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Prevarieties and Intertwined Completeness of Locally Convex Spaces

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Certain results about locally convex topological vector spaces (TVS) show that the properties of (semi-) reflexivity and completeness have many common links [22, p. 144]. We introduce property *HC* to examine the links between reflexivity and completeness. A Fréchet space has *HC* if, and only if, it is reflexive. In general, *HC* implies completeness and semireflexivity.

A prevariety is a collection of TVS's closed with respect to subspaces, products and isomorphic images. Prevarieties seem appropriate to the study of completeness and reflexivity properties (witness Theorem 1.1). Many natural examples of prevarieties are given in Section One. In particular, the spaces whose completion have *HC* form a prevariety **hC**. The notion of prevarieties is a weakening of the varieties of Diestel *et al.* [6]. For any *TVS* (E, ξ) with continuous dual E', and any prevariety $\boldsymbol{X}$, there is a strongest topology of the dual pair, ξ_X, weaker than ξ, with $(E, \xi_X) \in \boldsymbol{X}$ (Lemma 1.2).

Property *HC* is equivalent to Berezanskiĭ's inductive semi-reflexivity [3] (Proposition 2.2). In [3] it is shown that the TVS (E, ξ) has *HC*, exactly when (E, ξ_S) is complete. Here $\boldsymbol{S}$ is the (pre)variety of all Schwartz spaces and ξ_S is the topology given above. This result is strengthened, obtaining the Intertwined Completeness Theorem 4.1 (ICT). The ICT states that for certain prevarieties $\boldsymbol{X}$ and $\boldsymbol{Y}$ and for any TVS (E, ξ), (E, ξ_X) is complete if, and only if, (E, ξ_Y) is complete (ξ_X and ξ_Y may be incompatible) or if, and only if, (E, ξ) has *HC*.

As an application of the ICT, it is shown that a space is a Hausdorff inductive limit of Hilbert spaces if, and only if, it is the strong dual of a space with *HC*. Also using the ICT, every Banach space is shown to be an inductive limit of Hilbert spaces. Thus every inductive limit of Banach spaces (i.e., every ultrabornological space) is, in fact, an inductive limit of Hilbert spaces. These results are in Section 5.

Raman [17] considers a property *HC* which is similar to the property *HC* defined in Section 2. The two definitions agree in barrelled spaces. The definition of *HC* is chosen for its similarity to Grothendieck's Completeness Criterion, which is stated below for comparison and later use.

Grothendieck's Completeness Criterion [22, Corollary 2, p. 149].

The TVS (E, ξ) is complete if, and only if, every hyperplane in E', whose intersection with U^0 is $\sigma(E', E)$ closed whenever U is a ξ-neighborhood of the origin, is itself $\sigma(E', E)$ closed.

Here, and always, E is a vector space with the locally convex Hausdorff topological vector space topology ξ and E' is the continuous dual. Second duals will be, in general, algebraic and will be indicated by $E'^{\#}$. If $U \subset E$, then U^0 is the (absolute) polar of U in E'. The convention of ϱ_A representing the Minkowski gauge functional of the absolutely convex set A is used without further explanation.

Similarly, if V is an absolutely convex neighborhood for (E, ξ), then $(\hat{E}_V, \varrho_V)$ represents the completion of the normed space formed as the quotient of (E, ϱ_V) by the kernel of ϱ_V.

Finally, we mention that the ICT can be extended to include certain λ-nuclear prevarieties (see [2]).

§ 1. Prevarieties, Examples, and Basic Results

We define a *prevariety* to be a collection of TVS's closed with respect to subspaces, products and isomorphic images. A *variety* (in the sense of Diestel *et al.* [6] is a prevariety which, in addition, is closed with respect to quotients by closed subspaces. If $\mathscr{B}$ is a collection of TVS's, then the smallest variety (respectively, prevariety) containing $\mathscr{B}$ is $v(\mathscr{B})$ $(\varrho v(\mathscr{B}))$.

