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The $\bar{\partial}$ Problem with Uniform Bounds on Derivatives*

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Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with C^N boundary ($N \geqq 2$), i.e. there exists a C^N real-valued function ϱ on an open neighborhood $\tilde{\Omega}$ of Ω^- in C^n such that $\Omega = \{\varrho < 0\}$, $d\varrho$ is nowhere zero on $\partial\Omega$, and ϱ is strictly plurisubharmonic on $\partial\Omega$.

For $a \in \mathbb{C}^n$ the components of a are denoted by $a_1, \ldots, a_n$. $\mathbb{N}$ denotes the set of all nonnegative integers. For $\alpha \in \mathbb{N}^n$, $\zeta, z \in \mathbb{C}^n$, and $1 \leqq j \leqq n$,

$$|\alpha| = \alpha_1 + \cdots + \alpha_n$$

$$\alpha! = (\alpha_1!) \ldots (\alpha_n!)$$

$$(\zeta - z)^\alpha = (\zeta_1 - z_1)^{\alpha_1} \ldots (\zeta_n - z_n)^{\alpha_n}$$

$$D^\alpha = D_z^\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}}$$

$$\bar{D}^\alpha = \bar{D}_z^\alpha = \frac{\partial^{|\alpha|}}{\partial \bar{z}_1^{\alpha_1} \ldots \partial \bar{z}_n^{\alpha_n}}$$

$$D_j = D_{j,z} = \frac{\partial}{\partial z_j}.$$

When a differential operator is applied to a differential form, it is applied coefficientwise. $C^l(\Omega)$ [respectively $C^l_{(0,1)}(\Omega)$] denotes the set of all C^l functions [respectively (0, 1)-forms] on Ω. For $u \in C^l(\Omega)$ and $0 < \varepsilon < 1$ define

$$\|u\|_{\Omega,l} = \sup\{|D^\alpha \bar{D}^\beta f(z)| \,|\, z \in \Omega, |\alpha| + |\beta| \leqq l\}$$

$$\|u\|_{\Omega,l+\varepsilon} = \sup\left\{\frac{|D^\alpha \bar{D}^\beta f(z) - D^\alpha \bar{D}^\beta f(z')|}{|z - z'|^\varepsilon} \,\middle|\, z, z' \in \Omega, z \neq z', |\alpha| + |\beta| \leqq l\right\}.$$

For $f = \sum_{i=1}^n f_i(z)\, d\bar{z}_i \in C^l_{(0,1)}(\Omega)$ define $\|f\|_{\Omega,l} = \max_{1 \leqq i \leqq n} \|f_i\|_{\Omega,l}$.

In this paper we prove the following

Main Theorem. *Let* $0 \leqq k \leqq \infty$. *If* $N \geqq k + 4$ *and* $f \in C^\infty_{(0,1)}(\Omega)$ *with* $\bar{\partial} f = 0$, *then there exists* $u \in C^\infty(\Omega)$ *such that* $\bar{\partial} u = f$ *and* $\|u\|_{\Omega, l+\frac{1}{2}}$

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$\leqq C_l \|f\|_{\Omega,l}$ for all nonnegative integers $l \leqq k$, where C_l is a positive number independent of f (and small perturbations of Ω)[1].

The Main Theorem is proved by showing that the solution u constructed by Henkin [3] has the required bounds on its derivatives. By taking the Taylor expansion of f and using Stoke's theorem, we first derive a formula for the derivative of the Henkin solution which facilitates the estimation of its derivatives. Then we obtain the estimates of the derivatives of u for the special case where the coefficients of f are polynominals by using Stoke's theorem and the fact that a function differentiable of a high order on a curve in a domain in $\mathbb{C}$ can be extended to a function on the domain whose $\bar{\partial}$ derivative vanishes to a high order on the curve. The rest of the estimation is done in the same standard way as in the uniform bound case.

Lieb told me that Alt, by constructing by Frobenius theorem and other techniques a symmetric part of the singularity of the Ramirez kernel [6], could also show that the $\bar{\partial}$ problem can be solved with uniform bounds on derivatives and he could obtain the Hölder estimates of any exponent $<\frac{1}{2}$ for the derivatives of the highest order.

