq k$$

$$\delta f(a_{k+1})=\sum_{i=1}^{k} a_i\wedge b_i$$

$\delta f=0$ on remaining basis elements.

Finally, we have:

Lemma 4B. *If* (γ,δ) *is a* BP *bounding a surface* S *of genus* k *and* a_i, b_i *are chosen as above, then,*

$$\tau(T_\gamma T_\delta^{-1})=\left(\sum_{i=1}^{k} a_i\wedge b_i\right)\wedge b_{k+1}.$$

Proof. τ_f is given by

$$\sum_{i=1}^{g} (\delta f(a_i)\otimes b_i-\delta f(b_i)\otimes a_i)$$

$$=\sum_{i=1}^{k} [(b_{k+1}\wedge a_i)\otimes b_i-(b_{k+1}\wedge b_i)\otimes a_i]+\left(\sum_{i=1}^{k} a_i\wedge b_i\right)\otimes b_{k+1}$$

$$=\sum_{i=1}^{k} [(b_i\wedge b_{k+1})\otimes a_i+(b_{k+1}\wedge a_i)\otimes b_i+(a_i\wedge b_i)\otimes b_{k+1}].$$

But as we saw at the beginning of this section, this is just

$$\sum_{i=1}^{k} a_i\wedge b_i\wedge b_{k+1},\quad \text{QED.}$$

There is an alternate way of describing the element $\tau(T_\gamma T_\delta^{-1})$ which is sometimes useful. The inclusion $S \subset M$ induces a map on homology $H_1(S) \to H_1(M)$ which is actually injective (an easy proof may be given geometrically or by using the Mayer-Vietoris sequence). Choose a *maximal symplectic subspace* U of $H_1(S)$, that is, a subspace with basis a_i, b_i $(i=1, \ldots, k)$ satisfying the symplectic laws and which is maximal in this respect. Then $H_1(S)$ has the basis a_i, b_i, c, where c is the homology class of γ given the natural orientation induced by that of S (this orientation of γ puts S on its left as we move around γ). The above lemma may be (taking care of the signs!) translated into:

Corollary. $\tau(T_\gamma T_\delta^{-1}) = \left(\sum_{i=1}^{k} a_i \wedge b_i\right) \wedge c$, *the latter expression being independent of the choices involved.*

We are now ready to prove:

Theorem 1. *For* $g \geqq 2$, $\operatorname{Im} \tau = \Lambda^3 H$. *Thus* τ *gives an abelian representation of* $\mathscr{I}$ *onto a free abelian group of* rank $\binom{2g}{3}$.

Proof. We have already shown that τ takes Powell's generators into $\Lambda^3 H$, so we need only show that τ is surjective. If f is a BP map of genus 1 and a_i, b_i is a symplectic basis of H chosen as in Lemma 4B and its corollary, then we have $a_1 \wedge b_1 \wedge b_2$ is in $\operatorname{Im} \tau$. Using this element and the fact that $\operatorname{Im} \tau$ is an Sp-submodule of $\Lambda^3 H$ will give us the theorem.

First let $g=2$; we apply the Sp map $a_2 \to b_2 \to -a_2$ (basis elements not mentioned are assumed fixed) to $a_1 \wedge b_1 \wedge b_2$ and get $-a_1 \wedge b_1 \wedge a_2 \in \operatorname{Im} \tau$. The Sp map $a_1 \leftrightarrow a_2$, $b_1 \leftrightarrow b_2$ applied to these two elements gives us $a_2 \wedge b_2 \wedge b_1$ and $-a_2 \wedge b_2 \wedge a_1$, and we now have a basis of $\Lambda^3 H$ in $\operatorname{Im} \tau$.

If $g \geqq 3$, we apply instead the map $a_1 \to a_1 + b_1 - b_3$, $a_3 \to a_3 - b_1 + b_3$ to $a_1 \wedge b_1 \wedge b_2$ and get $a_1 \wedge b_1 \wedge b_2 - b_1 \wedge b_2 \wedge b_3 \in \operatorname{Im} \tau$, showing that $b_1 \wedge b_2 \wedge b_3 \in \operatorname{Im} \tau$. If now we apply to the two elements $a_1 \wedge b_1 \wedge b_2$ and $b_1 \wedge b_2 \wedge b_3$ the Sp maps of the type

a) $a_1 \leftrightarrow a_i$, $b_1 \leftrightarrow b_i$

and

b) $a_i \to b_i \to -a_i$

we easily see that all of the following are in $\operatorname{Im} \tau$:

$$\left.\begin{array}{l} 1)\ a_i \wedge b_i \wedge b_j \\ 2)\ a_i \wedge b_i \wedge a_j \end{array}\right\} i \neq j$$

$$\left.\begin{array}{l} 3)\ a_i \wedge a_j \wedge a_k \\ 4)\ a_i \wedge a_j \wedge b_k \\ 5)\ a_i \wedge b_j \wedge b_k \\ 6)\ b_i \wedge b_j \wedge b_k \end{array}\right\} i, j, k \text{ distinct.}$$

These form a basis for $\Lambda^3 H$, QED.