There are many examples of prevarieties. We list a few examples of prevarieties below and at the same time develop notation for later use. The first eight examples are actually varieties; in fact, using Theorem 1.4 of [6, p. 210], we note that $\varrho v(\mathscr{B}) = v(\mathscr{B})$, if $\mathscr{B}$ is closed with respect to quotients by closed subspaces of finite products of spaces in $\mathscr{B}$.

1. The simplest and smallest prevariety or variety is $\boldsymbol{K}$, the one generated by the scalar field [6, Theorem 3.6, p. 216].

2. The variety of nuclear spaces, $\boldsymbol{N}$ [6, p. 209], [22, p. 103].

3. The variety of Schwartz spaces, $\boldsymbol{S}$ [11, pp. 278—9], [6, p. 209].

4. The variety of strongly nuclear spaces, $\boldsymbol{sN}$ [6, p. 209].

5. Let $\mathscr{H}$ be the class of all Hilbert spaces and let $\boldsymbol{H} = v(\mathscr{H}) = \varrho v(\mathscr{H})$ be the variety generated by $\mathscr{H}$. $\boldsymbol{H}$ contains those TVS's whose topology can be defined by means of semi-norms ϱ_V such that $(\tilde{E}_V, \varrho_V)$ is a Hilbert space.

6. Let $\mathscr{S}\mathscr{R}$ be the class of all super-reflexive Banach spaces [12, p. 896] (or equivalently, the class of Banach spaces isomorphic to any uniformly convex Banach space [8]). Let $\boldsymbol{SR} = v(\mathscr{S}\mathscr{R}) = \varrho v(\mathscr{S}\mathscr{R})$.

7. Let $\mathscr{R}$ be the class of all reflexive Banach spaces and let $\boldsymbol{R} = \varrho v(\mathscr{R}) = v(\mathscr{R})$.

8. Under suitable conditions on the sequence space λ, the class of all λ-nuclear spaces form a variety, $\boldsymbol{\lambda N}$. See [7], [18], and [20] for examples of such sequence spaces λ.

The remaining examples are instances of prevarieties which are not varieties. In fact, they can all be shown to contain a Fréchet Montel Space, constructed by Grothendieck and Köthe [13, p. 433] (see also [6, p. 219]), that has l_1 as a quotient, but none of these prevarieties contains l_1.

9. Let $\mathscr{F}\mathscr{M}$ be the collection of all Fréchet Montel spaces and let $\boldsymbol{FM} = \varrho v(\mathscr{F}\mathscr{M})$.

10. Let $\mathscr{F}\mathscr{R}$ be the class of all reflexive Fréchet spaces and let $\boldsymbol{FR} = \varrho v(\mathscr{F}\mathscr{R})$.

11. Let $\boldsymbol{PB}$ be the class of all TVS's in which every (weakly) bounded set is precompact. A proof that $\boldsymbol{PB}$ is a prevariety is essentially [19, Proposition 14, p. 85].

12. The prevariety $\boldsymbol{hC}$ of all TVS's whose completion has property *HC*. Both *HC* and $\boldsymbol{hC}$ are defined and discussed at length in the next section.

13. Although we make no use of nonstandard analysis, it is worth mentioning that the collection of TVS's with invariant nonstandard hulls [9], [10] forms a prevariety ***IH***. See [2] for results that deal with ***IH*** and for nonstandard results dealing with the ICT.

The main distinction between prevarieties and varieties is perhaps best described in Theorem 1.1; varieties do not enjoy any similar theorem [6, pp. 219, 225]. The proof of Theorem 1.1 is straight forward and will not be given.

