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Special Coordinate Coverings of Riemann Surfaces

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1. Introduction

The general uniformization theorem ensures that any Riemann surface can be represented as the quotient space of a subset of the complex projective line by a discontinuous group of projective (linear fraction) transformations. There are generally a great many different ways in which a given surface can be so represented, an observation which has been of some use in recent work on the moduli of Riemann surfaces, [3]. Any such representation provides a special coordinate covering of the Riemann surface, with the property that the local analytic coordinates are related to one another by projective transformations. The aim of the present paper is to investigate some aspects of the structures determined by special coordinate coverings of this sort, principally for compact Riemann surfaces. An underlying motivation, in addition to the possibility of gaining some further insights to an old topic by varying the point of view, is the quest for phases of the classical uniformization theorem which might suggest generalizations to several complex variables; there are manifolds of several dimensions which admit special coordinate coverings of a similar sort, for instance, quotient spaces of the Siegel upper half-spaces by discontinuous groups of transformations.

The following is a brief outline of the contents of this paper. Section 2 is devoted to establishing the notation and terminology to be used, and to defining the structures to be investigated. Section 3 contains a discussion of relationships between these special coordinate structures and certain canonically associated flat fibre bundles; the principal result is that the bundles completely describe the structures to which they are associated. Section 4 contains a differential-geometric discussion of these structures and their associated bundles. Section 5, as a slight digression, briefly sketches the role of these structures in an intrinsic formulation of the Eichler cohomology groups on Riemann surfaces. Section 6 contains a discussion of the classical uniformizations of Riemann surfaces, from the point of view adopted here. There is also included an appendix, containing a review of some relevant results about complex vector bundles over Riemann surfaces, a discussion of flat bundles associated to a complex vector bundle, and a sketch of an alternative approach to the topics treated in Section 4.

2. Notation and terminology

A *coordinate covering* $\{U_\alpha, z_\alpha\}$ of a two-dimensional manifold M consists of an open covering $\{U_\alpha\}$ of M, together with homeomorphisms $z_\alpha: U_\alpha \to V_\alpha$ (the *coordinate mappings*) from the sets U_α to open subsets V_α of the complex plane $\mathbf{C}$. On the intersections $U_\alpha \cap U_\beta \subset M$ there are defined two homeomorphisms into $\mathbf{C}$; the compositions

$$f_{\alpha\beta} = z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cap U_\beta) \to z_\alpha(U_\alpha \cap U_\beta),$$

which are homeomorphisms between open subsets of the two domains V_α, $V_\beta \subset \mathbf{C}$, are called the *coordinate transition functions* for the given coordinate covering. Note that the union of two coordinate coverings, consisting of all the open sets and coordinate mappings from the two, is again a coordinate covering; the set of coordinate transition functions for this union of course contains more than just the sets of coordinate transition functions from the two separate coordinate coverings.

A *complex analytic coordinate covering* of M is a coordinate covering in which the coordinate transition functions are complex analytic mappings; two such coordinate coverings are called *equivalent* if their union is again a complex analytic coordinate covering, and an equivalence class of such coordinate coverings is called a *complex structure* on M. In the traditional terminology, a two-dimensional manifold with a fixed complex structure is called a *Riemann surface*. Similarly, a *complex projective* (or *affine*) *coordinate covering* of M is a coordinate overing in which the coordinate transition functions are complex projective (or affine) mappings; recall that a complex projective mapping f is a complex analytic mapping of the form $f(z) = (az+b)(cz+d)^{-1}$ for complex constants a, b, c, d with $ad - bc \neq 0$, and a complex affine mapping is the special case of a complex projective mapping in which $c = 0$. Two such coverings are called *equivalent* if their union is a coordinate covering of the same kind, and an equivalence class of these coverings is called a *complex projective* (or *affine*) *structure* on M. In defining a complex projective structure, it is at times more convenient to envisage the sets V_α as lying in the projective line, for then the points z for which $cz + d = 0$ do not require special attention. The complex projective or affine structures are special cases of the locally homogeneous structures introduced by EHRESMANN, [6]. Note that a complex projective structure on M belongs to a unique complex analytic structure; the complex projective structure is said to be *subordinate* to that complex structure. Similarly, a complex affine structure on M is subordinate to a unique complex projective structure.

If $\{U_\alpha, z_\alpha\}$ is a complex projective (or affine) coordinate covering of M, the coordinate transition functions $\{f_{\alpha\beta}\}$ are elements of the one-dimensional complex projective group $\mathscr{P}$ (or of the one-dimensional complex affine group $\mathscr{A}$) associated to the intersections $U_\alpha \cap U_\beta \subset M$; and for any triple intersection $U_\alpha \cap U_\beta \cap U_\gamma \subset M$, these group elements satisfy the condition $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$. Therefore, the set $\{f_{\alpha\beta}\}$ of all these transformations define a coordinate bundle over M, the group of the bundle being the complex projective group $\mathscr{P}$ (or the complex affine group $\mathscr{A}$) and the fibre being the complex projective line **P** (or the complex affine line **C**); the terminology used here follows STEENROD [13]. Note that the element $f_{\alpha\beta}$ is constant, when considered as a mapping from the intersection $U_\alpha \cap U_\beta$ into the group $\mathscr{P}$ (or $\mathscr{A}$); hence the bundle is a *flat coordinate bundle*, or in the terminology of [13, section 13], a coordinate bundle with totally disconnected group. The coordinate bundles associated to two equivalent complex projective (or affine) coordinate coverings are equivalent, in the category of flat coordinate bundles; hence to each complex projective (or affine) structure on M there corresponds a unique flat fibre bundle, which will be called the *fibre bundle associated to the complex projective* (or *affine*) *structure.* A flat fibre bundle associated to some complex projective (or affine) structure on the surface M will be called an *indigenous* bundle on M; and a flat fibre bundle associated to some complex projective (or affine) structure subordinate to a fixed complex structure on the surface M will be called an indigenous bundle to that complex structure on M (or to the Riemann surface M, for short).

3. The associated bundles

The first topic for more detailed consideration is the relationship between complex projective (or affine) structures and their associated fibre bundles. It is an immediate consequence of the definitions that if φ is the fibre bundle associated to a complex projective structure on M, then the coordinate mappings in a coordinate covering representing that structure compose a cross-section of the bundle φ, indeed, a cross-section of a very special sort. For any fibre bundle over M, the set of points in the bundle space lying over a suitably small open neighborhood in M is homeomorphic to the Cartesian product of the fibre with that neighborhood in M, hence that set admits a projection onto the fibre; and for a flat fibre bundle, any two such projections differ only by a fixed homeomorphism of the fibre. Now the compositions of these local projections with a cross-section of the bundle associates to the cross-section a family of local mappings into the fibre, which are determined uniquely up to homeomorphisms of the fibre. The cross-section of φ arising from a complex projective structure has the particular property that its associated family of mappings into the fibre consists entirely of local homeomorphisms; moreover, any such cross-section of φ determines a complex projective structure to which is associated the bundle φ. It is thus a trivial assertion that a flat fibre bundle φ is indigenous to M if and only if it admits a continuous cross-section which determines local homeomorphisms into the fibre. The further investigation of this matter by purely topological methods, to determine directly the class of indigenous bundles on M, appears to be rather difficult; but the problem is quite amenable analytically, and will be discussed further in the following sections, the present section being devoted to the uniqueness of the structure to which a bundle on a compact Riemann surface is associated.

For the sake of completeness, we begin with the rather trivial case of the affine structures.

Lemma 1. *If a compact topological surface admits a complex affine structure, then the surface has genus* 1.

Proof. Let $\{U_\alpha, z_\alpha\}$ be a complex affine coordinate covering of the surface; in each intersection $U_\alpha \cap U_\beta$ the coordinate mappings are related by $z_\alpha = f_{\alpha\beta}(z_\beta) = a_{\alpha\beta} z_\beta + b_{\alpha\beta}$ for some complex constants $a_{\alpha\beta} \neq 0, b_{\alpha\beta}$. The canonical bundle of the complex structure defined on the surface by this affine covering has the transition functions $k_{\alpha\beta} = (dz_\alpha/dz_\beta)^{-1} = 1/a_{\alpha\beta}$, [2]; since these are constant, the canonical bundle has zero Chern class, hence the surface has genus 1.

Theorem 1. *An indigenous flat affine bundle to a compact Riemann surface is associated to a unique affine structure on the surface.*

Proof. Let $\{U_\alpha, z_\alpha\}$ and $\{U_\alpha, w_\alpha\}$ be two complex affine coordinate coverings of the Riemann surface, having the same associated affine coordinate bundles; thus in each intersection $U_\alpha \cap U_\beta$ the coordinate mappings are related by $z_\alpha = a_{\alpha\beta} z_\beta + b_{\alpha\beta}$ and $w_\alpha = a_{\alpha\beta} w_\beta + b_{\alpha\beta}$ for some complex constants $a_{\alpha\beta} \neq 0$, $b_{\alpha\beta}$. Since the two affine structures are by assumption subordinate to the same complex structure, in each neighborhood U_α the coordinate mappings are also related by $w_\alpha = f_\alpha(z_\alpha)$ for some complex analytic function f_α. Now the derivatives dw_α/dz_α are also analytic functions in the neighborhoods U_α; and it is readily seen that $dw_\alpha/dz_\alpha = dw_\beta/dz_\beta$ in each intersection $U_\alpha \cap U_\beta$, so that these derivatives define a global analytic function on the Riemann surface. Since the surface is compact, this function must be a constant c, and hence $w_\alpha = cz_\alpha + d_\alpha$ in each neighborhood U_α. The two affine coordinate coverings are therefore necessarily equivalent, and this suffices to prove the desired result.