Because of a number of analogies with differential geometry and mechanics, we shall call the element $\tau_f \in \Lambda^3 H$ the *torsion of* f.

5. Some Applications of τ

Let $\mathcal{T}$ be the subgroup of $\mathcal{I}$ generated by all BSCC twists. Birman has conjectured that $\mathcal{T}$ is finite index in $\mathcal{I}$. For the surface $M_{g,1}$, $g \geqq 2$, the homomorphism τ shows that this conjecture is false: for by Lemma 4A, $\mathcal{T} \subset \operatorname{Ker} \tau$, and yet $\operatorname{Im} \tau$ is infinite, being free abelian of rank $\binom{2g}{3} > 0$. In fact, any map f with non-zero torsion is of infinite order in $\mathcal{I}/\mathcal{T}$. The disproof of Birman's conjecture for open surfaces has also been obtained by Wagoner (using algebraic K-theory) and, implicitly (as we describe below), by Chillingworth in [C 1, C 2]. In [C 2] Chillingworth exploits the concept of the "winding number" of a curve on a surface w.r.t. some vector field on the surface to obtain a necessary condition for a given map $f \in \mathcal{I}$ to be in $\mathcal{T}$, and conjectures that this condition is sufficient. Using the torsion we will show this conjecture is false also. First we need a short resume of Chillingworth's work.

For an open surface M and a non-singular (that is, nowhere zero) vector field X on M, the *winding number* (w.r.t. X) of an oriented regular curve γ is defined to be the number of times its tangent rotates w.r.t. the framing which X induces along the curve; it is notated $\omega_X(\gamma)$. This concept is then extended to the winding number of a free homotopy class of curves by selecting suitable representative regular curves for each class. If X_1, X_2 are two vector fields, Chillingworth defines an integral cohomology class $d(X_1, X_2)$ such that $\omega_{X_2}(\gamma) - \omega_{X_1}(\gamma) = \langle d(X_1, X_2), \gamma \rangle$ (originally defined in [R], p. 274).

Suppose now $f \in \mathcal{I}$. We introduce the function $e_{f,X}(\gamma) = \omega_X(f\gamma) - \omega_X(\gamma)$; it measures the change in winding numbers produced by f. Clearly $\omega_{fX}(f\gamma) = \omega_X(\gamma)$, so $\omega_X(f\gamma) = \omega_{f^{-1}X}(\gamma)$, and hence $e_{f,X}(\gamma) = \omega_{f^{-1}X}(\gamma) - \omega_X(\gamma) = \langle d(X, f^{-1}X), \gamma \rangle$, showing that the function $e_{f,X}$ is actually a function on homology classes and can be identified with the cohomology class $d(X, f^{-1}X)$. Now

$$\begin{aligned} e_{f,X_2}(\gamma) - e_{f,X_1}(\gamma) &= \omega_{X_2}(f\gamma) - \omega_{X_2}(\gamma) - \omega_{X_1}(f\gamma) + \omega_{X_1}(\gamma) \\ &= \langle d(X_1, X_2), f\gamma \rangle - \langle d(X_1, X_2), \gamma \rangle = \langle d(X_1, X_2), f\gamma - \gamma \rangle = 0, \end{aligned}$$

since $f\gamma$ is homologous to γ. We have thus shown:

Lemma 5A. *The cohomology class $e_{f,X} = d(X, f^{-1}X)$ defined by $e_{f,X}(\gamma) = \omega_X(f\gamma) - \omega_X(\gamma)$ is independent of the vector field X. We write simply e_f henceforth. We also have:*

Lemma 5B. $e_{fg} = e_f + e_g$ *for* $f, g \in \mathcal{I}$.

Proof.

$$\begin{aligned} e_{fg}(\gamma) &= \omega_X(fg\gamma) - \omega_X(\gamma) = \omega_X f(g\gamma) - \omega_X(g\gamma) + \omega_X(g\gamma) - \omega_X(\gamma) \\ &= e_f(g\gamma) + e_g(\gamma). \end{aligned}$$

But $g\gamma$ is homologous to γ so $e_f(g\gamma) = e_f(\gamma)$, QED. [Note: Wagoner (in unpublished work) has produced, by entirely different means, an isotopy invariant of homeomorphisms which seems to be the same as e_f.]