Theorem 1.1. *Let $\mathscr{C}$ be any collection of complete TVS's and $\mathscr{B}$ any collection of complete semi-reflexive TVS's. The completion of any space in $\varrho v(\mathscr{C})$ is in $\varrho v(\mathscr{C})$. The completion of any space in $\varrho v(\mathscr{B})$ is semi-reflexive.*

The two following lemmas are important for our use of prevarieties. We adopt the convention if (E, ξ) is a TVS and ***X*** is a prevariety, then the topology given by Lemma 1.2 is ξ_X. It could be said that ξ_X majorizes all "***X***" TVS topologies on E weaker than ξ. The proof of Lemma 1.2 is essentially [22, p. 52].

Lemma 1.2. *If (E, ξ) is a TVS with continuous dual E' and* ***X*** *is a prevariety, then there exists a unique TVS topology on E, ξ_X, such that*

(i) $\sigma(E, E') \leqq \xi_X \leqq \xi$,

(ii) $(E, \xi_X) \in \boldsymbol{X}$,

(iii) *if η is any TVS topology on E with $\sigma(E, E') \leqq \eta \leqq \xi$ and $(E, \eta) \in \boldsymbol{X}$, then $\eta \leqq \xi_X$.*

Lemma 1.3 will be used to construct examples of spaces which have the property that many of their "***X***"-topologies are distinct. The proof is an easy exercise from the definitions.

Lemma 1.3. *Let $(E, \xi), (F, \eta)$ and (G, ζ) be TVS's and* ***X*** *a prevariety. If $(E, \xi) = (F, \eta) \oplus (G, \zeta)$ (i.e. the direct sum), then $(E, \xi_X) = (F, \eta_X) \oplus (G, \zeta_X)$.*

§ 2. The Prevariety *hC* and Inductive Semireflexivity

Property *HC* and inductive semi-reflexivity are equivalent properties that many known TVS's enjoy. Berezanskiĭ [3] first considered the notion of inductive semi-reflexivity, while property *HC* is a slight perversion of Raman's [17] property *HC*. The name *HC* stands for Hyperplane Closure. The following definitions differ in some cases, from the usual notions of the same name.

Definition. A sequence $\{x_n\}$ in the continuous dual E' of the TVS (E, ξ) is said to converge *locally* to $y \in E'$ if there exists an absolutely convex *ξ-equicontinuous* set U in E' such that $\{x_n\} \cup \{y\}$ is in the span of U and $\varrho_U(x_n - y) \to 0$ as $n \to \infty$.

This definition is given by Terzioglu in [23]. The usual notion is Mackey convergence defined below. (See [13, p. 382].)

Definition. A sequence $\{x_n\}$ in (E, ξ) is said to be *Mackey* convergent to $y \in E$ if there exists a (weakly) bounded absolutely convex set U in E such that $\{x_n\} \cup \{y\}$ is in the span of U and $\varrho_U(x_n - y) \to 0$ as $n \to \infty$.

The following proposition relates Schwartz spaces and local null sequences and will be needed later. The topology ξ_S has received attention before, for instance, Berezanskiĭ [3] calls it k_0. The proposition is a very easy consequence of [23, § 2(4), p. 237].

Proposition 2.1. *Let* (E, ξ) *be a TVS with continuous dual* E'. *Then the topology* ξ_S *is the topology of uniform convergence on local null sequences.*

Definition. We say that a TVS (E, ξ) has (property) HC if each hyperplane in E', which is closed for local convergence, is $\sigma(E', E)$ closed.

Though this definition is *not* that of Raman's given in [17], the two definitions agree on ω-barreled spaces [21].

Using Grothendieck's Completeness Criterion [22, Corollary 2, 149], we note that a TVS with HC is complete.

Definition. The prevariety ***hC*** is the collection of all TVS's whose completion has property HC.

We still have to show that ***hC*** is a prevariety. It suffices to show that a closed subspace of a product of spaces with HC has property HC. A proof of this fact can be constructed similar to the proof of Raman's theorem 16 [17, p. 193] with care paid to the existence of equicontinuous sets as opposed to bounded sets. This is due to our distinction between local and Mackey convergence.