It should be remarked that this theorem is false without the restriction that the affine structures be subordinate to the same complex structure.

Next we turn to the rather more interesting case of the projective structures. Suppose that φ^0 is a one-dimensional flat complex projective bundle over a topological space M, and is defined by transition functions $\varphi^0_{\alpha\beta}$ in terms of a coordinate covering $\{U_\alpha, z_\alpha\}$ for the space M. Each projective transformation $\varphi^0_{\alpha\beta}$ can be represented by a matrix

$$\varphi_{\alpha\beta} = \begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{pmatrix}$$

of complex constants; and if this matrix is normalized by requiring that det $\varphi_{\alpha\beta} = 1$, it is then uniquely determined up to sign. For any intersection $U_\alpha \cap U_\beta \cap \cap U_\gamma$ these matrices must satisfy the relation that $\varphi_{\alpha\beta}\varphi_{\beta\gamma} = \pm \varphi_{\alpha\gamma}$. If the signs of the various matrices can be so chosen that the positive sign holds in all of these relations, then this collection of matrices defines a two-dimensional flat complex vector bundle φ over the space M; the projective bundle φ^0 will then be said to be *associated* to the vector bundle φ. Although not all flat projective bundles are associated to flat vector bundles in this manner, the indigenous projective bundles always are. (A further discussion of complex vector bundles over a Riemann surface, giving those properties which will be used in the subsequent discussion, will be found in the appendix to the present article.)

Theorem 2. *Let M be a compact Riemann surface of genus $g>1$. A flat complex projective bundle φ^0 is indigenous to M if and only if φ^0 is associated to a flat complex vector bundle φ for which* $\det\varphi=1$ *and* $|\operatorname{div}\varphi|=g-1$. *Further, if $\{U_\alpha, z_\alpha\}$ is a complex projective coordinate covering of M for which the mapping functions z_α compose a cross-section of φ^0, then there is a complex analytic cross-section $h_\alpha=(h_{1\alpha}, h_{2\alpha})$ of $\chi\otimes\varphi$, for some complex line bundle χ with $|\chi|=0$, such that $z_\alpha=h_{1\alpha}/h_{2\alpha}$ in U_α.*

Proof. First, suppose that φ^0 is a flat complex projective bundle indigenous to M; then there will be a complex projective coordinate covering $\{U_\alpha, z_\alpha\}$ of M for which the coordinate mappings in an intersection $U_\alpha\cap U_\beta$ satisfy

$$z_\alpha=\varphi^0_{\alpha\beta}(z_\beta)=\frac{a_{\alpha\beta}z_\beta+b_{\alpha\beta}}{c_{\alpha\beta}z_\beta+d_{\alpha\beta}},$$

where $\{\varphi^0_{\alpha\beta}\}$ are transition functions defining the bundle φ^0. Note that the canonical bundle κ of M is defined by the transition functions

$$k_{\alpha\beta}=(c_{\alpha\beta}z_\beta+d_{\alpha\beta})^2.$$

Since $|\kappa|=2g-2$, there exists a complex line bundle ξ with $|\xi|=0$ such that the bundle $\xi\otimes\kappa$ has a holomorphic cross-section $\{g_\alpha\}$, the divisor of which consists precisely of $g-1$ double zeros. Since $|\xi|=0$, that line bundle can be defined by transition functions $\{\xi_{\alpha\beta}\}$ which are complex constants of modulus 1; and the functions $\{g_\alpha\}$ therefore satisfy

$$g_\alpha=\xi_{\alpha\beta}(c_{\alpha\beta}z_\beta+d_{\alpha\beta})^2 g_\beta \quad \text{in} \quad U_\alpha\cap U_\beta.$$

Now in each neighborhood U_α select a branch of the function $h_{2\alpha}=(g_\alpha)^{1/2}$. These functions $\{h_{2\alpha}\}$ are well-defined holomorphic functions in U_α, since $\{g_\alpha\}$ has only double zeros; moreover they satisfy relations of the form

$$h_{2\alpha}=\chi_{\alpha\beta}(c_{\alpha\beta}z_\beta+d_{\alpha\beta})\,h_{2\beta} \quad \text{in} \quad U_\alpha\cap U_\beta,$$

where $\chi_{\alpha\beta}$ are complex constants of modulus 1, and their global divisor on M consists of $g-1$ simple zeros. The pair of functions $h_{1\alpha}=z_\alpha h_{2\alpha}$ and $h_{2\alpha}$ therefore satisfy

$$h_{1\alpha}=\chi_{\alpha\beta}(a_{\alpha\beta}h_{1\beta}+b_{\alpha\beta}h_{2\beta})$$
$$h_{2\alpha}=\chi_{\alpha\beta}(c_{\alpha\beta}h_{1\beta}+d_{\alpha\beta}h_{2\beta}).$$

On the one hand, it follows therefrom that the matrices

$$\varphi^*_{\alpha\beta}=\chi_{\alpha\beta}\begin{pmatrix}a_{\alpha\beta} & b_{\alpha\beta}\\ c_{\alpha\beta} & d_{\alpha\beta}\end{pmatrix}$$

define a flat complex vector bundle φ^* over M, to which the projective bundle φ^0 is associated; the desired vector bundle φ is then the one defined by the transition functions $\varphi_{\alpha\beta}=(\det\varphi^*_{\alpha\beta})^{-1}\,\varphi^*_{\alpha\beta}$. Furthermore, the functions $h_{1\alpha}, h_{2\alpha}$ compose a holomorphic cross-section of the bundle φ^* and the common divisor of this cross-section consists in the $g-1$ simple zeros of $h_{1\alpha}$; therefore, $|\operatorname{div}\varphi|=g-1$. Finally, since $z_\alpha=h_{1\alpha}/h_{2\alpha}$, the last statement of the theorem follows as well.

Next, suppose that φ^0 is a flat complex projective bundle associated to a flat complex vector bundle φ for which $\det\varphi = 1$ and $|\operatorname{div}\varphi| = g-1$. Select a complex analytic coordinate covering $\{U_\alpha, z_\alpha\}$ of M, in terms of which the bundle φ is defined by transition functions

$$\varphi_{\alpha\beta} = \begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{pmatrix}.$$

Since $|\operatorname{div}\varphi| = g-1$, there is a complex line bundle ξ over M such that $|\xi| = g-1$ and that $\xi^{-1}\otimes\varphi$ has a holomorphic cross-section $(h_{1\alpha}, h_{2\alpha})$. There must also exist a complex line bundle η over M such that $|\eta| = 0$ and that $\eta\otimes\xi$ has a holomorphic cross-section g_α; the bundle η can be defined by transition functions $\eta_{\alpha\beta}$ which are complex constants of modulus 1. The functions $f_{i\alpha} = g_\alpha h_{i\alpha}$ $(i = 1, 2)$ then compose a holomorphic cross-section of the flat complex vector bundle $\eta\otimes\varphi$, that is,

$$f_{1\alpha} = \eta_{\alpha\beta}(a_{\alpha\beta} f_{1\beta} + b_{\alpha\beta} f_{2\beta}), \quad f_{2\alpha} = \eta_{\alpha\beta}(c_{\alpha\beta} f_{1\beta} + d_{\alpha\beta} f_{2\beta}).$$