§ 1. The Integral Formula and the Solution Kernel

1.1. Let

$$F(\zeta, z) = \sum_{i=1}^{n} (D_i\varrho)(\zeta)(z_i - \zeta_i) + \frac{1}{2} \sum_{i,j=1}^{n} (D_i D_j \varrho)(\zeta)(z_i - \zeta_i)(z_j - \zeta_j).$$

Since ϱ is strictly plurisubharmonic on $\partial\Omega$, there exists $\lambda_0 > 0$ such that

$$-2\operatorname{Re} F(\zeta, z) \geqq \varrho(\zeta) - \varrho(z) + \lambda_0 |\zeta - z|^2 + o(|\zeta - z|^2) \tag{I}$$

for $\zeta \in \partial\Omega$. By solving the Cousin II problem with differentiable parameters (cf. [2]), (after shrinking $\tilde{\Omega}$) we obtain a C^{N-2} function $\Phi(\zeta, z)$ on $\tilde{\Omega} \times \tilde{\Omega}$ holomorphic in z with the following properties:

i) $\Phi(\zeta, z) \neq 0$ for $\zeta, z \in \tilde{\Omega}$ with $\varrho(\zeta) > \varrho(z)$,

ii) for $\zeta^0 \in \partial\Omega$ there exist an open neighborhood U of ζ^0 in $\tilde{\Omega}$ and a nowhere vanishing C^{N-2} function $H(\zeta, z)$ on $U \times U$ holomorphic in z such that $\Phi(\zeta, z) = H(\zeta, z)\, F(\zeta, z)$ on $U \times U$,

iii) there exist C^{N-2} functions $P_i(\zeta, z)$ on $\tilde{\Omega} \times \tilde{\Omega}$ holomorphic in z such that $\Phi(\zeta, z) = \sum_{i=1}^{n} (z_i - \zeta_i)\, P_i(\zeta, z)$.

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[1] A small perturbation of Ω means a domain $\{\varrho' < 0\}$ where $\varrho' \in C^N(\tilde{\Omega})$ with $D^\alpha \varrho'$ close to $D^\alpha \varrho$ on $\tilde{\Omega}$ for $|\alpha| \leqq N$.

1.2. Let

$$\Theta(Z_1, \ldots, Z_n) = \sum_{i=1}^{n} (-1)^i Z_i \wedge \left(\bigwedge_{\nu \neq i} dZ_\nu\right),$$

where $Z_1, \ldots, Z_n$ are interminates. Define

$$C(\zeta, z) = \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \Theta\left(\frac{P_1}{\Phi}, \ldots, \frac{P_n}{\Phi}\right) \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n$$

$$K'(\zeta, z, \lambda) = \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \Theta\left(\lambda \frac{\bar z_1 - \bar\zeta_1}{|z-\zeta|^2} + (1-\lambda)\frac{P_1}{\Phi}, \ldots, \lambda \frac{\bar z_n - \bar\zeta_n}{|z-\zeta|^2} + (1-\lambda)\frac{P_n}{\Phi}\right) \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n$$

(where $\lambda \in \mathbb{R}$) and

$$L(\zeta, z) = \frac{(n-1)!}{(2\pi\sqrt{-1})^n} \Theta\left(\frac{\bar z_1 - \bar\zeta_1}{|z-\zeta|^2}, \ldots, \frac{\bar z_n - \bar\zeta_n}{|z-\zeta|^2}\right) \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n.$$