Definition (Berezanskiĭ [3]). A TVS (E, ξ) is said to be *inductively semi-reflexive* if for every linear functional f on E', which is bounded on every ξ-equicontinuous set, there exists an $e \in E$, such that, for all $x \in E'$, we have $f(x) = \langle x, e \rangle$.

Proposition 2.2. *A TVS* (E, ξ) *has HC if and only if, it is inductively semi-reflexive.*

Proof. For a linear functional on a normed space, sequential continuity, continuity, and boundness are equivalent to the sequential closeness of the kernal hyperplane. It is now easy to show that both HC and inductive semi-reflexivity are equivalent to the following: $(*)$ Every linear functional on E', which is continuous on every span U^0 with norm ϱ_{U^0}, for every ξ-neighborhood of the origin U, is in the canonical image of E in $E'^{\#}$.

We note that many of Berezanskiĭ's results in [3] can be proved simply via the use of $(*)$ and well known results about normed spaces. In particular, his Theorem 2.4 [3, p. 1081] (Proposition 2.3 below) follows from $(*)$ and the fact that a linear functional on a normed space is continuous if it is bounded on all everywhere non-zero sequences $\{x_n\}$ with $\|x_n\| \leqq \lambda_n$, for any everywhere positive null sequence of reals $\{\lambda_n\}$.

Definition (Berezanskiĭ [3]). A *ξ-rotor topology* on the TVS (E, ξ) is any TVS topology η, on E, such that:

(i) $\sigma(E, E') \leqq \eta \leqq \xi$,

(ii) For each ξ-neighborhood of the origin U, there exists a null sequence of everywhere positive reals $\{\lambda_n\}$ such that any sequence $\{x_n\}$ in E' with $\varrho_{U^0}(x_n) \leqq \lambda_n$ is η-equicontinuous.

Proposition 2.3. (Berezanskiĭ [3, p. 1081]). *If* (E, ξ) *has HC and if* η *is a* ξ*-rotor topology, then* (E, η) *has HC and is thus complete.*

We note that if η is a ξ-rotor topology and ζ is a TVS topology on E with $\eta \leqq \zeta \leqq \xi$, then ζ is a ξ-rotor topology. The topology ξ_{sN} is ξ-rotor. It is sufficient for every ξ-neighborhood of zero U to chose $\{\lambda_n\}$ to be rapidly decreasing to zero [14, p. 158], [2]. Thus it will follow from Table 3.1 that the topologies ξ_X, for $\boldsymbol{X} = \boldsymbol{N}, \boldsymbol{S}, \boldsymbol{FM}, \boldsymbol{IH}, \boldsymbol{PB}, \boldsymbol{H}, \boldsymbol{SR}, \boldsymbol{R}, \boldsymbol{FR}$, and ***hC*** are all ξ-rotor topologies.

§ 3. Inclusion Relations Among Prevarieties

We list the results of this section in two tables. The purpose of Table 3.2 is to allow the construction of a TVS (E, ξ) in which all the topologies ξ_X, $\boldsymbol{X}$ one of our prevarieties in the first section, are distinct. For Table 3.1, $\boldsymbol{X} \to \boldsymbol{Y}$ means $\boldsymbol{X} \supset \boldsymbol{Y}$.

Table 3.1

$$\begin{array}{ccccccccccccc} & & \boldsymbol{hC} & \to & \boldsymbol{FR} & \to & \boldsymbol{R} & \to & \boldsymbol{SR} & \to & \boldsymbol{H} & & \\ & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\ \boldsymbol{PB} & \to & \boldsymbol{IH} & \to & \boldsymbol{FM} & \to & \boldsymbol{S} & & \longrightarrow & & \boldsymbol{N} & \to \boldsymbol{sN} \to & \boldsymbol{K} \end{array}$$

All the inclusions except the following are well known (see [14] and [22]): $\boldsymbol{R} \supset \boldsymbol{S}$ follows from the recent result [4] that weakly compact operators factor through reflexive Banach spaces; for $\boldsymbol{hC} \supset \boldsymbol{FR}$ we note that a Fréchet space has HC if, and only if, it is reflexive [3, Corollary 3.6, p. 1082]; $\boldsymbol{PB} \supset \boldsymbol{IH} \supset \boldsymbol{FM}$ can be found in [9] and [10]; for $\boldsymbol{hC} \supset \boldsymbol{IH}$ see [2]; and, finally, $\boldsymbol{FM} \supset \boldsymbol{S}$ follows from a general principle that yields Corollary 3.2.

