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Math. Annalen 174, 1—7 (1967)

Regular Convergence Spaces

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Introduction

The method of defining a topology by specifying what nets converge to what points is due to BIRKHOFF [1]. BIRKHOFF also showed that regular topological spaces may be characterized in terms of nets. The first three axioms of BIRKHOFF have obvious analogues for filters. However, his axiom (δ), the iterated limit axiom does not have such an immediate interpretation. This paper gives a set of axioms that define a topology by specifying what filters converge to what points which includes a workable iterated limit axiom in terms of filters. It will be shown that the converse of this axiom characterizes regular topological spaces. We should note that DIEUDONNÉ [3] has characterized regular topological spaces in terms of filters. However, the condition that we give, as we shall show, is at least as weak as DIEUDONNÉ's and seems to be more flexible.

SONNER [6] has given a set of two axioms for limit spaces and establishes polarity between limit spaces and topological spaces. His axioms suffer from the handicap that he must explicitly mention in his axiom (2) which sets are eventually to be the open sets in the topology. For our purposes entirely too much structure is couched in axiom (2) which makes his study inapplicable for more general convergence spaces. By using the filter analogues of BIRKHOFF's first three axioms and the iterated limit axiom we are able to obtain a greater unity between topological spaces and more general convergence spaces.

It is also of some importance to find a notion of regularity for a convergence space in the sense of KOWALSKY [5]. We shall produce evidence that the definition of regularity we give is the axiom that should be adopted in this setting. For example, a regular T_1 convergence space is T_2 and an inverse limit of regular convergence spaces is regular.

We take the liberty of making free use of the results and terminology of [2] and [4]. Some of the most frequently used results will be repeated here, however, for easy reference. $\mathbf{F}(X)$ will always denote the collection of proper filters on the set X. If f is a function on X and $\mathscr{F} \in \mathbf{F}(X)$ then $f(\mathscr{F})$ is the filter generated by the sets $f(F)$, $F \in \mathscr{F}$. $[\{F\}]$ is used to denote the filter generated by the filter base $\{F\}$; when $\{F\}$ is merely $\{\{x\}\}$ for some $x \in X$, then $[\{\{x\}\}]$ will be denoted by $\dot{x}$. If I is a directed set then $\mathscr{S}(I)$ will denote the filter of sections of I.
KOWALSKY [5] has defined the notion of a diagonal convergence structure and proved that the closure operator associated with such a convergence is

idempotent. Diagonal convergence structures are defined by means of a "compression operator" κ which is defined here. If $\Phi \in \mathbf{F}(\mathbf{F}(X))$, denote by $\kappa(\Phi)$ the filter on X defined as follows:

$$\kappa(\Phi) = \sup_{A \in \Phi} \inf_{\mathscr{F} \in A} \mathscr{F} = \bigcup_{A \in \Phi} \inf A .$$

A convergence structure is then said to be *diagonal* if at each point $x_0 \in X$ the following condition is satisfied: for each map $\eta : X \to \mathbf{F}(X)$ that satisfies $\eta x \in \tau x$ and for each filter $\mathscr{F} \in \tau x_0$, $\kappa\eta(\mathscr{F}) \in \tau x_0$. Our iterated limit axiom is a strengthening of the notion of a compression operator. In order to express this axiom we define a filter $\mathscr{G}$ on X by:

$$\text{if } I \text{ is a set, } \mathscr{F} \in \mathbf{F}(I) \text{ and } \sigma \in \mathbf{F}(X)^I$$

$$\mathscr{G} = \left\{ \bigcup_{b \in F} \phi(b) \,|\, F \in \mathscr{F} \;\; \phi \in \bigtimes_{b \in I} \sigma(b) \right\} .$$