We now dualize the class e_f to a homology class t_f, so that t_f is thus defined by $\gamma \cdot t_f = e_f(\gamma)$; we call t_f the *Chillingworth class* of f. As in Lemma 5B, t is a homomorphism from $\mathcal{I}$ to H.

Lemma 5C. *Let $f \in \mathscr{I}$ and $h \in \mathscr{M}$; then $t_{hfh^{-1}} = h(t_f)$. Thus t is a homomorphism commuting with the action of $\mathscr{M}$ on $\mathscr{I}$ and H.*

Proof.

$$\begin{aligned} \gamma \cdot t_{hfh^{-1}} &= \omega_X(hfh^{-1}(\gamma)) - \omega_X(\gamma) = \omega_X(hfh^{-1}(\gamma)) - \omega_X(hh^{-1}(\gamma)) \\ &= \omega_{h^{-1}X}(fh^{-1}\gamma) - \omega_{h^{-1}X}(h^{-1}\gamma) \\ &= (\text{by Lemma 5A})\ \omega_X(fh^{-1}\gamma) - \omega_X(h^{-1}\gamma) \\ &= (h^{-1}\gamma) \cdot t_f = \gamma \cdot h(t_f), \quad \text{QED.} \end{aligned}$$

In order to establish the connection between the torsion and the Chillingworth class, we need to borrow the concept of a *contraction* from tensor calculus. This is a homomorphism $C_{ij}: H^{k+2} \to H^k$ (i, j distinct and $\leqq k+2$) defined by

$$C_{ij}(x_1 \otimes \ldots \otimes x_{k+2}) = (x_i \cdot x_j) \bigotimes_{r \neq i,j} x_r .$$

It is easily seen that C_{ij} is actually an Sp-module map. Applying this definition to $\Lambda^3 H \subset H^3$, we find all contractions on $\Lambda^3 H$ differ only by a sign, so we put simply $C = C_{12}$ and get:

$$C(x \wedge y \wedge z) = 2[(x \cdot y)z + (y \cdot z)x + (z \cdot x)y] .$$

In particular, $C(a_1 \wedge b_1 \wedge b_2) = 2b_2$.

Lemma 6. *Let A be an $\mathscr{M}$-module and $F, G: \mathscr{I} \to A$ two homomorphisms commuting with the action of $\mathscr{M}$ on $\mathscr{I}$, A; then $F = G$ iff $F(f) = G(f)$, where f is some* BP *map $T_\gamma T_\delta^{-1}$ of genus* 1.

Proof. Clearly the condition is necessary. Conversely, since all BP maps of genus 1 generate $\mathscr{I}$ (see [J 1]) and all such maps are conjugate in $\mathscr{M}$, we can express any $g \in \mathscr{I}$ in the form $g = \prod_{i=1}^{r} (h_i f h_i^{-1})$ $(h_i \in \mathscr{M})$. Then $F(g) = \sum_{i=1}^{r} F(h_i f h_i^{-1}) = \sum_{i=1}^{r} h_i(F(f))$, and likewise $G(g) = \sum_{i=1}^{r} h_i(G(f))$, so $F(f) = G(f) \Rightarrow F(g) = G(g)$, QED.

Theorem 2. *For $f \in \mathscr{I}$, $t_f = C(\tau_f)$.*

$$\begin{tikzcd} \mathscr{I} \arrow[rr, "\tau"] \arrow[dr, "t"'] & & \Lambda^3 H \arrow[dl, "C"] \\ & H & \end{tikzcd}$$

Proof. We are asked to prove that the diagram commutes. We will apply Lemma 6, with $F = t$, $G = C \circ \tau$, to the map $f = T_{\beta_2} T_{\beta_2'}^{-1}$ defined by Fig. 5a. By the corollary to Lemma 4B, we have $\tau_f = a_1 \wedge b_1 \wedge b_2$, and hence $C(\tau_f) = 2b_2$, so we need only show that $t_f = 2b_2$. Now f leaves all the curves α_i, β_i unaltered except α_2, which is tranformed into $f\alpha_2 = \alpha_2'$ as shown in Fig. 5b, so $b_i \cdot t_f = a_i \cdot t_f = 0$ for all the homology classes a_i, b_i excepting a_2. To find $a_2 \cdot t_f$, note that $\alpha_2 \cup -\alpha_2'$ is the