Now introduce the holomorphic matrix-valued functions

$$H_\alpha = \begin{pmatrix} h_{1\alpha} & h'_{1\alpha} \\ h_{2\alpha} & h'_{2\alpha} \end{pmatrix}, \quad F_\alpha = \begin{pmatrix} f_{1\alpha} & f'_{1\alpha} \\ f_{2\alpha} & f'_{2\alpha} \end{pmatrix},$$

the primes denoting the derivatives with respect to the local coordinates z_α. The matrices F_α clearly satisfy

$$F_\alpha = \eta_{\alpha\beta}\varphi_{\alpha\beta}F_\beta \begin{pmatrix} 1 & 0 \\ 0 & k_{\alpha\beta} \end{pmatrix} \quad \text{in} \quad U_\alpha \cap U_\beta,$$

where $k_{\alpha\beta} = (dz_\alpha/dz_\beta)^{-1}$ are transition functions defining the canonical bundle κ; and consequently

$$\det F_\alpha = \eta^2_{\alpha\beta} k_{\alpha\beta} \det F_\beta.$$

It is easy to verify that $\det F_\alpha$ is not identically zero. Recalling that $f_{i\alpha} = g_\alpha h_{i\alpha}$, it follows that $\det F_\alpha = g^2_\alpha \det H_\alpha$, and therefore that

$$\det H_\alpha = \xi^{-2}_{\alpha\beta} k_{\alpha\beta} \det H_\beta;$$

that is to say, the functions $\det H_\alpha$ compose a non-trivial holomorphic cross-section of the bundle $\xi^{-2}\otimes\kappa$. Since $|\xi^{-2}\otimes\kappa| = -2|\xi| + (2g-2) = 0$, the functions $\det H_\alpha$ are nowhere vanishing on the Riemann surface M. The desired result follows readily from this. For introduce the local meromorphic functions $w_\alpha = h_{1\alpha}/h_{2\alpha}$ in the neighborhoods U_α; these compose an analytic cross-section of the flat projective bundle φ^0, since also $w_\alpha = f_{1\alpha}/f_{2\alpha}$. Recall that the functions $h_{1\alpha}$ and $h_{2\alpha}$ have no common zeros, and observe that $w'_\alpha = dw_\alpha/dz_\alpha = -(h_{2\alpha})^{-2} \det H_\alpha$ and that $(1/w_\alpha)' = (h_{1\alpha})^{-2} \det H_\alpha$; since $\det H_\alpha \neq 0$, the mapping w_α is actually a local homeomorphism into projective space, hence the bundle φ^0 is indigenous, which completes the proof.

Remark. A more detailed description of the possible vector bundles φ satisfying the hypotheses of the preceding theorem will be found in the appendix to this paper; see, in particular, Proposition A 4.

Theorem 3. *An indigenous flat projective bundle to a compact Riemann surface of genus $g > 1$ is associated to a unique projective structure on the surface.*

Proof. Let $\{U_\alpha, z_\alpha\}$ and $\{U_\alpha, w_\alpha\}$ be two complex projective coordinate coverings of the Riemann surface, having the same associated projective coordinate bundles. By Theorem 2 there will be a flat complex vector bundle φ, with $\det\varphi = 1$ and $|\operatorname{div}\varphi| = g - 1$, to which this projective bundle is associated; and there will be complex line bundles ξ and η, with $|\xi| = |\eta| = 0$, and holomorphic cross-sections $(g_{1\alpha}, g_{2\alpha})$ of $\xi \otimes \varphi$ and $(h_{1\alpha}, h_{2\alpha})$ of $\eta \otimes \varphi$ such that $z_\alpha = g_{1\alpha}/g_{2\alpha}$ and $w_\alpha = h_{1\alpha}/h_{2\alpha}$. Introducing the matrix functions

$$F_\alpha = \begin{pmatrix} g_{1\alpha} & h_{1\alpha} \\ g_{2\alpha} & h_{2\alpha} \end{pmatrix},$$

observe that these satisfy

$$F_\alpha = \varphi_{\alpha\beta} F_\beta \begin{pmatrix} \xi_{\alpha\beta} & 0 \\ 0 & \eta_{\alpha\beta} \end{pmatrix} \quad \text{in} \quad U_\alpha \cap U_\beta,$$

hence that $\det F_\alpha = \xi_{\alpha\beta}\eta_{\alpha\beta} \det F_\beta$; that is, the functions $\det F_\alpha$ compose a cross-section of the line bundle $\xi \otimes \eta$. Since $|\xi \otimes \eta| = |\xi| + |\eta| = 0$, either $\det F_\alpha$ is nowhere vanishing, or $\det F_\alpha \equiv 0$. In the first case, in which the matrices F_α are non-singular, the bundle φ is analytically equivalent to the bundle $\xi^{-1} \otimes \eta^{-1}$, hence $|\operatorname{div}\varphi| = 0$, which is a contradiction. Therefore $\det F_\alpha \equiv 0$, in which case the two vectors $(g_{1\alpha}, g_{2\alpha})$ and $(h_{1\alpha}, h_{2\alpha})$ are linearly dependent, and hence $z_\alpha = w_\alpha$. This shows the uniqueness of the projective structure, which was the desired result.

As in the affine case, this theorem is false without the restriction that the projective structures be subordinate to the same complex structure; for examples, see the discussion in [3].

4. Complex analytic connections

A differential-geometric description of the possible complex projective (or affine) structures and their associated bundles on a Riemann surface M can be obtained rather easily from an analysis of the formal properties of the differential operators defining projective or affine mappings. Suppose that $f: U \to V$ is a complex analytic local homeomorphism between two open subsets $U, V \subset \mathbf{C}$, so that $f'(z) \neq 0$ for all points $z \in U$. Introduce the differential operators θ_1, θ_2 defined by

(1) $$\theta_1 f(z) = f''(z)/f'(z),$$

(2) $$\theta_2 f(z) = (2f'(z)\, f'''(z) - 3f''(z)^2)/2f'(z)^2\,;$$

thus, $\theta_r f$ is a complex analytic function defined in the same open subset $U \subset \mathbf{C}$, $(r = 1, 2)$. If $g: V \to W$ is another complex analytic local homeomorphism, then so is the composition $h = g \circ f: U \to W$; and a straightforward calculation shows that

(3) $$\theta_r h(z) = \theta_r g(f(z))\, f'(z)^r + \theta_r f(z), \; r = 1, 2.$$

Introducing the families $\mathscr{F}_r$ of complex analytic local homeomorphisms f such that $\theta_r f(z) \equiv 0$, it is clear from (3) that each family is closed under composition of mappings, whenever composition is defined; that is to say, the families $\mathscr{F}_r$ are Lie pseudogroups of complex analytic transformations, defined by the differential equations (1) and (2). Indeed, the family $\mathscr{F}_1$ clearly consists of all complex affine transformations; and, since θ_2 is the classical Schwarzian derivative [4], the family $\mathscr{F}_2$ consists of all complex projective transformations. It is not difficult to show that $\mathscr{F}_1$ and $\mathscr{F}_2$ are essentially the only such pseudogroups in one complex variable, thus explaining their special role in the subject; but that matter belongs to the classification theory of pseudogroups, [5].

Select a complex analytic coordinate covering $\{U_\alpha, z_\alpha\}$ of the Riemann surface M, and consider the coordinate transition functions $z_\alpha = f_{\alpha\beta}(z_\beta)$ defined in the intersections $U_\alpha \cap U_\beta$. The expressions $\sigma_{r\alpha\beta} = \theta_r f_{\alpha\beta}$ are also complex analytic functions defined in the intersections $U_\alpha \cap U_\beta$, and it follows from (3) by a simple calculation that $\sigma_{r\alpha\gamma} = (f'_{\beta\gamma})^r \sigma_{r\alpha\beta} + \sigma_{r\beta\gamma}$ in $U_\alpha \cap U_\beta \cap U_\gamma$; that is to say, the collection of functions $\{\sigma_{r\alpha\beta}\}$ defines a one-cocycle of the covering $\{U_\alpha\}$ with coefficients in the sheaf of germs of holomorphic cross-sections of the line bundle κ^r. A zero-cochain $\{h_\alpha\}$ of the covering having $\{\sigma_{r\alpha\beta}\}$ as its coboundary will be called a *complex analytic projective connection* on the Riemann surface M if $r = 2$, and a *complex analytic affine connection* if $r = 1$; thus such a connection consists of a collection of functions $\{h_\alpha\}$ holomorphic in the various neighborhoods $\{U_\alpha\}$ and satisfying

$$(f'_{\alpha\beta})^r h_\alpha - h_\beta = \theta_r f_{\alpha\beta} \quad \text{in} \quad U_\alpha \cap U_\beta . \tag{4}$$

For an explanation of the terminology, see [10]. Note that the existence of a holomorphic such connection amounts to the condition that the cocycle $\{\sigma_{r\alpha\beta}\}$ be cohomologous to zero; thus the vanishing of the first cohomology group of M with coefficients in the sheaf of germs of holomorphic cross-sections of the line bundle κ^r, the group denoted by $H^1(M, \Omega(\kappa^r))$, guarantees the existence of at least one connection. Having one connection given, the most general connection is clearly the given connection plus an arbitrary quadratic differential (if $r = 2$) or abelian differential (if $r = 1$) on M; there is thus established a one-to-one correspondence, albeit not a natural correspondence, between projective connections (if they exist) and quadratic differentials, and a correspondence between affine connections and abelian differentials. The role of these connections is indicated by the following result.