Since the infimum of a set $\{\mathscr{F}_\alpha : \alpha \in I\}$ of filters can be expressed as

$$\inf \mathscr{F}_\alpha = \left\{ \bigcup_{\alpha} \phi(\alpha) \,|\, \phi \in \bigtimes_{\alpha} \mathscr{F}_\alpha \right\} ,$$

it is easy to see, subject to the above conditions, that

$$\mathscr{G} = \kappa(\sigma(\mathscr{F})) .$$

Axioms for a topological space in terms of filters

Let X be a set and τ be a relation on X to $\mathbf{F}(X)$; i.e. $\tau \subset X \times \mathbf{F}(X)$. The set $\tau x = \{\mathscr{F} \,|\, (x, \mathscr{F}) \in \tau\}$ of relatives of x can be thought of as the collection of all filters that converge to x (similarly $\tau^{-1}\mathscr{F} = \{x \,|\, (x, \mathscr{F}) \in \tau\}$ can be thought of as the collection of all points in X to which $\mathscr{F}$ converges). For each subset A of X one may then define the closure $\bar{A}$ of A as

$$\bar{A} = \{x \in X \,|\, \exists \mathscr{F} \in \tau x \ni A \in \mathscr{F}\} .$$

$\bar{A}$ is called the closure operator associated with τ. Clearly $\bar{\emptyset} = \emptyset$. If one requires that $\dot{x} \in \tau x$ for each $x \in X$ then it is easy to see that $\bar{A} \supset A$, $\overline{A \cup B} \supseteqq \bar{A} \cup \bar{B}$. Another relation σ may now be defined on $X \times \mathbf{F}(X)$ as follows:

$$(x, \mathscr{F}) \in \sigma \overset{\text{df}}{\cdot = \cdot} \quad \text{if} \quad V \subset X \quad \text{and} \quad x \notin \overline{V'} \quad \text{then} \quad V \in \mathscr{F} ,$$

where V' denotes the complement of V with respect to X. In general $\sigma \supset \tau$.

Let τ be a relation on X to $\mathbf{F}(X)$. In what follows τ will be assumed to satisfy any or all of the following axioms.

(α) $\dot{x} \in \tau x$,

(β) if $\mathscr{F} \in \tau x$ and $\mathscr{G} \geq \mathscr{F}$, then $\mathscr{G} \in \tau x$,

(γ) if $\mathscr{F} \notin \tau x$ then $\exists\, \mathscr{G} \geq \mathscr{F}$ such that if $\mathscr{H} \geqq \mathscr{G}$ then $\mathscr{H} \notin \tau x$,

(δ') for a given set I, $\mathscr{F} \in \mathbf{F}(I)$ and a $\sigma \in \mathbf{F}(X)^I$, if

(i) $\forall\, b \in I,\ \sigma(b) \in \tau\psi(b)$,

(ii) $\psi(\mathscr{F}) \in \tau x$

then

$$\kappa(\sigma(\mathscr{F})) \in \tau x .$$

We shall refer to (δ') as the iterated limit axiom. It should be noted that (α), (β), (γ) do not define a convergence space in the sense of KOWALSKY since one cannot conclude that if $\mathscr{F}\in\tau x$, $\mathscr{G}\in\tau x$ then $\mathscr{F}\wedge\mathscr{G}\in\tau x$. Sometimes we will write $\mathscr{F}\to x$ instead of $\mathscr{F}\in\tau x$.

In order to point up the significance of the individual axioms with regard to the associated closure operator, we prove several lemmas.

Lemma 1. *If* τ *satisfies* (α) and (β), then

(K1) $\bar{\emptyset}=\emptyset$,

(K2) $A\subset X\Rightarrow A\subset\bar{A}$,

(K3) $A\subset X,\quad B\subset X\Rightarrow\overline{A\cup B}=\bar{A}\cup\bar{B}$.

Proof. The proof of this lemma is trivial and hence omitted.

Lemma 2. *If* τ *satisfies* (α) *and* (δ') *then*

(K4) $A\subset X\Rightarrow\bar{A}=\bar{\bar{A}}$.