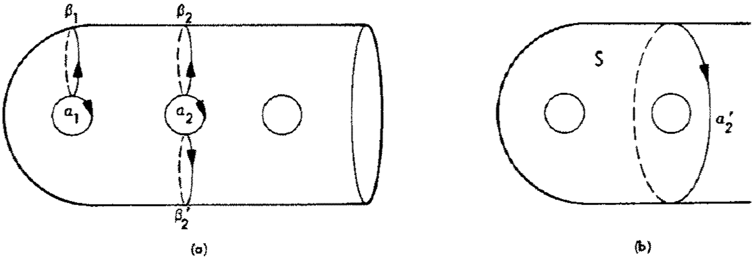

Fig. 5a and b

oriented boundary of the genus 1 subsurface S, and thus according to Chillingworth's Lemma 5.7, $a_2 \cdot t_f = \omega_X(\alpha_2') - \omega_X(\alpha_2)$ is ± 2. In fact, a more careful book-keeping of signs shows that we actually get $a_2 \cdot t_f = +2$. [For example, choose a non-singular vector field X on S which is orthogonal to α_2, so that $\omega_X(\alpha_2) = 0$. If we close up S to a genus 1 surface $\bar{S}$ by filling in α_2, α_2' with discs D, D' respectively, then X extends to $\bar{S}$ so as to have a zero of index $+1$ in D and a single zero in D', which by Hopf's theorems connecting vector fields and the Euler characteristic must have index -1. This means that along $\partial D' = +\alpha_2'$ the field X rotates -1 times w.r.t. the standard parallel field on D'; the tangent field of α_2' rotates $+1$ times w.r.t. the standard field, so we get $\omega_X(\alpha_2'') = 1 - (-1) = 2$.] This shows that $t_f = 2b_2$, QED.

Analogous to the corollary to Lemma 4B, we have:

Corollary 1. *Let $f = T_\gamma T_\delta^{-1}$ with (γ, δ) bounding S of genus k, and put $c =$ the homology class of γ with orientation acquired from S; then $t_f = 2kc$.* Im t *is just $2H \subset H$.*

Proof. If a_i, b_i is a basis for a maximal symplectic subspace of $H_1(S)$ then by the aforementioned corollary $\tau_f = \sum_{i=1}^{k} a_i \wedge b_i \wedge c$, and we find then that $C(\tau_f) = 2kc$. The latter statement follows from this, since Im $t \subset 2H$ and $2H$ is an Sp-module of H generated over Sp by, for example, $2b_2$.

Since Im $t = 2H$ is infinite and $t(\mathcal{T}) = 0$ we see how Chillingworth's methods disprove Birman's conjecture for open surfaces. It will be shown in the next section that this method fails, however, for a closed surface.

In [C 2] Chillingworth conjectures that if $f \in \mathcal{I}$ and f preserves all winding numbers, then $f \in \mathcal{T}$. In our terminology this conjecture takes the form: $t_f = 0 \Rightarrow f \in \mathcal{T}$. We have, however:

Corollary 2. *There exists homeomorphisms $f \in \mathcal{I}$ of $M_{g,1}$ $(g \geqq 3)$ with $t_f = 0$ but $f \notin \mathcal{T}$.*

Proof. H, and hence $2H$, is free abelian of rank $2g$. $\Lambda^3 H$ is free abelian of rank $\binom{2g}{3}$, and this is greater than $2g$ for $g \geqq 3$, so C must have non-trivial kernel. If k is any non-zero element of Ker C then by Theorem 1 there is an $f \in \mathcal{I}$ with $\tau_f = k$, and then $t_f = 0$. But $f \notin \mathcal{T}$ since $\tau_f \neq 0$.

Note. For $g=2$ the map $C:\Lambda^3H\to 2H$ is actually an isomorphism and the above argument fails.

We now make the obvious conjecture by strengthening Chillingworth's hypotheses:

Conjecture. *For* $f\in\mathscr{I}$, $\tau_f=0\Rightarrow f\in\mathscr{T}$.

6. Closed Surfaces

We investigate how the above results can be adapted to the case of a closed surface. Let now $M=M_g$ be closed, and suppose D is a (closed) disk in M. Then we put $M_D=M-\operatorname{Int}D$, $\mathscr{M}_D$=mapping class group of M_D, with obvious definition for $\mathscr{I}_D$. M_D is an $M_{g,1}$, and if f is any homeomorphism of M_D, then we may extend it by the identity on D to get a well defined homeomorphism of M. In this way we get a surjective homomorphism $\mathscr{M}_D\xrightarrow{p_D}\mathscr{M}$. Suppose E is any other disc in M. Then there exists a homeomorphism j of M which is isotopic to 1 on M and such that $j(D)=E$. The map j then induces a homeomorphism $j:M_D\to M_E$, as well as a map $j*:\mathscr{M}_D\to\mathscr{M}_E$ defined by $j*(f)=jfj^{-1}$ for $f\in\mathscr{M}_D$. The fact that $j=1$ in $\mathscr{M}$ implies the commutativity of the diagram.

$$\begin{tikzcd}
\mathscr{M}_D \arrow[dr, "p_D"] \arrow[dd, "j*"'] & \\
& \mathscr{M} \\
\mathscr{M}_E \arrow[ur, "p_E"'] &
\end{tikzcd}$$

We shall exploit this fact to define the torsion of maps in $\mathscr{I}=\mathscr{I}_g$.