Theorem 4. *For an arbitrary Riemann surface M, there is a natural one-to-one correspondence between the set of complex projective (or affine) structures on M subordinate to the given complex structure, and the set of complex analytic projective (or affine) connections on M.*

Proof. Any complex analytic coordinate covering $\{U_\alpha, z^*_\alpha\}$ of M, defined with respect to the given open covering $\{U_\alpha\}$ and equivalent to the given coordinate covering $\{U_\alpha, z_\alpha\}$, must be of the form $z^*_\alpha = g_\alpha(z_\alpha)$ for some complex analytic local homeomorphisms g_α; the coordinate transition functions for this new coordinate covering are of the form $f^*_{\alpha\beta} = g_\alpha \circ f_{\alpha\beta} \circ g_\beta^{-1}$. The condition that the new coordinate covering be a complex projective (or affine) coordinate covering is just that $\theta_r f^*_{\alpha\beta}=0$ for $r=2$ (or $r=1$, in the affine case). Using (3), the last condition can be written

$$\theta_r f_{\alpha\beta} = -(f'_{\alpha\beta})^r\, \theta_r g_\alpha + \theta_r g_\beta\,. \tag{5}$$

It is convenient to introduce the intermediary functions

$$h_\alpha = -\theta_r g_\alpha\,, \tag{6}$$

in terms of which condition (5) reduces to condition (4); thus, the set of all complex projective (or affine) coordinate coverings of M subordinate to the given complex structure of M can be put into one-to-one correspondence with the set consisting of those collections of pairs of analytic functions $\{g_\alpha, h_\alpha\}$ satisfying (4) and (6). Two collections of such pairs $\{g_\alpha, h_\alpha\}$ and $\{g^*_\alpha, h^*_\alpha\}$ define equivalent complex projective (or affine) coordinate coverings precisely when the complex analytic local homeomorphisms t_α, defined by $g^*_\alpha = t_\alpha \circ g_\alpha$, are complex projective (or affine) mappings; and since $\theta_r g^*_\alpha = (g'_\alpha)^r\, \theta_r t_\alpha + \theta_r g_\alpha$, this condition merely amounts to the condition that $h^*_\alpha = h_\alpha$. Therefore, to conclude the proof of the theorem, it is merely necessary to show that, given any collection of complex analytic functions $\{h_\alpha\}$, there exist local solutions $\{g_\alpha\}$ of the differential equation (6); this is clear for $r=1$, and becomes clear for $r=2$ upon putting $g'_\alpha = y_\alpha^{-2}$ and observing that (6) then has the form $2y''_\alpha + h_\alpha y_\alpha = 0$.

Corollary 1. *Any open Riemann surface admits a complex affine structure subordinate to the given complex structure; the set of all such structures can be put into one-to-one correspondences with the set of abelian differentials on the surface.*

Proof. Since $\Omega(\kappa)$ is a coherent analytic sheaf, and an open Riemann surface M is a Stein manifold, it follows that $H^1(M, \Omega(\kappa))=0$, [9]. Therefore there exist complex analytic affine connections on M as noted above, and the Corollary follows immediately from the preceding theorem.

Corollary 2. *Any Riemann surface admits a complex projective structure subordinate to the given complex structure; the set of all such structures can be put into one-to-one correspondences with the set of quadratic differentials on the surface.*

Proof. For an open surface, the argument proceeds as in the preceding Corollary. For a compact Riemann surface, it follows from the Serre duality theorem that $H^1(M, \Omega(\kappa^2)) \cong H^0(M, \Omega(\kappa^{-1}))=$ the space of holomorphic cross-sections of κ^{-1}; since $|\kappa^{-1}| = 2-2g$, it follows that $H^1(M, \Omega(\kappa^2))=0$ if $g>1$, g being the genus of the surface M. The existence of projective structures on surfaces of genus 0 or 1 being obvious, the Corollary follows.

Each projective connection on a Riemann surface M leads to a complex projective structure on M by Theorem 4, and this structure in turn has an associated flat complex projective bundle over M; the bundle can be determined directly from the projective connection as follows. (The corresponding affine case is quite trivial, so it will not be discussed further.) Given a complex analytic coordinate covering $\{U_\alpha, z_\alpha\}$ of M, let $\{h_\alpha\}$ be a complex projective connection;

and let $\{\xi_{\alpha\beta}\}$ be transition functions defining a complex line bundle ξ such that $\xi^2=\kappa$. In each neighborhood U_α select any two linearly independent complex analytic functions $f_{1\alpha}, f_{2\alpha}$ which are solutions of the linear differential equation

$$f''_{i\alpha}=\frac{1}{2}h_\alpha f_{i\alpha}, \qquad i=1,2. \tag{7}$$

A straightforward verification shows that in $U_\alpha\cap U_\beta$ the functions $(f_{i\beta}\circ f_{\beta\alpha})\xi_{\beta\alpha}$ are also solutions of the differential equation (7); so, introducing the vector-valued functions

$$F_\alpha=\begin{pmatrix} f_{1\alpha}\\ f_{2\alpha}\end{pmatrix},$$

there is a unique complex constant matrix $\varphi_{\beta\alpha}$ such that

$$F_\beta(z_\beta(z_\alpha))\xi_{\beta\alpha}(z_\alpha)=\varphi_{\beta\alpha}F_\alpha(z_\alpha) \quad \text{in} \quad U_\alpha\cap U_\beta. \tag{8}$$

Theorem 5. *The matrices $\{\varphi_{\beta\alpha}\}$ are transition functions defining the flat complex projective bundle over M associated to the projective structure of the projective connection $\{h_\alpha\}$.*

Proof. In each neighborhood U_α introduce the meromorphic function $g_\alpha=f_{1\alpha}/f_{2\alpha}$, noting that these are non-trivial functions. Indeed, since $f_{i\alpha}$ are linearly independent solutions of (7), their Wronskian W_α is nowhere vanishing in U_α; and since $g'_\alpha=W_\alpha f_{2\alpha}^{-2}$, it follows that $g'_\alpha\neq 0$ at all points where $f_{2\alpha}\neq 0$. Similarly, $(1/g_\alpha)'\neq 0$ at all points at which $f_{1\alpha}\neq 0$, so since $f_{1\alpha}$ and $f_{2\alpha}$ have no common zeros, the functions g_α are local homeomorphisms into projective space. Thus a new complex analytic coordinate covering can be introduced by putting $w_\alpha=g_\alpha(z_\alpha)$ in U_α, and refining the covering if necessary. Now in $U_\alpha\cap U_\beta$ it is clear from (8) that $w_\beta=\varphi_{\beta\alpha}(w_\alpha)$, considering $\varphi_{\beta\alpha}$ as a projective transformation; that is to say, $\{U_\alpha, w_\alpha\}$ defines a complex projective structure on M for which $\varphi=\{\varphi_{\alpha\beta}\}$ is the associated flat projective bundle. To complete the proof it is only necessary to observe that this projective structure corresponds to the connection $\{h_\alpha\}$ as in Theorem 4, that is, that $\theta_2 g_\alpha+h_\alpha=0$; but this follows readily enough from (2) and (7), and the proof is therewith concluded.

This same result can be derived in another manner, as is outlined in the appendix to the present paper.

5. The Eichler cohomology groups

In some recent investigations of automorphic functions, the cohomology groups introduced by EICHLER have played an important role, [7, 12]. These cohomology groups are associated to a discontinuous group Γ of projective transformations of a domain $D\subset \mathbf{C}$; hence they can also be considered as being associated to the Riemann surface D/Γ, with the natural projective structure on that surface. Now this cohomology theory can be associated, in a natural intrinsic manner, to any complex projective structure on any Riemann surface. This matter will not be considered in detail here; rather, a special case will be discussed to illustrate the role of the projective structures in the theory.

Let $\{U_\alpha, z_\alpha\}$ be a complex analytic coordinate covering of a Riemann surface M, and let $\{h_\alpha\}$ be a complex analytic projective connection on M. Consider the family of linear differential operators $D=\{D_\alpha\}$ defined in the various neighborhoods $\{U_\alpha\}$ by

$$D_\alpha = d^3/dz_\alpha^3 - 2h_\alpha\, d/dz_\alpha - h'_\alpha. \tag{9}$$

As before, for any line bundle ξ, let $\Omega(\xi)$ denote the sheaf of germs of holomorphic cross-sections of the bundle ξ; and let κ denote the canonical bundle of M. The cross-sections of the sheaf $\Omega(\xi)$, the holomorphic cross-sections of ξ, will be denoted by $\Gamma(\xi)=H^0(M,\Omega(\xi))$.

Lemma 2. *The operator D defines a sheaf homomorphism $D:\Omega(\kappa^{-1})\to\Omega(\kappa^2)$*

Proof. Suppose given a germ of a cross-section $f_\alpha\in\Omega(\kappa^{-1})$, in a neighborhood U_α; so in any other neighborhood U_β in which the same germ has a representation f_β, it follows that $f_\alpha=\kappa_{\alpha\beta}^{-1} f_\beta$. A straightforward calculation shows that $D_\alpha f_\alpha = D_\alpha(\kappa_{\alpha\beta}^{-1} f_\beta)=\kappa_{\alpha\beta}^2 D_\beta f_\beta$; but this is just the condition that $D_\alpha f_\alpha\in\Omega(\kappa^2)$, as desired.

The kernel of the homomorphism D in the above lemma is a subsheaf $\Omega_h(\kappa^{-1})\subset\Omega(\kappa^{-1})$, the subscript indicating the dependence of this sheaf on the choice of the projective connection $h=\{h_\alpha\}$.