We say that the prevariety $\boldsymbol{X}$ is *operator-defined* if there exists an ideal of operators $\mathscr{I}$ (in the sense of Pietsch [15], [16]) such that $(E, \xi) \in \boldsymbol{X}$ if and only if there exists a fundamental base of absolutely convex ξ-neighborhoods of the origin $\mathscr{U}$ such that, for all $U \in \mathscr{U}$, there exists $V \in \mathscr{U}$ with $V \subset U$ and the canonical map between the Banach space $(\hat{E}_V, \varrho_V) \to (\hat{E}_U, \varrho_U)$ belongs to $\mathscr{I}$.

Proposition 3.1. *If $\boldsymbol{X}$ is an operator defined prevariety and $\boldsymbol{FX}$ is the prevariety generated by all Fréchet spaces in $\boldsymbol{X}$, then $\boldsymbol{FX} = \boldsymbol{X}$.*

Proof. This is modeled on the well known canonical embedding of a TVS into a product of Banach spaces [22, Corollary 2, p. 54]. All we need to show is that any TVS in $\boldsymbol{X}$ can be embedded into a product of Fréchet spaces in $\boldsymbol{X}$. These Fréchet spaces can be constructed as projective limits of sequences $\ldots \to (\hat{E}_{U(n+1)}, \varrho_{U(n+1)}) \to (\hat{E}_{U(n)}, \varrho_{U(n)}) \to \ldots \to (\hat{E}_U, \varrho_U)$, where all arrows represent maps in $\mathscr{I}$ for $U \in \mathscr{U}$.

Corollary 3.2. $\boldsymbol{FM} \supset \boldsymbol{FS} = \boldsymbol{S}$.

In Table 3.2, examples are listed with references to where they are defined. The statements can be easily checked, with the exception of the last (see below). If (E, ξ) is the direct sum of all the spaces in Table 3.2, then each of the topologies ξ_X, $\boldsymbol{X}$ in Table 3.1, are distinct. Furthermore, the following pairs of topologies are incompatible: ξ_S and ξ_H, ξ_{FM} and ξ_R, ξ_{IH} and ξ_{FR}, and ξ_{hC} and ξ_{PB}. These statements are easy consequences of Lemma 1.3.

The rest of this section is devoted to justifying the last line of Table 3.2. The fact that $\phi_d \in \boldsymbol{IH}$ can be found in [10, § 2]. We will indicate that if ϕ_d is embedded in a product of Fréchet spaces, then one of these Fréchet factor spaces is not reflexive.

For each function $\theta: d \to$positive reals, we define the neighborhood $U(\theta)$ of the origin in ϕ_d to be the absolutely convex hull of the union, $i \in d$, of the $\theta(i)$-discs in the ith copy of the scalar field [19, p. 89]. If θ is the constant function 1, we write

Table 3.2

Space	Does not Belong to	Belongs to
ϕ [6, p. 216]	***K***	***sN***
Σ [14, p. 177] (often called s)	***sN***	***N***
l_2	***PB***	***H***
Λ_0 [1]	***H***	***S***
l_4	***H***	***SR***
$\oplus_{l_2} c_0^n$, $n=1,2,\ldots$[5, p. 146]	***SR***	***R***
E_0 [13, § 31,5, p. 433] [25, p. 240]	***R***	***FM***
$(E, \lvert\sigma\rvert(E,F))$ [10, § 1] [2]	***hC***	***PB***
ϕ_d [6, p. 216]	***FR***	***IH***

d is the cardinality of the power set of the continuum

$U(\theta)$ as just U. We will need to know that if $b\subset d$ is countably infinite and $W=\phi_b\cap U$, then the completion of (ϕ_b, p_W) is l_1.