Proof. Since $\bar{A}$ satisfies (K2) it suffices to show that $\bar{A}\supset\bar{\bar{A}}$. Let $x\in\bar{\bar{A}}$ and $\mathscr{F}\in\tau x$ be such that $\bar{A}\in\mathscr{F}$, $\mathscr{F}_{\bar{A}}$ be the trace of $\mathscr{F}$ on $\bar{A}$. Let $I=\bar{A}$, $\psi=1_X$ and for $b\in I$ let $\sigma(b)\in\tau(1_X(b))=\tau b$ where $A\in\sigma(b)$. Clearly $1_X(\mathscr{F}_I)=\mathscr{F}\in\tau x$ since $I\in\mathscr{F}$. Define $\phi\in\bigtimes_{b\in I}\sigma(b)$ by $\phi(b)=A$. It follows that if $F\in\mathscr{F}$ then $\bigcup_{b\in F\cap I}\phi(b)=A$ which implies $A\in\kappa(\sigma(\mathscr{F}_I))$. $\kappa(\sigma(\mathscr{F}_I))\in\tau x$ by (δ') so that $x\in\bar{A}$.

Lemmas 1 and 2 show that the closure operator $\bar{A}$ is a Kuratowski closure operator. Our next task to answer the question, "Does $\mathscr{F}\in\tau x$ if $\mathscr{F}$ converges to x in the topology generated by the closure operator $\bar{A}$?" The following lemma is the key which provides an affirmative answer to this question.

Lemma 3. *Let* τ *satisfy* (α) *and* (δ'). *Then*

$$\dot{x}\geqq\bar{\mathscr{F}}\Leftrightarrow x\in\alpha_\tau(\mathscr{F}).$$

Proof. Recall that $\bar{\mathscr{F}}=[\{\bar{F}|F\in\mathscr{F}\}]$ and $\alpha_\tau(\mathscr{F})=\{x|\exists\mathscr{G}\geqq\mathscr{F},\ \mathscr{G}\in\tau x\}$. Suppose $x\in\alpha_\tau(\mathscr{F})$ and let $\mathscr{G}\geqq\mathscr{F}$, $\mathscr{G}\in\tau x$. If $F\in\mathscr{F}$ then $F\in\mathscr{G}$; $\mathscr{G}\in\tau x$ and $F\in\mathscr{G}$ imply $x\in\bar{F}$. Hence $x\in\bar{F}$ for each $F\in\mathscr{F}$ so that $\dot{x}\geqq\bar{\mathscr{F}}$. Conversely, let $\dot{x}\geqq\bar{\mathscr{F}}$ or equivalently $x\in\cap\{\bar{F}|F\in\mathscr{F}\}$. For each $F\in\mathscr{F}$ there exists a $\mathscr{G}_F\in\tau x$ and $F\in\mathscr{G}_F$. Let $I=\{F|F\in\mathscr{F}\}$, $\sigma(F)=\mathscr{G}_F$, $\psi:I\to X::F\to x$. For *any* $\Phi\in\mathbf{F}(I)$, $\psi(\Phi)=\dot{x}\in\tau x$. Axiom ($\delta'$) now assures that $\kappa(\sigma(\Phi))\in\tau x$ for any choice of $\Phi\in\mathbf{F}(I)$. Choose Φ as $\mathscr{S}(\mathscr{F})=[\{S_F|F\in\mathscr{F}\}]$ where $S_F=\{H\in\mathscr{F}|H\subset F\}$. We shall show that $\kappa(\sigma(\mathscr{S}(\mathscr{F})))\geqq\mathscr{F}$. Let F_0 be an arbitrary element of $\mathscr{F}$. If $H\in S_{F_0}$ then $H\subset F_0$ and $H\in\mathscr{G}_H$. Choose $\phi\in\bigtimes_{F\in\mathscr{F}}\sigma(F)$ such that $\phi(H)=H$. Then $\bigcup_{H\in S_{F_0}}\phi(H)=\bigcup_{H\subset F_0}H=F_0$ which shows that $\kappa(\sigma(\mathscr{S}(\mathscr{F})))\geqq\mathscr{F}$ so that $x\in\alpha_\tau(\mathscr{F})$.