Theorem 6. *If M is a compact Riemann surface of genus $g>1$ and $h=\{h_\alpha\}$ is a complex analytic projective connection on M, there is a natural exact sequence of complex vector spaces*

$$0\to\Gamma(\kappa^2)\to H^1(M,\Omega_h(\kappa^{-1}))\to\Gamma(\kappa^2)\to 0. \tag{10}$$

Proof. By Lemma 2 the operator D yields the exact sequence of sheaves

$$0\to\Omega_h(\kappa^{-1})\to\Omega(\kappa^{-1})\xrightarrow{D}\Omega(\kappa^2)\to 0;$$

that D is surjective is a familiar property of ordinary differential equations. Corresponding to this exact sequence of sheaves is the exact cohomology sequence, a part of which has the form

$$\Gamma(\kappa^{-1})\to\Gamma(\kappa^2)\to H^1(M,\Omega_h(\kappa^{-1}))\to H^1(M,\Omega(\kappa^{-1}))\to H^1(M,\Omega(\kappa^2)).$$

Since $|\kappa^{-1}|=2-2g<0$, it follows that $\Gamma(\kappa^{-1})=0$. By the Serre duality theorem, $H^1(M,\ \Omega(\kappa^{-1}))\cong H^0(M,\ \Omega(\kappa^2)) = \Gamma(\kappa^2)$ and $H^1(M,\ \Omega(\kappa^2))\cong$ $\cong H^0(M,\Omega(\kappa^{-1}))=\Gamma(\kappa^{-1})=0$. The desired result then follows immediately.

The groups $H^1(M,\Omega_h(\kappa^{-1}))$ will be called *Eichler cohomology groups* of the Riemann surface M. The groups depend upon the choice of a projective connection h. In addition to the particular case mentioned here, one can introduce similar groups $H^1(M,\Omega_h(\kappa^{-n}))$ for all positive integers n, by suitable extensions of the differential operator (9); higher dimensional cohomology groups do not occur for Riemann surfaces. The more general case will be discussed elsewhere.

At each point of M, the stalk of the sheaf $\Omega_h(\kappa^2)$ is a three-dimensional complex vector space, consisting of the germs of functions in the kernel of the differential operator (9); the Eichler cohomology groups are cohomology groups of M with a suitable system of local coefficients, [13]. The description of these coefficients becomes simplest in terms of a complex projective coordinate covering $\{U_\alpha, z_\alpha\}$ representing the projective structure associated to the projective connection $h=\{h_\alpha\}$. For in such a coordinate system $h_\alpha \equiv 0$ in each U_α, and hence the kernel of operator (9) in U_α consists of all polynomials of degree at most two in the local coordinate z_α. These observations, together with the exact sequence (10) relating the cohomology groups and the space $\Gamma(\kappa^2)$ of quadratic differentials on M, exhibit the cohomology groups introduced here as reasonable extensions of the groups appearing in the study of automorphic functions.

6. The geometric realization

The complex projective (or affine) structures on a surface lead to rather explicit geometric representations of the surface, which are closely related to the various classical uniformizations of Riemann surfaces. Suppose that M is a compact topological surface with a given complex projective (or affine) structure. Under the covering mapping $\tilde{M} \to M$, the universal covering space $\tilde{M}$ of M inherits in the obvious manner a complex projective (or affine) structure. Since $\pi_1(\tilde{M})=0$, the flat fibre bundle associated to the complex projective (or affine) structure of $\tilde{M}$ is trivial, [13]; hence there is a complex projective (or affine) coordinate covering $\{U_\alpha, z_\alpha\}$ of $\tilde{M}$, representing the given structure, such that all the coordinate transition functions $\tilde{z}_\alpha = \tilde{f}_{\alpha\beta}(\tilde{z}_\beta)$ are identity mappings. For such a coordinate covering, the various coordinate mappings actually define a global mapping $\varrho: \tilde{M} \to D$ from the surface $\tilde{M}$ onto a subset D of the complex projective line $\mathbf{P}$ (or complex affine line $\mathbf{C}$); this mapping will be called the *geometric realization* of the given complex projective (or affine) structure. The realization mapping ϱ is of course a local homeomorphism, since locally it coincides with a coordinate mapping; the set D is then a connected open subset of $\mathbf{P}$ (or $\mathbf{C}$). Any other such coordinate covering representing the same structure will differ at most by the same complex projective (or affine) mapping applied to each coordinate mapping; thus the geometric realization $\varrho: \tilde{M} \to D$ is unique up to a projective (or affine) mapping applied to D.

Let $\Pi \cong \pi_1(M)$ be the covering translation group for the universal covering mapping $\tilde{M} \to M$; and note that each $T^* \in \Pi$ is a complex projective (or affine) mapping in terms of the induced projective (or affine) structure of $\tilde{M}$. Thus for any point $\tilde{p} \in \tilde{M}$ and any element $T^* \in \Pi$ there will be a complex projective (or affine) transformation T such that $\varrho(T^*\tilde{p}) = T\varrho(\tilde{p})$; this relation will of course hold for all points $\tilde{p}$ in a small open neighborhood on the surface $\tilde{M}$, so by continuity the transformation T must indeed be independent of the point $\tilde{p}$. That is, for each element $T^* \in \Pi$ there is a complex projective (or affine) transformation T such that $\varrho(T^*\tilde{p}) = T\varrho(\tilde{p})$ for all points $\tilde{p} \in \tilde{M}$. The set Γ of all these transformations $\{T\}$ is therefore a group of complex projective (or affine) automorphisms of the domain D, and $\Gamma \cong \Pi$ under an isomorphism commuting with the realization mapping ϱ. Henceforth, by the *geometric realization* of a complex projective (or affine) structure on M, we shall mean the realization mapping $\varrho: \tilde{M} \to D$ together with the transformation group Γ

on D. Note that changing the realization mapping by a projective (or affine) mapping of D has the effect of replacing Γ by the corresponding conjugate subgroup of $\mathscr{P}$ (or $\mathscr{A}$). Note further that the isomorphism $\Pi \cong \Gamma$ actually represents the characteristic class of the flat projective (or affine) bundle associated to the given structure on M, [13, section 13].

Since the realization mapping ϱ induces a mapping $\varrho^* : \tilde{M}/\Pi \to D/\Gamma$, it follows that the identification space D/Γ is compact in its natural topology whenever the space $M = \tilde{M}/\Pi$ is compact. The group Γ does not necessarily act in a discontinuous manner on the space D, however; the quite simple affine case, which we shall consider briefly next, illustrates this.

Let M be a compact surface with a given affine structure; by Lemma 1, M must have genus 1, so that Π is a free abelian group on two generators S^*, T^*. Thus Γ is an abelian group of affine transformations generated by two mappings S, T, which one easily sees must have one or the other of the following forms:

(i) $S(z) = z + 1$, $T(z) = z + \omega$, any complex ω;

(ii) $S(w) = aw$, $T(w) = bw$, any non-zero complex a, b.

For the quotient space D/Γ to be compact, in case (i) necessarily $D = \mathbf{C}$ and $\operatorname{Im}\omega \neq 0$; and in case (ii), necessarily $D = \mathbf{C} - 0$ and either $|a| \neq 1$ or $|b| \neq 1$. Case (i) gives the usual representation of a compact Riemann surface of genus 1 as the quotient of the complex plane by a discrete lattice subgroup, which evidently describes an affine structure on the surface as well. The mapping $w = e^{cz}$, for a non-zero complex constant c, takes the full plane onto the punctured plane and transforms the group action of case (i) to that of case (ii), with $a = e^c$ and $b = e^{c\omega}$; thus all of the groups in case (ii) actually do occur. It may be observed that ω (modulo the usual action of the modular group) serves as modulus for the complex structures of M in the representation (i), and the affine structures of case (ii) can be used to associate the same flat affine bundle to two inequivalent affine (indeed, inequivalent complex) structures. It should further be observed that, in case (ii), the group Γ does not always act discontinuously on D.

Next, turning to the projective case, we have the following simple result.

Theorem 7. *Let M be a compact topological surface of genus $g > 1$ with a complex projective structure; and let $\varrho : \tilde{M} \to D$ be its geometric realization, where $D \subset \mathbf{P}$, $D \neq \mathbf{P}$. Then the realization mapping ϱ is a covering map; and either D is analytically equivalent to the unit disc, or its complement in $\mathbf{P}$ has infinitely many components.*

Proof. If $\mathbf{P} - D$ consists of a single point, that point can be taken as the point at infinity in $\mathbf{P}$; and since the transformations $T \in \Gamma$ preserve D, they must then be affine mappings. Now the isomorphism $\Pi \cong \Gamma$ represents the characteristic class of the flat fibre bundle associated to the given projective structure, and if Γ is affine, it follows that the projective structure can be reduced to an affine structure; but by Lemma 1, the surface M must then have genus 1, a contradiction. Next, if $\mathbf{P} - D$ consists of two points, they can be taken as the points 0 and ∞ in $\mathbf{P}$; the transformations $T \in \Gamma$ then have one or the other of the forms: $Tz = az$, $Tz = a/z$, $(a \in \mathbf{C}, a \neq 0)$. The subgroup $\Gamma_0 \subset \Gamma$ consisting of affine transformations is easily seen to be a normal subgroup of finite index. The corresponding subgroup $\Pi_0 \subset \Pi$ of covering translations then determines a finite-sheeted covering space of M which has an affine structure, hence which is a surface of genus 1; it then follows that M itself must have genus 1, again a contradiction. Therefore the complement of D must contain at least three points. If D is simply connected, it must be conformally equivalent to the unit disc; otherwise, recalling that D/Γ is compact and hence that D must have an infinite group of automorphisms, the complement of D must have infinitely many components, [11].