For any C^1 function u on Ω^-, since

$$d(u(\zeta)\, K'(\zeta, z, \lambda)) = \bar\partial u(\zeta) \wedge K'(\zeta, z, \lambda) \quad \text{on} \quad \Omega^- \times [0,1] \quad \text{with} \quad \varrho(\zeta) > \varrho(z)$$

$$d(u(\zeta)\, L(\zeta, z)) = \bar\partial u(\zeta) \wedge L(\zeta, z) \quad \text{on} \quad \Omega^- \quad \text{with} \quad \zeta \neq z,$$

it follows from Stoke's theorem (cf. [2]) that

$$u(z) = \int_{\zeta \in \partial\Omega} u(\zeta)\, C(\zeta, z) + \int_{\substack{\zeta \in \partial\Omega \\ \lambda \in [0,1]}} \bar\partial u(\zeta) \wedge K'(\zeta, z, \lambda) - \int_{\zeta \in \Omega} \bar\partial u(\zeta) \wedge L(\zeta, z).$$

By integrating over $\lambda \in [0, 1]$ terms of $K'(\zeta, z, \lambda)$ containing $d\lambda$, we obtain a form $K(\zeta, z)$ and the following integral formula

$$u(z) = \int_{\zeta \in \partial\Omega} u(\zeta)\, C(\zeta, z) + \int_{\zeta \in \partial\Omega} \bar\partial u(\zeta) \wedge K(\zeta, z) - \int_{\zeta \in \Omega} \bar\partial u(\zeta) \wedge L(\zeta, z). \qquad \text{(II)}$$

1.3. For a C^∞ $\bar\partial$-closed (0, 1)-form f defined on an open neighborhood of Ω^-, we define

$$T_\Omega(f) = \int_{\zeta \in \partial\Omega} f \wedge K(\zeta, z) - \int_{\zeta \in \Omega} f \wedge L(\zeta, z).$$

Since there exists a C^∞ function u defined on an open neighborhood of Ω^- satisfying $\bar\partial u = f$, it follows from (II) that $\bar\partial T_\Omega(f) = f$ on Ω.

§ 2. Formula for First-Order Derivatives

Let $f = \sum_{i=1}^{n} f_i(z)\, d\bar z_i$ be a C^∞ $\bar\partial$-closed (0, 1)-form defined on some open neighborhood of Ω^-.

2.1. On Ω we have

$$T_\Omega(f) = \sum_{i=1}^{n} f_i(z)\, T_\Omega(d\bar{\zeta}_i) + \int_{\zeta\in\partial\Omega} (f(\zeta) - f(z)) \wedge K(\zeta, z) - \int_{\zeta\in\Omega} (f(\zeta) - f(z)) \wedge L(\zeta, z). \qquad \text{(III)}_f$$

Using $\frac{\partial}{\partial z_j} = \left(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \zeta_j}\right) - \frac{\partial}{\partial \zeta_j}$, we obtain

$$\begin{aligned}\frac{\partial}{\partial z_j} T_\Omega(f) = {} & \sum_{i=1}^{n} \frac{\partial f_i(z)}{\partial z_j} T_\Omega(d\bar{\zeta}_i) + \sum_{i=1}^{n} f_i(z) \frac{\partial}{\partial z_j} T_\Omega(d\bar{\zeta}_i) \\ & + \int_{\zeta\in\partial\Omega} \left(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \zeta_j}\right) [(f(\zeta) - f(z)) \wedge K(\zeta, z)] \\ & - \int_{\zeta\in\partial\Omega} \frac{\partial}{\partial \zeta_j} [(f(\zeta) - f(z)) \wedge K(\zeta, z)] \\ & - \int_{\zeta\in\Omega} \left(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \zeta_j}\right) [(f(\zeta) - f(z)) \wedge L(\zeta, z)] \\ & + \int_{\zeta\in\Omega} \frac{\partial}{\partial \zeta_j} [(f(\zeta) - f(z)) \wedge L(\zeta, z)].\end{aligned}$$

By comparing it with $\text{(III)}_{D_j f}$, we obtain

$$\begin{aligned}\frac{\partial}{\partial z_j} T_\Omega(f) = T_\Omega(D_j f) & + \sum_{i=1}^{n} f_i(z) \frac{\partial}{\partial z_j} T_\Omega(d\bar{\zeta}_i) \\ & + \int_{\zeta\in\partial\Omega} (f(\zeta) - f(z)) \wedge \left(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \zeta_j}\right) K(\zeta, z) \\ & - \int_{\zeta\in\partial\Omega} \frac{\partial}{\partial \zeta_j} [(f(\zeta) - f(z)) \wedge K(\zeta, z)] \\ & - \int_{\zeta\in\Omega} (f(\zeta) - f(z)) \wedge \left(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \zeta_j}\right) L(\zeta, z) \\ & + \int_{\zeta\in\Omega} \frac{\partial}{\partial \zeta_j} [(f(\zeta) - f(z)) \wedge L(\zeta, z)].\end{aligned}$$