Let $\psi\tau$ denote the topology generated by the closure operator $\bar{A}$. The neighborhood filter of a point x for $\psi\tau$ is of course given by

$$\mathscr{V}(x)=[\{V|x\notin\overline{V'}\}].$$

Lemma 4. *If* τ *satisfies* (α), (β), (γ) *and* (δ') *then*

$$\mathscr{F}\geqq\mathscr{V}(x)\Leftrightarrow\mathscr{F}\in\tau x.$$

Proof. Suppose $\mathscr{F} \ngeq \mathscr{V}(x)$ and $\mathscr{F} \in \tau x$. Then there exists a set $V \in \mathscr{V}(x)$ such that $V \not\supset F$ for every $F \in \mathscr{F}$, or $V' \cap F \neq \emptyset$. $\mathscr{F}_{V'} = [\{V' \cap F \mid F \in \mathscr{F}\}]$ is then a filter on X which is finer than $\mathscr{F}$ so that $\mathscr{F}_{V'} \in \tau x$ by axiom (β). This implies that $x \in \overline{V'}$ which contradicts the fact that $V \in \mathscr{V}(x)$. Conversely, suppose $\mathscr{F} \notin \tau x$ and $\mathscr{F} \geqq \mathscr{V}(x)$. Then there exists a $\mathscr{G} \geqq \mathscr{F}$ such that $x \notin \alpha_\tau(\mathscr{G})$ by axiom (γ). By Lemma 3 $\dot{x} \ngeq \overline{\mathscr{G}}$ so that there is a $\overline{G} \in \overline{\mathscr{G}}$ which does not contain x. Hence $G' \in \mathscr{V}(x)$ which is a contradiction since then both G and G' would belong to $\mathscr{G}$.

Lemmas 1 through 4 now produce the following theorem.

Theorem 1. *If the relation τ on X to $\mathbf{F}(X)$ satisfies (α)—(δ') then the associated closure operator is a Kuratowski closure operator. Furthermore, a filter $\mathscr{F}$ converges with respect to the closure topology iff $\mathscr{F} \in \tau x$.*

Regular topological spaces

In this section we shall show that the converse of axiom (δ') characterizes regular topological spaces. We shall indicate that a filter $\mathscr{F}$ on a topological space converges to the point x by $\mathscr{F} \in \tau x$. Of course Theorem 1 says that τ is a relation on $X \times \mathbf{F}(X)$ which satisfies (α)—(δ').

Let τ be a relation on $X \times \mathbf{F}(X)$. τ is said to satisfy *condition* (δ'') if:

for a given set I, $\mathscr{F} \in \mathbf{F}(I)$, $\sigma \in \mathbf{F}(X)^I$, if

$$(\delta'') \quad \begin{array}{ll} \text{(i)} & \forall\, b \in I, \quad \sigma(b) \in \tau \psi(b), \\ \text{(ii)} & \kappa(\sigma(\mathscr{F})) \in \tau x, \end{array}$$

then $\psi(\mathscr{F}) \in \tau x$.

Theorem 2. *Every regular topological space satisfies* (δ'').

Proof. Let V be a neighborhood of x, W a closed neighborhood of x, $W \subset V$. By (ii) there exist an $F \in \mathscr{F}$ and a $\phi \in \bigtimes_{b \in I} \sigma(b)$ such that $\bigcup_{b \in F} \phi(b) \subset W$. It follows that $\bigcup \overline{\phi(b)} \subset W$. However, by (i) $\psi(b) \in \overline{\phi(b)}$ for each $\phi(b) \in \sigma(b)$ and $b \in F$. Hence $\psi(F) \subset W \subset V$, which proves the theorem.