Since ϱ is a local homeomorphism, to show that it is a covering mapping it suffices to prove the following: given any closed path λ in D, from a point p to a point q, and any point $\tilde{p} \in \tilde{M}$ for which $\varrho(\tilde{p}) = p$, there exists a path $\tilde{\lambda}$ in $\tilde{M}$ beginning at the point $\tilde{p}$ and satisfying $\varrho(\tilde{\lambda}) = \lambda$. The path $\tilde{\lambda}$ will indeed be constructed quite explicitly as follows. The quotient space $M = \tilde{M}/\Pi$ being compact and $\varrho: \tilde{M} \to D$ being a local homeomorphism, there are a finite number of pairs of sets $\tilde{K}_i \subset \tilde{L}_i \subset \tilde{M}$ such that:

(i) each $\tilde{L}_i$ is topologically a closed disc, and $\tilde{K}_i$ is a closed subset of the interior of $\tilde{L}_i$;

(ii) $\varrho: \tilde{L}_i \to L_i$ is a homeomorphism between $\tilde{L}_i$ and its image $L_i \subset D$;

(iii) for any point $\tilde{s} \in \tilde{M}$ there is a transformation $T^* \in \Pi$ such that $T^*\tilde{s} \in \tilde{K} = \cup_i \tilde{K}_i$.

Without loss of generality, we may suppose that $\tilde{p} \in \tilde{K}_1$; thus $p = \varrho(\tilde{p}) \in K_1 = \varrho(\tilde{K}_1) \subset L_1$. Let λ_1 be that segment of the path λ from p to the first point $p_2 \in \lambda$ at which λ meets the boundary of L_1. By (ii) there is a path $\tilde{\lambda}_1$ from $\tilde{p}$ in $\tilde{L}_1$ such that $\varrho(\tilde{\lambda}_1) = \lambda_1$. By (iii) there is a transformation $T_2^* \in \Pi$ such that $T_2^* \tilde{p}_2 \in \tilde{K}_{i_2}$ for some index i_2; if $T_2 \in \Gamma$ corresponds to T_2^* under the isomorphism $\Pi \cong \Gamma$, then $T_2 p_2 \in K_{i_2}$, and $T_2 \lambda$ will be a path passing through $T_2 p_2$. Let λ_2 be that segment of the path λ from p_2 to the first point p_3 further along λ at which the path $T_2 \lambda$ meets the boundary of L_{i_2}. By (ii) there is a path $(T_2^* \tilde{\lambda}_2)$ from $T_2^* \tilde{p}_2$ in $\tilde{L}_{i_2}$ such that $\varrho(T_2^* \tilde{\lambda}_2) = T_2 \lambda_2$; and so $\tilde{\lambda}_2 = (T_2^*)^{-1}(T_2^* \tilde{\lambda}_2)$ is a path in $\tilde{M}$ beginning at $\tilde{p}_2$ (the end of $\tilde{\lambda}_1$) and such that $\varrho(\tilde{\lambda}_2) = \lambda_2$. The process can obviously be continued in this manner; the result is a sequence of subsegments $\lambda_1, \lambda_2, \ldots$ of the path λ, λ_j running from p_j to p_{j+1}, together with transformations $T_j \in \Gamma$ such that:

(iv) $T_j(p_j) \in K_{i_j}$, for some specified index i_j;

(v) p_{j+1} is the first point of λ, further along the path than p_j, at which $T_j \lambda$ meets the boundary of L_{i_j}.

Moreover, the union $\lambda_1 \cup \lambda_2 \cup \ldots$ is an initial segment of the path λ which can be lifted to a path in $\tilde{M}$ beginning at the specified point p. To complete the proof, we shall show that the preceding procedure stops after a finite number of steps, necessarily therefore with an exhaustion of the entire path λ. Suppose, in contradiction to the desired result, that the sequence $\{T_j\}$ is infinite. Since these mappings T_j map D into itself, and since the complement of D contains at least three points as noted above, it follows from Montel's theorem that the sequence $\{T_j\}$ is a normal family in D; there is thus a subsequence $\{T_{j_k}\}$ which is uniformly convergent on the compact subset $\lambda \subset D$ to a limit mapping T. The sequence of points $\{p_j\}$ has a unique limit point r along the path λ in the natural order topology; of course, r is also a limit point of the sequence $\{p_j\}$ in the topology of $\mathbf{P}$. Note that $Tr \in K = \cup_i K_i$. (For if $Tr \notin K$, then there would be an open segment $\mu \subset \lambda$ containing r and such that $T\mu \cap K = \emptyset$. From the uniformity of the convergence it would follow that $T_{j_k}\mu \cap K = \emptyset$ for all sufficiently large values of k; but since $p_{j_k} \in \mu$ for large values of k, and since $T_{j_k}p_{j_k} \in K$ by (iv), this would be an immediate contradiction.) Let L_0 be the intersection of all those sets L_i such that $Tr \in K_i$. There is an open segment $\nu \subset \lambda$ containing r and such that the closure of $T\nu$ is in the interior of L_0. From the uniformity of the convergence it follows that $T_{j_k}\nu$ is contained in the interior of L_0 for all sufficiently large values of k. Select some such value of k which is also large enough that $p_{j_k} \in \nu$. Then the entire segment of $T_{j_k}\lambda$ from $T_{j_k}p_{j_k}$ to $T_{j_k}r$ is contained within the interior of L_0, so in particular $T_{j_k}p_{j_k+1}$ is in the interior of L_0; but this contradicts (v), and the proof of the theorem is therewith concluded.

In the preceding theorem, if D is simply connected the covering mapping $\varrho : \tilde{M} \to D$ must actually be a homeomorphism; therefore, in this case, the group Γ acts discontinuously on D and $M \cong D/\Gamma$. This is the familiar uniformization of Riemann surfaces, although it should be pointed out that D need not be a proper disc, there being many other projective structures possible on the same surface, [3]. The Schottky uniformization [8] furnishes examples of the other case in the theorem.

7. Appendix: On complex vector bundles

Let M be a compact Riemann surface (the complex structure of which will be fixed throughout), and let φ be a complex vector bundle over M. The *dimension* of φ, written $\dim\varphi$, refers to the dimension of the fibre. A bundle φ with $\dim\varphi = 1$ is called a *line bundle*; the *Chern class* of a line bundle φ, considered as an integer under the natural identification $H^2(M, \mathbf{Z}) \cong \mathbf{Z}$, will be denoted by $|\varphi|$. In terms of a complex analytic coordinate covering $\{U_\alpha, z_\alpha\}$ of M, an n-dimensional complex vector bundle φ is defined by transition functions $\varphi_{\alpha\beta}: U_\alpha \cap U_\beta \to GL(n, \mathbf{C})$ which are complex analytic in the given complex structure on M. The functions $\det\varphi_{\alpha\beta}$ then define a complex line bundle, which will be denoted by $\det\varphi$. If φ is an n-dimensional complex vector bundle and ξ is a complex line bundle, the tensor product $\xi \otimes \varphi$ is also an n-dimensional complex vector bundle. For further general properties of complex vector bundles, see [1].

We shall here consider in somewhat greater detail vector bundles φ with $\dim\varphi = 2$, $\det\varphi = 1$, (where 1 stands for the trivial line bundle). For any fixed such bundle φ, consider the set $\Delta(\varphi)$ of those complex line bundles ξ over M such that the bundle $\xi^{-1} \otimes \varphi$ has non-trivial holomorphic cross-sections; this is a non-empty set, and the Chern classes $|\xi|$ of the bundles $\xi \in \Delta(\varphi)$ are bounded from above, [1]. The *divisor class* of the bundle φ is defined to be the following set of complex line bundles: $\operatorname{div}\varphi = \{\xi \in \Delta(\varphi);\ |\xi| = \max_{\eta\in\Delta(\varphi)}|\eta|\}$. Note that $|\operatorname{div}\varphi| = |\xi|$, $\xi \in \operatorname{div}\varphi$, is well-defined, indeed is merely the maximum Chern class of the bundles of $\Delta(\varphi)$. Note further that if $\xi \in \operatorname{div}\varphi$ is defined by transition functions $\xi_{\alpha\beta}$ in terms of a complex analytic coordinate covering $\{U_\alpha, z_\alpha\}$ of M, then the bundle φ can be defined by transition functions of the form

$$\varphi_{\alpha\beta} = \begin{pmatrix} \xi_{\alpha\beta} & \sigma_{\alpha\beta} \\ 0 & \xi_{\alpha\beta}^{-1} \end{pmatrix} \tag{11}$$

for some holomorphic functions $\sigma_{\alpha\beta}$, [1]. Thus, if $\eta \otimes \varphi$ admits a holomorphic cross-section for some line bundle η with $|\eta| = 0$, the order of the divisor of that cross-section is just $|\operatorname{div}\varphi|$.