Note first that the inclusion map $i_D:M_D\to M$ induces an isomorphism $H_1(M_D)\to H_1(M)=H$. We may thus identify $H_1(M_D)$ with H and hence also $\Lambda^3H_1(M_D)$ with Λ^3H.

Next, consider the exact sequence

$$1\to\operatorname{Ker}p_D\to\mathscr{I}_D\to\mathscr{I}\to 1\,.$$

If $f\in\mathscr{I}$, a lifting $\tilde{f}$ of it to $\mathscr{I}_D$ is well defined up to an arbitrary multiple by $k\in\operatorname{Ker}p_D$ and hence $\tau_{\tilde{f}}\in\Lambda^3H$ is well defined up to the addition of an arbitrary term of the form τ_k, $k\in\operatorname{Ker}p_D$. In other words, $\tau_{\tilde{f}}$ is well defined in the quotient $\Lambda^3H/\tau(\operatorname{Ker}p_D)$. We designate this quotient by V and the image of $\tau_{\tilde{f}}$ in it by τ_f. To justify this notation we need to show that V and τ_f do not depend on the choice of D. But $\operatorname{Ker}p_E=j*(\operatorname{Ker}p_D)=j(\operatorname{Ker}p_D)j^{-1}$ and so $\tau(\operatorname{Ker}p_E)=\tau(j(\operatorname{Ker}p_D)j^{-1})=$(by an easy generalization of Lemma 2D, as applied to τ) $j(\tau(\operatorname{Ker}p_D))=\tau(\operatorname{Ker}p_D)$, since $j=1$ on Λ^3H. Thus $\tau(\operatorname{Ker}p_D)\subset\Lambda^3H$ is independent of D, showing V also to be independent of D. If $\tilde{f}$ lifts $f\in\mathscr{I}$ to $\mathscr{I}_D$, then $j\tilde{f}j^{-1}$ lifts it to $\mathscr{I}_E$, and $\tau_{j\tilde{f}j^{-1}}=j(\tau_{\tilde{f}})=\tau_{\tilde{f}}$, so $\tau_f\in V$ is also independent of D. This is our definition of the torsion of $f\in\mathscr{I}_g$. It should be noted that, just as for an open surface, V is an Sp-module and

$\tau:\mathscr{I}\to V$ a homomorphism commuting with the action of $\mathscr{M}$ on $\mathscr{I}$, V. Our next problem is, then, to find $\tau(\operatorname{Ker} p_D)$ and the quotient V.

Let a_i, b_i be a symplectic basis of H. The element $\theta=\sum_{i=1}^{g} a_i\wedge b_i$ of $\Lambda^2 H$ is independent of the choice of symplectic basis and is thus *invariant* under the action of Sp on $\Lambda^2 H$. Consider then the map $u:H\to\Lambda^3 H$ given by $u(x)=\theta\wedge x$. Then u is clearly a homomorphism, and is also an Sp-module map since $u(gx)=\theta\wedge gx = g\theta\wedge gx = g(\theta\wedge x)$ for $g\in \mathrm{Sp}$. If $g\geqq 2$ we also have u is *injective*. For suppose $x\neq 0$: we may assume $x\in H$ is primitive (that is, not a non-trivial multiple of another vector in H), and hence we may use x as the first vector a_1 of some symplectic basis of H. In terms of this basis then,

$$\theta\wedge x=\left(\sum_{i=1}^{g} a_i\wedge b_i\right)\wedge a_1\neq 0 \quad \text{for} \quad g\geqq 2.$$