Theorem 3. *If a topological space satisfies* (δ'') *then it is regular.*

Proof. Suppose X is not regular. Then there exist an $x \in X$ and an open neighborhood V of x such that $\overline{U} \cap V' \neq \emptyset$ for each neighborhood U of x. Let $I = \mathscr{V}(x)$ ($\mathscr{V}(x)$ the neighborhood filter of x), $\mathscr{F} = \mathscr{S}(I)$. For each $U \in \mathscr{V}(x)$ let $\psi(U) \in \overline{U} \cap V'$ and $\sigma(U) \in \tau \psi(U)$ be such that $U \in \sigma(U)$. Clearly (i) is satisfied. To verify (ii) let $W \in I$. Choose a $S_W = \{V \in I \mid V \subset W\}$ from $\mathscr{F}$ and let $\phi(V) = V$ for each $V \in S_W$, then $\bigcup_{V \in S_W} \phi(V) = W$ which is included in W. Thus (ii) is satisfied. On the other hand it is clear that $\psi(\mathscr{F}) \notin \tau x$ since V is an open neighborhood of x.

Dieudonné [3] has characterized regular spaces as follows: Let E be a topological space, E_1, E_2 sets, $\phi: E_1 \times E_2 \to E$ a map, $\mathscr{F} \in \mathbf{F}(E_1)$, $\mathscr{G} \in \mathbf{F}(E_2)$. If

(1) $$\phi(\dot{x} \times \mathscr{G}) \to \psi(x)$$

(2) $$\phi(\mathscr{F} \times \mathscr{G}) \to a$$

implies

(3) $$\psi(\mathscr{F}) \to a$$

then E is regular. For brevity, we call this the D regularity condition.

The D condition will make sense if we only ask that E be a convergence space. We can then compare D with (δ'') in a convergence space setting.

Theorem 4. *In a convergence space E if D holds then (δ'') also holds.*

Proof. Assume D holds and let (i), (ii) of (δ'') hold. Let $E_1 = I$, $E_2 = E^I$, $\phi : E_1 \times E_2 \to E :: (b, F) \to F(b)$.

If we let

$$\mathscr{G} = \left[\left\{ \bigtimes_{a \in I} \lambda(a) \,\middle|\, \lambda \in \bigtimes_{a \in I} \sigma(a) \right\}\right]$$

then $\phi(\dot{b} \times \mathscr{G}) = \sigma(b) \in \tau\psi(b)$ and a short computation shows that $\phi(\mathscr{F} \times \mathscr{G}) = \kappa\sigma(\mathscr{F})$. Hence (δ'') holds.

Regular convergence spaces

A *regular convergence structure* on the set X is a relation τ on $X \times \mathbf{F}(X)$ that satisfies (α), (β), (δ'') and

(ε) $\quad \mathscr{F} \in \tau x, \quad \mathscr{G} \in \tau x \Rightarrow \mathscr{F} \wedge \mathscr{G} \in \tau x.$

A *regular convergence space* is a pair (X, τ) where X is a set, τ a regular convergence structure on X.

The first observation we make is that if $\{\tau_\alpha\}$ is an arbitrary collection of regular convergence structures then $\sup \tau_\alpha$ is itself regular. Next we note that if τ is regular and $f \in Y^X$ then $f(\kappa(\sigma\mathscr{F}))) = \kappa(f \circ \sigma(\mathscr{F}))$ which is a filter on Y. f actually induces a map $\mathbf{F}(X) \to \mathbf{F}(Y)$ which we have again denoted by f. This observation allows us to show that regular convergence spaces behave decently under inverse limits.