By standard methods, it suffices to show that if $T:\phi_d\to M$ is a continuous one-one linear map onto the metric TVS M such that $T(U)$ is open, then there is an infinite subset $b\subset d$ with $T(U\cap\phi_b)$ the unit ball of a normed subspace of M. Clearly the completion of M would then contain l_1 as a subspace.

Since M has a countable basis $V_1\supset V_2\supset\ldots$ with $V_1\subset T(U)$, and since T is continuous, there exist functions θ_j such that $T(U(\theta_j))\subset V_j, j=1,2,\ldots$. For each increasing sequence of positive integers $P=\{p_j\}$, let P' be the set $\{i\in d: \theta_j(i)\geqq p_j^{-1}$ for $j=1,2,\ldots\}$. The union of all the P' is d, but there are only continuum many P' so there is an infinite P'. We can take this P' as b above. Then $T(U\cap\phi_b)$ is the unit ball of a normed subspace of M, since $p_j^{-1}T(U\cap\phi_b)\subset V_j, j=1,2,\ldots$.

This result can also be used to show that the prevarieties ***PB***, ***IH***, and ***hC*** are not operator defined [2].

§ 4. Intertwined Completeness

The aim of this section is to prove the Intertwined Completeness Theorem (ICT).

Theorem 4.1. Intertwined Completeness Theorem. *For a TVS (E,ξ), the following are equivalent:*

(i) *(E,ξ) has property HC*

(ii) *(E,ξ) is inductive semi-reflexive*

(iii) *E is complete in all ξ-rotor topologies*

(iv) *$\xi_{hC}=\xi$ and (E,ξ) is complete*

(v***X***) *one (all) of the statements: (E,ξ_X) is complete, where **X** is **sN**, **N**, **S**, **H**, **SR**, **R**, **FM**, **FR**, **IH**, or **hC**.*

The history of this theorem starts with Berezanskiǐ [3], where he shows the equivalence of (ii), (iii), (v***S***) and (v***N***). Raman [17] gives the equivalence of (i) and (v***R***) for barreled spaces. Raman further claims the equivalence of (ii) and (v***R***) for barreled spaces [17, p. 197], but he misquotes Berezanskiǐ.

Proof of ICT. We show (ii)⇔(i)⇒(iii)⇒(vX) for all X⇔(vsN)⇒(vS)⇒(ii), (i)⇔(iv) and $\exists X$(vX)⇒(vsN).

(ii)⇔(i). This is Proposition 2.2.

(i)⇒(iii). This is Proposition 2.3.

(iii)⇒(vX), for all X listed. This follows from the remarks after Proposition 2.3.

(vX), for all X⇒(vsN). This is formal.

(vsN)⇒(vX), for all X. This follows from $\xi_{sN} \leqq \xi_X$ and use of [19, Proposition 5, p. 107].

(vsN)⇒(vS). This is a special case of the immediately preceding implication.

(vS)⇒(ii). This is Berezanskiĭ's [3, Th. 1.5, p. 1081].

(i)⇔(iv) This is by the definition of the prevariety $\boldsymbol{hC}$ and the fact that a space with HC is complete.

$\exists X$(vX)⇒(vsN). Let $\eta = \xi_X$. By hypothesis, (E, η) is complete and since $X \subset \boldsymbol{hC}$, we have $\eta_{hC} = \eta$. It follows that (E, η) has HC from the equivalence of (i) and (iv). Now (E, η_{sN}) is complete, but, $sN \subset X$ so $\eta_{sN} = \xi_{sN}$. Q.E.D.