Lemma 7A. $\tau(\operatorname{Ker} p_D)=\operatorname{Im} u$, *and so we get an exact sequence*

$$0\to H\xrightarrow{u}\Lambda^3 H\to V\to 0.$$

Proof. The kernel of p_D is generated by:

a) twisting the boundary curve of M_D,

b) maps of the type $T_\gamma T_\delta^{-1}$, where (γ,δ) is a BP bounding S of genus $g-1$. (See, for example, [B], pp. 156–160.) The former has zero torsion, and the torsion of the latter is $\left(\sum_{i=1}^{g-1} a_i\wedge b_i\right)\wedge c$ with a_i, b_i, c as in the corollary to Lemma 4B. Since c is the class of an SCC, it must be *primitive*. Further, $c\cdot a_i=c\cdot b_i=0$ $(i=1,\dots,g-1)$ and hence we may find $d\in H$ such that $d\cdot a_i=d\cdot b_i=0$ and $c\cdot d=1$; a_i, b_i and c, d is then a full symplectic basis of H. But then

$$\tau(T_\gamma T_\delta^{-1})=\left(\sum_{i=1}^{g-1} a_i\wedge b_i+c\wedge d\right)\wedge c=\theta\wedge c=u(c).$$

Thus $\tau(\operatorname{Ker} p_D)\subset \operatorname{Im} u$. Conversely, note that since $\operatorname{Ker} p_D$ is normal in $\mathscr{M}_D$, $\tau(\operatorname{Ker} p_D)$ is an Sp-submodule of $\Lambda^3 H$ containing, e.g., $\theta\wedge b_g$, and so also contains $\theta\wedge x$ for all $x\in H$, QED.

Recall that $C:\Lambda^3 H\to H$, so $Cu:H\to H$.

Lemma 7B. $Cu(x)=(2g-2)x$, *all* $x\in H$.

Proof. Let a_i, b_i be an Sp-basis and suppose $x=a_1$; then

$$u(x)=\sum_{i=2}^{g} a_i\wedge b_i\wedge a_1,\ Cu(x)=2(g-1)a_1.$$

Likewise for all the other basis elements, and the lemma follows.

Corollary. *Let* $K = \operatorname{Ker} C$; *then* $K \cap \operatorname{Im} u = 0$.

Proof. Cu *is* $1-1$, so C is $1-1$ on $\operatorname{Im} u$.
The corollary also shows that $K \subset \Lambda^3 H$ injects into V.

We now consider the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & K \arrow[r, "1"] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H \arrow[r, "u"] \arrow[d, "\text{mult. by } (2g-2)"'] & \Lambda^3 H \arrow[r] \arrow[d, "C"] & V \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & (2g-2)H \arrow[r, "\text{incl.}"'] \arrow[d] & 2H \arrow[r, "\text{quot.}"'] \arrow[d] & 2H \bmod (2g-2) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The rows and first two columns are all exact, and the 3 complete squares commute, so the dotted arrow is uniquely defined so as to complete the last square commutatively; the last column is then also exact by the 9-lemma. Since the dotted arrow is defined by the contraction C, we name it C also. Note that the diagonal map of

$$\begin{tikzcd}
\Lambda^3 H \arrow[r] \arrow[d, "C"'] & V \arrow[d] \\
2H \arrow[r] & 2H \bmod (2g-2)
\end{tikzcd}$$

takes $\tau_f \in \Lambda^3 H$ to its Chillingworth class reduced $\bmod 2g-2$. To be more precise, Chillingworth showed in [C 1] that for a closed surface his winding numbers are defined $\bmod 2g-2$, and mimicking the arguments of the preceding section we find that for $f \in \mathcal{I}_g$ we get a well defined class t_f in $2H \bmod (2g-2)$. Furthermore, t is a homomorphism and, as for open surfaces, $t_f = C(\tau_f)$. Using the obvious isomorphism of $2H \bmod (2g-2)$ with $H \bmod (g-1)$ (namely, division by 2) we get finally:

Theorem 3. *For a closed surface of genus $g \geqq 2$, we have a commutative diagram*

$$\begin{tikzcd} & & \mathscr{I} \arrow[dl, "\tau"'] \arrow[dr, "\frac{1}{2}t"] & & \\ 0 \arrow[r] & K \arrow[r] & V \arrow[r, "\frac{1}{2}C"'] & H \bmod (g-1) \arrow[r] & 0 \end{tikzcd}$$

where $K = \operatorname{Ker}(C: \Lambda^3 H \to H)$. The row is exact, and τ is surjective. (Note that for $g=2$, $K=0$ and $V = H \bmod (g-1)$.)