Proposition A 1. *If φ is a complex vector bundle on the compact Riemann surface M with $\dim\varphi = 2$, $\det\varphi = 1$, and $|\operatorname{div}\varphi| > 0$, then $\operatorname{div}\varphi$ contains a single complex line bundle.*

Proof. Let ξ be a line bundle in $\operatorname{div}\varphi$; any other line bundle in $\operatorname{div}\varphi$ must be of the form $\xi \otimes \eta$, where η is a line bundle having Chern class $|\eta| = 0$. Then if $\xi \otimes \eta \in \operatorname{div}\varphi$, the bundle $(\xi \otimes \eta)^{-1} \otimes \varphi$ must admit a holomorphic cross-section (f_α, g_α); writing the transition functions of the bundle φ in the form (11), the holomorphic local functions f_α, g_α must satisfy the equations

$$\begin{aligned} f_\alpha &= \eta_{\alpha\beta}^{-1} f_\beta + \eta_{\alpha\beta}^{-1} \xi_{\alpha\beta}^{-1} \sigma_{\alpha\beta} g_\beta, \\ g_\alpha &= \qquad\qquad \eta_{\alpha\beta}^{-1} \xi_{\alpha\beta}^{-2} g_\beta. \end{aligned}$$

Since $|\eta^{-1} \otimes \xi^{-2}| = -2|\xi| < 0$, necessarily $g_\alpha \equiv 0$ and $\{f_\alpha\}$ compose a cross-section of the line bundle η; but since $|\eta| = 0$, it then follows that $\eta = 1$, hence $\xi \otimes \eta = \xi$, which was the desired result.

It is easy to see that necessarily $|\operatorname{div}\varphi| \geqq -g$, and that the bundle φ is fully reducible whenever $|\operatorname{div}\varphi| \geqq g$, [1]. For any given complex line bundle ξ on M with $|\xi| > 0$, there is a unique fully reducible complex vector bundle φ over M with $\dim\varphi = 2$, $\det\varphi = 1$, $\operatorname{div}\varphi = \xi$; it is of course the bundle $\varphi = \xi \oplus \xi^{-1}$. The indecomposable (that is to say, not fully reducible) such bundles are described as follows.

Proposition A 2. *Let ξ be a complex line bundle with $|\xi| > 0$, over the compact Riemann surface M. The set of indecomposable complex vector bundles φ on M with $\dim\varphi = 2$, $\det\varphi = 1$, $\operatorname{div}\varphi = \xi$, is in one-to-one correspondence with the points of a complex projective space of dimension $N-1$, where $N = \dim\Gamma(\xi^{-2} \otimes \kappa)$; (here Γ denotes the space of holomorphic cross-sections of the relevant bundle, and κ is the canonical bundle of the surface).*

Proof. Since $\operatorname{div}\varphi$ is uniquely defined in these circumstances, by Proposition A1, all such bundles φ can be described by transition functions of the form (11) for various values of the functions $\sigma_{\alpha\beta}$. A simple calculation shows that these functions must satisfy the condition that $\sigma_{\alpha\gamma} = \xi_{\alpha\beta}\sigma_{\beta\gamma} + \xi_{\beta\gamma}^{-1}\sigma_{\alpha\beta}$ in $U_\alpha \cap U_\beta \cap U_\gamma$, that is, that the functions $\xi_{\alpha\beta}^{-1}\sigma_{\alpha\beta}$ compose a one-cocycle of the covering $\{U_\alpha\}$ with coefficients in the sheaf of germs of holomorphic cross-sections of the bundle $\xi_{\alpha\beta}^2$, [2]. Further, if $\sigma_{\alpha\beta}$ and $\tau_{\alpha\beta}$ are two such cocycles, the bundles they define are equivalent precisely when there are holomorphic functions f_α in the sets U_α and non-zero constants a, b such that $\xi^2_{\beta\alpha} f_\alpha - f_\beta = a\xi^{-1}_{\alpha\beta}\sigma_{\alpha\beta} - b\xi^{-1}_{\alpha\beta}\tau_{\alpha\beta}$, that is, precisely when the cocycles $a\xi^{-1}_{\alpha\beta}\sigma_{\alpha\beta}$ and $b\xi^{-1}_{\alpha\beta}\tau_{\alpha\beta}$ are cohomologous. The desired set of complex vector bundles is therefore in one-one correspondence with the projective space associated to the complex vector space $H^1(M, \Omega(\xi^2))$; the zero element in the vector space corresponds to the fully reducible vector bundles, which are here being ignored. Since $H^1(M, \Omega(\xi^2)) \cong \Gamma(\kappa \otimes \xi^{-2})$, by Serre duality [2], the theorem is proved.

Example. Consider the particular case of the above proposition in which $|\xi| = g-1$. Since $|\xi^{-2} \otimes \kappa| = 0$, it follows that

$$\dim \Gamma(\xi^{-2} \otimes \kappa) = \begin{Bmatrix} 0 & \text{if} & \xi^2 \neq \kappa, \\ 1 & \text{if} & \xi^2 = \kappa. \end{Bmatrix}$$

In the first case ($\xi^2 \neq \kappa$) there are thus no indecomposable vector bundles φ with $\dim\varphi = 2$, $\det\varphi = 1$, $\operatorname{div}\varphi = \xi$. In the second case ($\xi^2 = \kappa$), there is a unique indecomposable vector bundle φ with $\dim\varphi = 2$, $\det\varphi = 1$, $\operatorname{div}\varphi = \xi$; it is a straightforward task to verify that this bundle is defined by transition functions

$$\varphi_{\alpha\beta} = \begin{pmatrix} \xi_{\alpha\beta} & \xi^{-1}_{\alpha\beta} \dfrac{d}{dz_\beta} \kappa_{\alpha\beta} \\ 0 & \xi^{-1}_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \xi_{\alpha\beta} & 2\dfrac{d}{dz_\beta} \xi_{\alpha\beta} \\ 0 & \xi^{-1}_{\alpha\beta} \end{pmatrix} \tag{12}$$

Referring back to Theorem 2 (in Section 3), it is of interest to determine the set of flat complex vector bundles φ for which $\dim\varphi = 2$, $\det\varphi = 1$, and $|\operatorname{div}\varphi| = g-1$; note that to any complex vector bundle φ there corresponds a set (perhaps empty) of *associated flat vector bundles*, consisting of those flat vector bundles which are analytically equivalent to φ.

To begin the discussion, let $\{U_\alpha, z_\alpha\}$ be a complex analytic coordinate covering of a compact Riemann surface M; and let $\{\varphi_{\alpha\beta}\}$ be a complex analytic coordinate bundle defined with respect to that covering and determining a complex vector bundle φ. A set $\{G_\alpha\}$ of holomorphic, matrix-valued functions defined in the various neighborhoods $\{U_\alpha\}$ will be called an *endomorphism* of φ if $G_\alpha \varphi_{\alpha\beta} = \varphi_{\alpha\beta} G_\beta$ in $U_\alpha \cap U_\beta$. Let $\Lambda(\varphi)$ be the set consisting of collections $\{\Lambda_\alpha\}$ of holomorphic, matrix-valued differential forms of type (1, 0) in the various neighborhoods $\{U_\alpha\}$ such that $d\varphi_{\alpha\beta} = \varphi_{\alpha\beta}\Lambda_\beta - \Lambda_\alpha\varphi_{\alpha\beta}$ in $U_\alpha \cap U_\beta$. Introduce an equivalence relation in the set $\Lambda(\varphi)$ by defining $\{\Lambda_\alpha\} \sim \{\Lambda^*_\alpha\}$ provided $dG_\alpha = G_\alpha\Lambda_\alpha - \Lambda^*_\alpha G_\alpha$ in each neighborhood U_α, for some endomorphism $\{G_\alpha\}$ of φ. The set of equivalence classes will be denoted by $\tilde{\Lambda}(\varphi)$; and the subset, consisting of those equivalence classes containing a representative $\{\Lambda_\alpha\}$ for which $\operatorname{tr}\Lambda_\alpha = 0$, will be denoted by $\tilde{\Lambda}_0(\varphi)$.

Proposition A 3. *On a compact Riemann surface M, the set of flat vector bundles associated to a complex vector bundle φ is in one-to-one correspondence* *with the elements of* $\tilde{A}(\varphi)$; *further, if* $\det\varphi = 1$, *the set of flat vector bundles of determinant* 1 *associated to* φ *is in one-to-one correspondence with the subset* $\tilde{A}_0(\varphi)$.