We note that the example (E, ξ) in section three (the direct sum of the spaces in Table 3.2) yields an example with each X topology distinct and incomplete. However, the completion of (E, ξ_{hC}) is an example with each X topology distinct and complete.

§ 5. Duality and Ultrabornological Spaces

In this section, we apply the ICT to characterize the strong duals of spaces with HC. The interplay between the topologies ξ_S and ξ_H allows us to prove the somewhat surprising fact that ultrabornological spaces are actually inductive limits of Hilbert spaces. (See Theorem 5.3.) The reader is reminded that an ultrabornological space is an inductive limit of Banach spaces [19, p. 160]. Ultrabornological spaces are the domain space for a closed graph theorem described in the appendix of [19].

Theorem 5.1. *(F, η) is a Hausdorff inductive limit of Hilbert spaces if, and only if, (F, η) is the strong dual of a space with HC.*

Proof. Suppose (E, ξ) has HC. Then (E, ξ_H) has HC (Proposition 2.3). It follows from [3] that the strong dual of (E, ξ_H) [and thus (E, ξ)] is the inductive limit of the normed spaces (span V^0, ϱ_{V^0}), V a ξ_H-neighborhood of the origin. We note that it suffices to restrict V to run over the ξ_H-neighborhoods of the origin such that $(\tilde{E}_V, \varrho_V)$ is Hilbert space, in which case (span V^0, ϱ_{V^0}) is also Hilbert space.

Conversely, suppose (F, η) is a Hausdorff inductive limit of Hilbert spaces. Let E be the continuous dual of (F, η) and ξ be the topology of uniform convergence on the unit balls of the Hilbert spaces (i.e. ξ is the projective limit of Hilbert spaces given by the adjoints of the inductive limits forming (F, η) [19, Proposition 15, p. 85]).

We note that $\xi = \xi_H$ and ξ is complete [2.2, 5.3, p. 52] and so (E, ξ) has HC by the ICT. Furthermore, since each ball in Hilbert space is weakly compact, ξ is a topology of the dual pair (E, F). Hence F is the continuous dual of (E, ξ). Since (F, η) is barreled [19, Proposition 6, p. 81], η is the strong topology on F [19, Corollary 1, p. 66]. The proof is complete.

The following theorem is based on Proposition 2.1. It is the main step in Theorem 5.3.

Theorem 5.2. *If* (B, ξ) *is a Banach space, then* (B, ξ) *is an inductive limit of Hilbert spaces.*

Proof. We note that if (B, ξ) were reflexive, this would be a trivial consequence of the ICT and Theorem 5.1. In any case, let B' be the continuous dual of (B, ξ). Let η be the topology on B', of uniform convergence on null sequences of (B, ξ). Since the closed absolutely convex hull of a null sequence is compact and hence $\sigma(B, B')$-compact, η is a topology of the dual pair (B', B). In fact, by Proposition 2.1, $\eta = \eta_S$ is the strongest Schwartz topology of the dual pair (B', B).

Since a linear functional on B is continuous if and only if it is bounded on null sequences, (B', η) has *HC* (see definition of inductive semireflexivity.) Thus, by Theorem 5.1, (B, ξ), the strong dual of (B', η), is an inductive limit of Hilbert spaces.

Theorem 5.3. *An ultrabornological space is an inductive limit of Hilbert spaces. In particular, every sequentially complete bornological space is an inductive limit of Hilbert spaces.*

Proof. An ultrabornological space is an inductive limit of Banach spaces, which, in turn, are inductive limits of Hilbert spaces, by Theorem 5.2.

The author recently discovered a direct proof of Theorem 5.2, which will appear elsewhere.

Note added in proof: Since submission the author has learned that Theorems 5.2 and 5.3 have appeared in [25] (and are implicit in [24]). Valdivia, in [26], improves Theorem 5.2 to: if the Banach space E has a weak-star separable dual, then every Banach space is an inductive limit of spaces isomorphic to E.

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