We are now ready to examine Birman's and Chillingworth's conjectures for closed surfaces. As before, $\tau(\mathscr{T})$ and $t(\mathscr{T})$ are zero. Here, however, $t(\mathscr{I})$ is finite and thus the use of t alone will not tell us that $\mathscr{I}/\mathscr{T}$ is infinite. On the other hand, V remains infinite for $g \geqq 3$ since it contains K, and this disproves Birman's conjecture for closed surfaces. In [C 2] Chillingworth produces a homeomorphism $f \in \mathscr{I}$ such that $t_f = 0$ in $H \bmod 2(g-1)$, but f is not the product of BSCC maps. A close examination of his proof reveals that it does not, however, prove that f is not *isotopic* to such a product: the reduction by isotopy leaves his invariant defined only $\bmod 2(g-1)$. Again, the group $K \subset V$ gives examples of $f \notin \mathscr{T}$ having $t_f = 0$, showing that, for closed surfaces (of genus $\geqq 3$) as well, $t_f = 0$ is insufficient to insure that $f \in \mathscr{T}$.

7. Other Abelian Quotients of $\mathscr{I}$

In this section we return to the open surface $M_{g,1}$, and will assume implicitly that $g \geqq 3$ henceforth.

The representation of $\mathscr{I}$ onto $\Lambda^3 H$ suggests the problem of finding other abelian quotients of $\mathscr{I}$, and this naturally leads one to the universal such quotient, namely $\mathscr{I}/\mathscr{I}'$. A full knowledge of this group would be interesting in several ways, as mentioned in the introduction. As an obvious first conjecture, we might ask if $\mathscr{I}/\mathscr{I}'$ is $\Lambda^3 H$. This is not the case. In [BC], Birman and Craggs produced homomorphisms from $\mathscr{I}$ to Z_2 using the Rochlin invariant. Although it is a priori possible that these homomorphisms factor through $\Lambda^3 H$, we shall see below that in fact they do not, so $\mathscr{I}/\mathscr{I}'$ cannot be $\Lambda^3 H$.

First, we should draw attention to the important and useful fact that $\mathscr{I}/\mathscr{I}'$ has a natural Sp-module structure induced by the conjugation of $\mathscr{M}$ on $\mathscr{I}$. For let $\alpha \in \mathscr{I}/\mathscr{I}'$ be represented by $f \in \mathscr{I}$; f is determined up to a multiple by $k \in \mathscr{I}'$. Let also $v \in \mathrm{Sp} = \mathscr{M}/\mathscr{I}$ be represented by $h \in \mathscr{M}$: h is determined up to a multiple by $g \in \mathscr{I}$. Then define $v(\alpha)$ to be the reduction of hfh^{-1} to $\mathscr{I}/\mathscr{I}'$. Changing f to fk changes hfh^{-1} to $hfh^{-1} \cdot hkh^{-1} \equiv hfh^{-1} \bmod \mathscr{I}'$, and changing h to gh changes hfh^{-1} to $g(hfh^{-1})g^{-1} = [g, hfh^{-1}] \cdot hfh^{-1} \equiv hfh^{-1} \bmod \mathscr{I}'$. Thus $v(\alpha)$ is well defined, and it is easy to see that it defines an Sp-module structure on $\mathscr{I}/\mathscr{I}'$. In seeking abelian quotients of $\mathscr{I}$, then, those with Sp-module structure are of primary importance. They correspond to those quotients of $\mathscr{I}$ by a kernel which is normal not only in $\mathscr{I}$, but in $\mathscr{M}$ as well.

We describe now the abelian quotient of $\mathscr{I}$ derived from the Birman-Craggs homomorphisms. The homomorphisms themselves are from $\mathscr{I}$ to Z_2 and are finite in number. The intersection of all their kernels is denoted by $\mathscr{C}$ and a simple argument shows that the group $U=\mathscr{I}/\mathscr{C}$ is a finite dimensional vector space over Z_2. In [J 2] the author has given an explicit representation of this vector space as a space B_3 of cubic "Boolean" polynomials over a certain space closely related to H mod 2. Formally, these polynomials are ordinary Z_2-polynomials in the symbols $\bar{x}$ ($x \in H$ mod 2), subject to the relations:

1) $\overline{x+y}=\bar{x}+\bar{y}+x\cdot y$ (here $x\cdot y$ is the Z_2-valued intersection) and the "Boolean" relations:

2) $\bar{x}^2=\bar{x}$.

The Sp-action on the polynomials is given by $g(\bar{x})=\overline{g(x)}$, extended in the usual way. If e_i is a basis of H, then a basis for B_3 is given by:

$$\left.\begin{array}{l}\left.\begin{array}{l}\left.\begin{array}{l}\text{a) } 1 \text{ (constant polynomials } (B_0))\\ \text{b) } \bar{e}_i\end{array}\right\}\text{linear polynomials }(B_1)\\ \text{c) } \bar{e}_i\bar{e}_j,\ i<j\end{array}\right\}\text{quadratics }(B_2)\\ \text{d) } \bar{e}_i\bar{e}_j\bar{e}_k,\ i<j<k\end{array}\right\}\text{cubics }(B_3)$$

Thus $\dim B_3=\sum_{i=0}^{3}\binom{2g}{i}$. Note the common term $\binom{2g}{3}$ in the dimension (rank) of both B_3 and $\Lambda^3 H$; as this might suggest, the homomorphisms $\sigma:\mathscr{I}\to B_3$ (defined in [J2]) and $\tau:\mathscr{I}\to\Lambda^3 H$ are not independent. To make the relation between them precise, we need:

Lemma 8. *B_3/B_2 is naturally* Sp-*isomorphic to $\Lambda^3 H$* mod 2.