Proof. Since any coordinate bundle representing the vector bundle φ is defined by transition functions of the form $\varphi^*_{\alpha\beta} = F_\alpha \varphi_{\alpha\beta} F_\beta^{-1}$ for some holomorphic, non-singular matrix-valued functions $\{F_\alpha\}$ defined in the various neighborhoods $\{U_\alpha\}$, these bundles can be described by the functions $\{F_\alpha\}$. The coordinate bundle is flat precisely when $d\varphi^*_{\alpha\beta} = 0$; this is readily seen to be equivalent to the condition that $d\varphi_{\alpha\beta} = \varphi_{\alpha\beta} A_\beta - A_\alpha \varphi_{\alpha\beta}$ where $A_\alpha = F_\alpha^{-1}\, dF_\alpha$. Note that $\{F^*_\alpha\}$ and $\{F_\alpha\}$ define equivalent flat bundles whenever $F^*_\alpha F_\alpha^{-1}$ is a constant, that is, whenever $A^*_\alpha = A_\alpha$; and that any analytic differential form A_α of type (1, 0) can be written locally as $A_\alpha = F_\alpha^{-1}\, dF_\alpha$. Therefore, the flat vector bundles associated to φ can be described by the elements of $A(\varphi)$. Now, two sets $\{F^*_\alpha\}$ and $\{F_\alpha\}$ determine equivalent flat bundles precisely when $F^*_\alpha \varphi_{\alpha\beta} F^{*-1}_\beta = C_\alpha F_\alpha \varphi_{\alpha\beta} F_\beta^{-1} C_\beta^{-1}$ in $U_\alpha \cap U_\beta$ for some non-singular constant matrices C_α; this condition amounts to the same thing as the condition that $G_\alpha = F^{*-1}_\alpha C_\alpha F_\alpha$ is an endomorphism of φ, or equivalently, that $d(F^*_\alpha G_\alpha F_\alpha^{-1}) = 0$ for some endomorphism G_α of φ. Since the last condition reduces to the condition that $\{A^*_\alpha\} \sim \{A_\alpha\}$, this completes the proof of the first assertion of the proposition. For the second assertion, if $\det\varphi_{\alpha\beta} = 1$, then $\varphi^*_{\alpha\beta}$ is readily seen to be equivalent to a flat vector bundle of determinant 1 precisely where $\det F_\alpha$ is constant; and since $d(\det F_\alpha) = (\det F_\alpha)\operatorname{tr} A_\alpha$, this is equivalent to the condition $\operatorname{tr} A_\alpha = 0$, which completes the proof. (Note that actually the condition $\operatorname{tr} A_\alpha = 0$ is preserved by the equivalence relation introduced in $A(\varphi)$.)

Proposition A 4. *Let* M *be a compact Riemann surface of genus* $g > 1$, *and* φ *be a complex vector bundle over* M *with* $\dim\varphi = 2$, $\det\varphi = 1$, *and* $|\operatorname{div}\varphi| = g - 1$. *Then* φ *has a non-empty set of associated flat vector bundles if and only if* $(\operatorname{div}\varphi)^2 = \kappa$ *and* φ *is indecomposable. (Thus,* φ *must be as in the example considered above.)*

Proof. As a consequence of Proposition A3, φ has a non-empty set of associated flat vector bundles if and only if $A(\varphi) \neq \theta$; so we need only consider the space $A(\varphi)$ in somewhat more detail. In view of the hypotheses, the vector bundle φ is defined by transition functions of the form

$$\varphi_{\alpha\beta} = \begin{pmatrix} \xi_{\alpha\beta} & \sigma_{\alpha\beta} \\ 0 & \xi_{\alpha\beta}^{-1} \end{pmatrix},$$

where $\{\xi_{\alpha\beta}\}$ define a complex line bundle ξ with $|\xi| = g - 1$; and, as in the example considered above, either $\sigma_{\alpha\beta} = 0$ or $\sigma_{\alpha\beta} = 2d\xi_{\alpha\beta}/dz_\beta$, (the latter case occurring only when $\xi^2 = \kappa$). An element $\{A_\alpha\} \in A(\varphi)$ is a set of holomorphic matrix differential forms satisfying

$$d\varphi_{\alpha\beta} = \varphi_{\alpha\beta} A_\beta - A_\alpha \varphi_{\alpha\beta}; \tag{13}$$

write these matrices out explicitly as

$$A_\alpha = \begin{pmatrix} f_{11\alpha}(z_\alpha)\, dz_\alpha & f_{12\alpha}(z_\alpha)\, dz_\alpha \\ f_{21\alpha}(z_\alpha)\, dz_\alpha & f_{22\alpha}(z_\alpha)\, dz_\alpha \end{pmatrix},$$

where z_α is the coordinate function in U_α, and $f_{ij\alpha}(z_\alpha)$ are analytic functions. Note first of all that, as a consequence of (13), the functions $\{f_{21\alpha}\}$ compose a cross-section of the complex line bundle $\xi^{-2}\otimes\kappa$; so if $\xi^2=\kappa$, these functions form an arbitrary constant on M, while otherwise necessarily $f_{21\alpha}\equiv 0$. Then, also as a consequence of (13), the functions $\{f_{11\alpha}\}$ satisfy the condition

(14) $$f_{11\alpha}\,dz_\alpha - f_{11\beta}\,dz_\beta = \xi_{\alpha\beta}^{-1}(\sigma_{\alpha\beta}f_{21\beta}\,dz_\beta - d\xi_{\alpha\beta}) \quad \text{in} \quad U_\alpha\cap U_\beta;$$

thus, the right-hand side of (14), considered as a one-cocycle of the covering $\{U_\alpha\}$ with coefficients in the sheaf of germs of holomorphic cross-sections of the canonical bundle, must be cohomologous to zero. By the Serre duality theorem this cohomology group is isomorphic to $H^0(M,\mathcal{O})\cong\mathbf{C}$; it is a straightforward calculation to show further that the one-cocycle $\{\xi_{\alpha\beta}^{-1}\,d\xi_{\alpha\beta}\}$ corresponds to the constant $2g-2\in\mathbf{C}$, hence is not cohomologous to zero. Now if $f_{21\alpha}\equiv 0$, there can hence exist no possible solutions $f_{11\alpha}$ to (14); for the same reason, there can exist no solutions when the bundle is decomposable ($\sigma_{\alpha\beta}\equiv 0$). However if $f_{21\alpha}\equiv c$ for some constant $c\neq 0$ and if the bundle is indecomposable, so that the elements $\sigma_{\alpha\beta}$ have the form given in the preceding example, the right-hand side of (14) is easily seen to reduce to the expression $(2c-1)\,\xi_{\alpha\beta}^{-1}\,\mathrm{d}\xi_{\alpha\beta}$; so there exist solutions $\{f_{11\alpha}\}$ precisely when $c=1/2$. It is therefore already apparent that the set $\Lambda(\varphi)$ can be non-empty only when $\xi^2=\kappa$ and the bundle φ is indecomposable. To show that $\Lambda(\varphi)$ is actually non-empty in this case, observe that $f_{21\alpha}\equiv 1/2$ means that the right-hand side of (14) vanishes, and hence that $f_{11\alpha}\,dz_\alpha$ can be taken to be an arbitrary abelian differential; similarly, $f_{22\alpha}\,dz_\alpha$ is an arbitrary abelian differential. The function $f_{12\alpha}$ finally must satisfy $f_{12\alpha}=\xi_{\alpha\beta}^4 f_{12\beta}+\tau_{\alpha\beta}$, where the expressions $\{\tau_{\alpha\beta}\}$ form a one-cocycle of the covering $\{U_\alpha\}$ with coefficients in the sheaf of germs of holomorphic cross-sections of the line bundle $\xi^4=\kappa^2$, the precise form of the functions $\tau_{\alpha\beta}$ not being important here; this cohomology group is always trivial, by Serre duality, hence such functions $f_{12\alpha}$ always exist, and the proof is concluded.

Remarks. The analysis begun in the proof of the preceding theorem can be continued to provide a description of the sets $\tilde\Lambda(\varphi)$ and $\tilde\Lambda_0(\varphi)$, and hence a description of the flat vector bundles associated to φ; since the final description of this set of flat bundles coincides with that provided by Theorem 4 in Section 4, there is no need to carry out the argument any further here. It may be of interest merely to note that, when φ is the indecomposable bundle with $(\operatorname{div}\varphi)^2=\kappa$ as described in the previous example, the elements of $\tilde\Lambda_0(\varphi)$ have unique representatives in $\Lambda_0(\varphi)$ of the form

$$\Lambda_\alpha=\begin{pmatrix} 0 & h_\alpha\,dz_\alpha \\ \frac{1}{2}\,dz_\alpha & 0\end{pmatrix}$$

where $\{h_\alpha\}$ are holomorphic functions in U_α which satisfy $h_\alpha=\xi_{\alpha\beta}^4 h_\beta - 2\xi_{\alpha\beta}^3(d^2\xi_{\alpha\beta}/dz_\beta^2)$ in $U_\alpha\cap U_\beta$; note that the latter equation is equivalent to the assertion that the functions $\{h_\alpha\}$ define a complex analytic projective connection on the surface M, as discussed in Section 4.

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