Let (E_α, τ_α) be a family of convergence spaces and $\theta_\alpha : E \to E_\alpha$ a family of maps. The projective limit convergence structure τ in E is defined as follows (cf. [4], p. 288); $\tau = \varprojlim(\tau_\alpha, \theta_\alpha)$ is the coarsest convergence structure on E which makes all maps θ_α continuous. A filter $\mathscr{F}$ or E τ-converges to x if and only if for every α, $\theta_\alpha(\mathscr{F})$ τ_α-converges to $\theta_\alpha x$.

Theorem 5. *If every τ_α is regular, then so is τ.*

Proof. Let again I be any non-empty set, $\mathscr{F} \in \mathbf{F}(I)$, $\sigma \in \mathbf{F}(E)^I$ such that for each $b \in I$, $\sigma(b) \to \psi(b)$ in E and assume that $\kappa\sigma(\mathscr{F}) \in \tau x$. Thus, we have that for every $\alpha \in A$, $\theta_\alpha\sigma(b) \to \theta_\alpha\psi(b)$ and that $\theta_\alpha\kappa\sigma(\mathscr{F}) \to \theta_\alpha x$. Since

$$\kappa\sigma(\mathscr{F}) = \sup_{F \in \mathscr{F}} \inf_{b \in F} \sigma(b),$$

the filter generated by sets of the form $\bigcup_{b \in F} S_b$, $S_b \in \sigma(b)$, it follows that $\theta_\alpha\kappa\sigma(\mathscr{F}) = \kappa\theta_\alpha\sigma(\mathscr{F})$: the filter on the right is generated by the sets $\bigcup_{b \in F} T_b$, $T_b \in \theta_\alpha\sigma(\mathscr{F})$, so that we may assume $T_b = \theta_\alpha(S_b)$, $S_b \in \sigma(b)$. But then

$$\bigcup_{b \in F} \theta_\alpha(S_b) = \theta_\alpha\left(\bigcup_{b \in F} S_b\right),$$

which is a generator of $\theta_\alpha\kappa\sigma(\mathscr{F})$. The same argument yields inclusion the other way, so that the equality of the two filters is proved.

Since every τ_α is regular, we now conclude that $\theta_\alpha \psi(\mathscr{F}) \to \theta_\alpha x$ for every $\alpha \in A$. It follows that $\psi(\mathscr{F}) \to x$ in E, thus proving the theorem.

Remark. It my be useful to note the assertion proved in the preceding theorem more explicitly: Given any two sets E, F, a map $\theta: E \to F$ and $\sigma \in \mathbf{F}(E)^I$, we have

$$\theta \kappa \sigma(\mathscr{F}) = \kappa \theta \sigma(\mathscr{F}).$$

Theorem 6. *A product* $\mathsf{X}_\alpha \tau_\alpha$ *is regular if and only if each* τ_α *is regular.*

Proof. There only remains the necessity of the condition, the sufficiency being obvious from the theorem we just proved. Thus, suppose $\mathsf{X}_\alpha \tau_\alpha$ is regular and let $\sigma \in \mathbf{F}(E_\alpha)^I$ be given such that $\sigma(b) \to \psi(b) \in E_\alpha$ for each $b \in I$ and $\kappa\sigma(\mathscr{F}) \to x_\alpha$. We have to show that $\psi(\mathscr{F}) \to x_\alpha$ in E_α.

To this end, choose arbitrary points $x_\beta \in E_\beta$, $\beta \neq \alpha$, and define, for any filter $\mathscr{F}_\alpha \in \mathbf{F}(E_\alpha)$, a filter $\mathscr{F}_\alpha^* \in \mathbf{F}(E)$ by $\mathscr{F}_\alpha^* = \mathsf{X}\mathscr{G}_\beta$ with $\mathscr{G}_\alpha = \mathscr{F}_\alpha$ and, for $\beta \neq \alpha$, $\mathscr{G}_\beta = \dot{x}_\beta$. It is clear that $\mathscr{F}_\alpha^*$ converges in E *if and only if* $\mathscr{F}_\alpha$ converges in E_α and that is if $y_\alpha^* = (z_\beta)_{\beta \in A}$ (with $z_\alpha = y_\alpha$, $z_\beta = x_\beta$ for $\beta \neq \alpha$) is a limit of $\mathscr{F}_\alpha^*$, there y_α is a limit of $\mathscr{F}_\alpha$ and vice versa. Moreover, if w is a map $Z \to E_\alpha$, we may define $w^*: Z \to E$ by letting $w^*(z)$ be $w(z)^*$ for each $z \in Z$.