2.2. For the purpose of estimation, we are going to transform

$$\int_{\zeta\in\partial\Omega} \frac{\partial}{\partial \zeta_j} [(f(\zeta) - f(z)) \wedge K(\zeta, z)]$$

by Stoke's theorem. Let

$$(f(\zeta)-f(z))\wedge K(\zeta,z)=\sum_{\mu,\nu=1}^{n} b_{\mu\nu}(\zeta,z)(f_\nu(\zeta)-f_\nu(z))\,d\bar\zeta_1\wedge\cdots\wedge\widehat{d\bar\zeta_\mu}\wedge\cdots\wedge d\bar\zeta_n\wedge d\zeta_1\wedge\cdots\wedge d\zeta_n$$

where $b_{\mu\nu}(\zeta,z)$ is a $\mathbb{Z}$-linear combination of coefficients of $K(\zeta,z)$ and $\widehat{d\bar\zeta_\mu}$ means that $d\bar\zeta_\mu$ is omitted. For $z\in\Omega$, by applying Stoke's theorem to

$$\begin{aligned}
d_\zeta&\left((-1)^{n+j-1}\sum_{\mu,\nu=1}^{n} b_{\mu\nu}(\zeta,z)(f_\nu(\zeta)-f_\nu(z))d\bar\zeta_1\wedge\cdots\wedge\widehat{d\bar\zeta_\mu}\wedge\cdots\wedge d\bar\zeta_n\wedge d\zeta_1\wedge\cdots\wedge\widehat{d\zeta_j}\wedge\cdots\wedge d\zeta_n\right)\\
&=\frac{\partial}{\partial\zeta_j}[(f(\zeta)-f(z))\wedge K(\zeta,z)]\\
&\quad+\sum_{\mu,\nu=1}^{n}(-1)^{n+j+\mu}\frac{\partial}{\partial\bar\zeta_\mu}[b_{\mu\nu}(\zeta,z)(f(\zeta)-f(z))]\,d\bar\zeta_1\wedge\cdots\wedge d\bar\zeta_n\wedge d\zeta_1\wedge\cdots\wedge\widehat{d\zeta_j}\wedge\cdots\wedge d\zeta_n
\end{aligned}$$

on $\partial\Omega$, we obtain

$$\int_{\zeta\in\partial\Omega}\frac{\partial}{\partial\zeta_j}[(f(\zeta)-f(z))\wedge K(\zeta,z)]=\int_{\zeta\in\partial\Omega}\sum_{\mu,\nu=1}^{n}(-1)^{n+j+\mu}\frac{\partial}{\partial\bar\zeta_\mu}\cdot[b_{\mu\nu}(\zeta,z)(f(\zeta)-f(z))]\,d\bar\zeta_1\wedge\cdots\wedge d\bar\zeta_n\wedge d\zeta_1\wedge\cdots\wedge\widehat{d\zeta_j}\wedge\cdots\wedge d\zeta_n.$$