Remark. In general the proof goes through to show $B_r/B_{r-1}\simeq\Lambda^r H$ mod 2.

Proof. If e_i is a basis of H mod 2, the assignment $e_i\otimes e_j\otimes e_k\to\bar{e}_i\bar{e}_j\bar{e}_k \bmod B_2$ is easily seen to be trilinear by using the relations 1) and so defines a map r from H^3 mod 2 to B_3/B_2. Since $e_i\otimes e_i\otimes e_j$ goes to $\bar{e}_i^2\bar{e}_j=\bar{e}_i\bar{e}_j\in B_2$, r actually factors through $\Lambda^3 H$ mod 2 (here we are using the definition of $\Lambda^3 H$ mod 2 as a *quotient* of H^3 mod 2), and $r:\Lambda^3 H \bmod 2\to B_3/B_2$ is clearly onto. But $\Lambda^3 H$ mod 2 and B_3/B_2 are both of dimension $\binom{2g}{3}$, so r is actually an isomorphism. It is also an Sp-map since

$$\begin{aligned} jg(e_1\wedge e_2\wedge e_3)&=j(ge_1\wedge ge_2\wedge ge_3)=g\bar{e}_1 g\bar{e}_2 g\bar{e}_3=g(\bar{e}_1)g(\bar{e}_2)g(\bar{e}_3)\\ &=g(\bar{e}_1\bar{e}_2\bar{e}_3)=gj(e_1\wedge e_2\wedge e_3), \quad \text{QED.}\end{aligned}$$

This lemma gives us a surjective map $q:B_3\to\Lambda^3 H$ mod 2 defined by $B_3\to B_3/B_2\xrightarrow{j^{-1}}\Lambda^3 H$ mod 2 (i.e., $\bar{x}\bar{y}\bar{z}\to x\wedge y\wedge z$), and $\operatorname{Ker} q=B_2$.

Theorem 4. *The diagram commutes.*

$$\begin{tikzcd} & \mathscr{I} \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\ B_3 \arrow[dr, "q"'] & & \Lambda^3 H \arrow[dl, "\text{mod 2 reduction}"] \\ & \Lambda^3 H \text{ mod } 2 & \end{tikzcd}$$

Proof. As previously stated, $\mathscr{I}$ is generated by BP maps $T_\gamma T_\delta^{-1}$, where (γ, δ) bounds a surface S of genus 1. If $H_1(S) \subset H_1(M)$ has a maximal symplectic subspace with Sp-basis a, b, and c is the boundary class, then by Lemma 12b of [J 2] we get $\sigma(T_\gamma T_\delta^{-1}) = \bar{a}\bar{b}(\bar{c}+1) = \bar{a}\bar{b}\bar{c} + \bar{a}\bar{b}$, and $q\sigma(T_\gamma T_\delta^{-1}) = a \wedge b \wedge c$. But $\tau(T_\gamma T_\delta^{-1}) = a \wedge b \wedge c$ also; the theorem now follows from Lemma 6.

Corollary. *Neither of σ, τ factors through the other.*

Proof. Certainly τ cannot factor through σ since B_3 is finite but $\Lambda^3 H$ is not. That σ does not factor through τ follows from the existence of $f \in \mathscr{I}$ with $\tau_f = 0$ but $\sigma_f \neq 0$; in fact, any non-trivial BSCC twist f has these properties. (See Lemma 12a of [J 2].)

The two abelian quotients B_3 and $\Lambda^3 H$ are (aside from their quotients) the only ones known. We may combine them into an Sp-module A by means of the pullback construction

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ B_3 \arrow[dr, "q"'] & & \Lambda^3 H \arrow[dl, "\text{reduction}"] \\ & \Lambda^3 H \bmod 2 & \end{tikzcd}$$

and then A incorporates all known information about $\mathscr{I}/\mathscr{I}'$. It is natural that we

Conjecture. $\mathscr{I}/\mathscr{I}' \simeq A$.

A number of sporadic calculations done by the author lend some added support to this conjecture.

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