With these notations, we conclude that $\sigma(b)^* \to \psi^*(b)$ for each $b \in I$ and that $(\kappa\sigma(\mathscr{F}))^* \to x_\alpha^*$. Let $\sigma^*(b) = \sigma(b)^* \in \mathbf{F}(E)$ for each $b \in I$. Then $(\kappa\sigma(\mathscr{F}))^* \leqq \kappa\sigma^*(\mathscr{F})$: the left hand side is generated by all products of the form $\mathsf{X}A_\beta$ with $A_\alpha \in \kappa\sigma(\mathscr{F})$, $A_\beta \in \dot{x}_\beta$ for $\beta \neq \alpha$, and $A_\beta = E_\beta$ for almost all β. We may choose A_α to be $\bigcup_{b \in F} S_b$, $S_b \in \sigma(b)$. Obviously, then $\mathsf{X}A_\beta$ contains $\bigcup_{b \in F} S_b^*$ where $S_b^* \in \sigma(b)^*$, i.e. $S_b^* = \mathsf{X}T_\beta^b$, $T_\alpha^b \in \sigma(b)$, $T_\beta^b \in \dot{x}_\beta$ for $\beta \neq \alpha$ and $T_\beta = E_\beta$ for almost all β — provided we choose, say, $T_\alpha^b = S_b$ and $T_\beta^b = A_\beta$ for all $b \in F$. The union $\bigcup_{b \in F} S_b^*$ is a generator of $\kappa\sigma^*(\mathscr{F})$, which proves the inequality $(\kappa\sigma(\mathscr{F}))^* \leqq \kappa\sigma^*(\mathscr{F})$. From this we conclude that $\kappa\sigma^*(\mathscr{F}) \to x_\alpha^*$. The regularity of $\mathsf{X}\tau_\alpha$ now implies that $\psi^*(\mathscr{F}) \to x_\alpha^*$. Thus, in particular, $p_\alpha \psi^*(\mathscr{F}) \to x_\alpha$, p_α being the projection $E \to E_\alpha$. The construction of $\psi^*(\mathscr{F})$ immediately implies that $p_\alpha \psi^*(\mathscr{F}) = \psi(\mathscr{F})$, whence finally $\psi(\mathscr{F}) \to x_\alpha$. This proves the regularity of (E_α, τ_α).

If (E, τ) is a convergence space and A a subset of E then A may be considered as a subspace with the coarsest convergence structure on A such that the inclusion map is continuous. Theorem 5 then guarantees that a subspace of a regular convergence space is a regular convergence space.

Theorem 7. *If* τ *is a regular and* T_1 *convergence structure on* X, *then* τ *is* T_2.

Proof. Suppose there exist distinct points x, y of X and $\tau x \cap \tau y \neq \theta$. Let $\mathscr{F} \in \tau x \cap \tau y$, $I = \tau y$, $\sigma: \tau y \to \mathbf{F}(X) :: \mathscr{G} \to \mathscr{G}$ $\psi: \tau y \to X :: \mathscr{G} \to y$. If $\Phi \in \mathbf{F}(\tau y)$ then $\psi(\Phi) = \dot{y}$. If $\mathscr{F} \in \tau y$, $\mathscr{F} \in \mathbf{F}(\tau y)$ and $\kappa(\sigma(\mathscr{F})) = \mathscr{F} \in \tau x$. Since τ is regular $\psi(\mathscr{F}) \in \tau x$. But $\psi(\mathscr{F}) = \dot{y} \in \tau x$ and so τ is not T_1.