2.3. Combining the results of (2.1) and (2.2) we have

$$\begin{aligned}
\frac{\partial}{\partial z_j}T_\Omega(f)&=T_\Omega(D_jf)+\sum_{i=1}^{n}f_i(z)\frac{\partial}{\partial z_j}T_\Omega(d\bar\zeta_i)\\
&\quad+\int_{\zeta\in\partial\Omega}(f(\zeta)-f(z))\wedge\left(\frac{\partial}{\partial z_j}+\frac{\partial}{\partial\zeta_j}\right)K(\zeta,z)\\
&\quad-\int_{\zeta\in\partial\Omega}\sum_{\mu,\nu=1}^{n}(-1)^{n+j+\mu}\frac{\partial}{\partial\bar\zeta_\mu}[b_{\mu\nu}(\zeta,z)(f_\nu(\zeta)-f_\nu(z))]\\
&\qquad\cdot d\bar\zeta_1\wedge\cdots\wedge d\bar\zeta_n\wedge d\zeta_1\wedge\cdots\wedge\widehat{d\zeta_j}\wedge\cdots\wedge d\zeta_n \qquad (\mathrm{IV})_{\Omega,f}\\
&\quad-\int_{\zeta\in\Omega}(f(\zeta)-f(z))\wedge\left(\frac{\partial}{\partial z_j}+\frac{\partial}{\partial\zeta_j}\right)L(\zeta,z)\\
&\quad+\int_{\zeta\in\Omega}\frac{\partial}{\partial\zeta_j}[(f(\zeta)-f(z))\wedge L(\zeta,z)].
\end{aligned}$$

We will show later that $\frac{\partial}{\partial z_j}T_\Omega(d\bar\zeta_i)$ is $\frac{1}{2}$-Hölder bounded on Ω if $N\geqq 5$, and the bound can be chosen to be independent of small perturbation of Ω. Let us assume it first.

Observe that

$$\left|\frac{\partial}{\partial \bar{\zeta}_\mu} F(\zeta, z)\right| \leqq \text{const} \, |\zeta - z|$$

and

$$\left|\left(\frac{\partial}{\partial z_j} + \frac{\partial}{\partial \zeta_j}\right) F(\zeta, z)\right| \leqq \text{const} \, |\zeta - z| \, .$$

Let $\Omega_m = \{\varrho < -\frac{1}{m}\}$ and let $h \in C^\infty_{(0,1)}(\Omega)$ with $\|h\|_{\Omega,1} < \infty$. Using $(\text{IV})_{\Omega_m,h}$ and employing the standard method of estimation (i.e. using (I) and taking $\operatorname{Im} F(\zeta, z)$ as one local coordinate of $\partial \Omega_m$ in the evaluation of the integrals over $\partial \Omega_m$, just as in the case of getting uniform bounds and Hölder conditions for solutions of the $\bar{\partial}$ problem [3, 4, 7]), we conclude that, for m large enough,

$$\left\|\frac{\partial}{\partial z_j} T_{\Omega_m}(h)\right\|_{\Omega_m, \frac{1}{2}} \leqq C \|h\|_{\Omega_m, 1}$$

where C is independent of h and m. (Henkin and Romanov [4], in getting the Hölder condition of exponent $\frac{1}{2}$, makes use of $\bar{\partial} h = 0$. Actually this is not necessary, as is shown in [7].)

Since

$$\sup_m \|T_{\Omega_m}(h)\|_{\Omega_m, 0} < \infty$$

and $\bar{\partial} T_{\Omega_m}(h) = h$ on Ω_m, there exist $g \in C^\infty(\Omega)$ and a sequence $\{m_\nu\}$ such that the derivatives of $T_{\Omega_{m_\nu}}(h)$ converge uniformly on compact subsets of Ω to the derivatives of g. Then $\bar{\partial} g = h$ on Ω and $\|g\|_{\Omega, \frac{3}{2}} \leqq C \|h\|_{\Omega, 1}$. This is precisely the Main Theorem for the case $k = 1$.

2.4. In preparation of deriving the analog of $(\text{IV})_{\Omega,f}$ for the case $k > 1$ we have to make a transformation in one of the integrals in $(\text{IV})_{\Omega,f}$, because one integral involves

$$d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_1 \wedge \cdots \wedge \widehat{d\zeta_j} \wedge \cdots \wedge d\zeta_n$$

instead of

$$d\bar{\zeta}_{\nu_1} \wedge \cdots \wedge d\bar{\zeta}_{\nu_{n-1}} \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n$$

whose presence is necessary for the process of induction on k.

We cover $\partial \Omega$ by a finite number of open subsets $U_i (1 \leqq i \leqq l)$ of $\tilde{\Omega}$ such that $\frac{\partial \varrho}{\partial \bar{\zeta}_{\nu_i}}(\zeta)$ is nowhere zero on U_i for some $1 \leqq \nu_i \leqq n$. Since $\varrho \equiv 0$ on $\partial \Omega$,

$$d\bar{\zeta}_{\nu_i} = -\left(\frac{\partial \varrho}{\partial \bar{\zeta}_{\nu_i}}(\zeta)\right)^{-1} \left(\sum_{\mu=1}^{n} \frac{\partial \varrho}{\partial \zeta_\mu}(\zeta)\, d\zeta_\mu + \sum_{\mu \neq \nu_i} \frac{\partial \varrho}{\partial \bar{\zeta}_\mu}(\zeta)\, d\bar{\zeta}_\mu\right)$$

when pulled back to $U_i \cap \partial\Omega$. Let σ_i $(1 \leqq i \leqq l)$ be a nonnegative C^∞ function with compact support in U_i such that $\sum_{i=1}^{l} \sigma_i \equiv 1$ on a neighborhood of $\partial\Omega$ in $\tilde{\Omega}$. Hence, when pulled back to $\partial\Omega$,

$$\begin{aligned} d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_1 \wedge \cdots \wedge \widehat{d\zeta_j} \wedge \cdots \wedge d\zeta_n \\ = \sum_{i=1}^{l} (-1)^{n-\nu_i+j} \sigma_i(\zeta) \frac{\partial \varrho}{\partial \zeta_j}(\zeta) \left(\frac{\partial \varrho}{\partial \bar{\zeta}_{\nu_i}}(\zeta) \right)^{-1} \\ \cdot d\bar{\zeta}_1 \wedge \cdots \wedge \widehat{d\bar{\zeta}_{\nu_i}} \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n . \end{aligned}$$

§ 3. Estimation in the Polynomial Case

Fix $\zeta^0 \in \partial\Omega$ and let $\tilde{U}$ be an open neighborhood of ζ^0 in $\tilde{\Omega}$.

3.1. Lemma. *Let* $\phi_i, \phi_{ij} \in C^1(\tilde{U})$ *such that, for* $\zeta \in \tilde{U} \cap \partial\Omega$, $\phi_i(\zeta) = \frac{\partial \varrho}{\partial \zeta_i}(\zeta)$ *and* $\phi_{ij}(\zeta) = \frac{\partial^2 \varrho}{\partial \zeta_i \partial \zeta_j}(\zeta)$. *Let* $\tilde{F}(\zeta, z) = \sum_{i=1}^{n} \phi_i(\zeta)(\zeta_i - z_i) + \frac{1}{2} \sum_{i,j=1}^{n} \phi_{ij}(\zeta)(\zeta_i - z_i)(\zeta_j - z_j)$. *Then there exist* $c > 0$ *and an open neighborhood* U *of* ζ^0 *in* $\tilde{U}$ *such that* $|F(\zeta, z)| \geqq c|\zeta - z|^2$ *for* $\zeta, z \in U$ *and* $\varrho(z) \leqq 0 \leqq \varrho(\zeta)$.

Proof. There exist $A > 0$ and an open neighborhood U_1 of ζ^0 in $\tilde{U}$ with diameter $\leqq \frac{1}{2}$ such that

$$\left| \phi_i(\zeta) - \frac{\partial \varrho}{\partial \zeta_i}(\zeta) \right| \leqq A \varrho(\zeta) \qquad (1 \leqq i \leqq n)$$

$$\left| \phi_{ij}(\zeta) - \frac{\partial^2 \varrho}{\partial \zeta_i \partial \zeta_j}(\zeta) \right| \leqq A \varrho(\zeta) \qquad (1 \leqq i, j \leqq n)$$

for $\zeta \in U_1$ and $\varrho(\zeta) \geqq 0$. It follows that

$$|\tilde{F}(\zeta, z) - F(\zeta, z)| \leqq \left(n + \frac{n^2}{2} \right) A \varrho(\zeta) \, |\zeta - z|$$