Theorem 8. *Let* τ *be a convergence structure on* X. *If* $\mathscr{F} \in \tau x$ *implies* $\bar{\mathscr{F}} \in \tau x$, *then* τ *is a regular convergence structure.*

Proof. Let I be a set, $\mathscr{F} \in \mathbf{F}(I)$, $\sigma \in \mathbf{F}(E)^I$. Suppose $\sigma(b) \in \tau \psi(b)$ for $b \in I$ and that $\kappa(\sigma(\mathscr{F})) \in \tau x$. Let $\bigcup_{b \in F} \theta(b)$ be a generator of $\kappa(\sigma(\mathscr{F}))$. Since $\sigma(b) \in \tau \psi(b)$,

$\psi(b) \in \overline{\theta(b)}$ for all $b \in F$. Hence $\psi(F) \subset \bigcup_{b \in F} \overline{\theta(b)} \subset \overline{\bigcup_{b \in F} \theta(b)}$ and since $\overline{\bigcup_{b \in F} \theta(b)}$ is a generator of $\overline{\kappa(\sigma(\mathscr{F}))}$ it follows that $\psi(\mathscr{F}) \geqq \overline{\kappa(\sigma(\mathscr{F}))}$. By hypothesis $\overline{\kappa(\sigma(\mathscr{F}))} \in \tau x$ so that $\psi(\mathscr{F}) \in \tau x$.

In the following 2 examples let X be a topological space and Y a uniform space.

Example 1. Continuous convergence γ_c on the function space Y^X is defined by (cf. [2]) $\Phi \in \gamma_c(f) \cdot \overset{\text{df}}{=} \cdot \forall x \in X,\ \Phi(\mathscr{V}(x)) \geqq \mathscr{J}[f(x)]$ where $\mathscr{V}(x)$ denotes the neighborhood filter of x in X and $\mathscr{J}$ the entourage filter for the uniform structure on Y. Let $x \in X$, V be a closed entourage in $\mathscr{J}$. Then there exists an $A_x \in \Phi$ and a $U_x \in \mathscr{V}(x)$ satisfying $A_x(U_x) \subset V[f(x)]$. For $g \in \bar{A}_x^{\tau_s}$, there exists a $\Psi \in \mathbf{F}(Y^X)$ satisfying $\Psi \underset{\tau_s}{\rightarrow} g$ or $\Psi(y) \rightarrow g(y)$ for each $y \in U_x$. Hence $g(y) \in \overline{A_x(U_x)}$ so that $\bar{A}_x^{\tau_s}(U_x) \subset \overline{A_x(U_x)} \subset V[f(x)]$, V being a closed entourage. This shows that for each $x \in X$, $\bar{\Phi}^{\tau_s} \in \gamma_c(f)$. Since $\gamma_c \geqq \tau_s$ it follows that $\bar{\Phi}^{\gamma_c} \in \gamma_c(f)$.

Example 2. Uniform convergence η of a filter $\Phi \in \mathbf{F}(Y^X)$ at a point is defined by $\Phi \in \eta f \cdot \overset{\text{df}}{=} \cdot \forall x \in X$, $\forall$ entourage V in $\mathscr{J}$ $\exists\, U_x \in \mathscr{V}(x)$ and $A_x \in \Phi$ such that $(A_x(y), f(y)) \in V$ for all $y \in U_x$. Now let V be a closed entourage. Then $\overline{A_x(y)} \subset V[f(y)]$ for $y \in U_x$ and so $\bar{A}_x^{\tau_s}(y) \subset \overline{A_x(y)} \subset V[f(x)]$, $y \in U_x$. Hence $\bar{\Phi}^{\tau_s} \in \eta f$ and since $\eta \geqq \tau_s$, $\bar{\Phi}^{\eta} \in \eta f$.

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(Received August 16, 1965)