for $\zeta, z \in U_1$ and $\varrho(\zeta) \geqq 0$. By (I) there exists an open neighborhood U_2 of ζ^0 in U_1 such that

$$|F(\zeta, z)| \geqq \varrho(\zeta) + \frac{\lambda_0}{2} |\zeta - z|^2$$

for $\zeta, z \in U_2$ and $\varrho(z) \leqq 0 \leqq \varrho(\zeta)$. Let U be an open neighborhood of ζ^0 in U_2 with diameter $\leqq \left(2\left(n+\frac{n^2}{2}\right)A\right)^{-1}$. Then

$$\begin{aligned}|\tilde{F}(\zeta, z)| &\geqq |F(\zeta, z)| - |\tilde{F}(\zeta, z) - F(\zeta, z)| \\ &\geqq \varrho(\zeta)\left(1-\left(n+\frac{n^2}{2}\right)A|\zeta - z|\right) + \frac{\lambda_0}{2}|\zeta - z|^2 \\ &\geqq \frac{\lambda_0}{2}|\zeta - z|^2\end{aligned}$$

for $\zeta, z \in U$ and $\varrho(z) \leqq 0 \leqq \varrho(\zeta)$. Q.E.D.

3.2. Lemma. *Let $m \leqq l \leqq N$ be positive integers and let G be an open subset of $\mathbb{C}^n$. Suppose $1 \leqq j \leqq n$ and $\frac{\partial \varrho}{\partial \bar{\zeta}_j}(\zeta) \neq 0$ for $\zeta \in \tilde{U}$. Then every C^l function $h(\zeta, z)$ on $(\tilde{U} \cap \partial\Omega) \times G$ can be extended to a C^{l-m+1} function $\tilde{h}(\zeta, z)$ on $\tilde{U} \times G$ such that $\frac{\partial \tilde{h}}{\partial \bar{\zeta}_j}(\zeta, z) = \gamma(\zeta, z)\,\varrho(\zeta)^{m-1}$ for some C^{l-m} function $\gamma(\zeta, z)$ on $\tilde{U} \times G$.*

Proof. Let $h_0(\zeta, z)$ be a C^l function on $\tilde{U} \times G$ with extends $h(\zeta, z)$. For $0 \leqq \nu < m$ define by induction on ν a $C^{l-\nu}$ function $h_\nu(\zeta, z)$ on $\tilde{U} \times G$ such that

$$\frac{\partial h_\nu}{\partial \bar{\zeta}_j}(\zeta, z) = h_{\nu+1}(\zeta, z)\frac{\partial \varrho}{\partial \bar{\zeta}_j}(\zeta).$$

Let

$$\tilde{h}(\zeta, z) = \sum_{\nu=0}^{m-1}(-1)^\nu \frac{1}{\nu!} h_\nu(\zeta, z)\,\varrho(\zeta)^\nu.$$

Then

$$\frac{\partial h}{\partial \bar{\zeta}_j}(\zeta, z) = (-1)^{m-1}\frac{1}{(m-1)!}\frac{\partial h_{m-1}}{\partial \bar{\zeta}_j}(\zeta, z)\,\varrho(\zeta)^{m-1}. \qquad \text{Q.E.D.}$$

3.3. Proposition. *Let $l \geqq 0$ and $m \geqq n$ be intergers such that $l \leqq 2(m-n)+1$ and $2(m-n)-l+4 \leqq N$. Let $1 \leqq \sigma_1 \leqq \cdots \leqq \sigma_l \leqq n$ and*

$$g(\zeta, z) = \prod_{i=1}^{l}(\zeta_{\sigma_i} - z_{\sigma_i}).$$

If

$$\omega(\zeta, z) = \sum_{\nu=1}^{n} a_\nu(\zeta, z)\, d\bar{\zeta}_1 \wedge \cdots \wedge \widehat{d\bar{\zeta}_\nu} \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n$$

is a $C^{2(m-n)-l+2}$ $(n, n-1)$-form on $\tilde{\Omega} \times \tilde{\Omega}$, then $\int_{\zeta \in \partial\Omega} \frac{g(\zeta, z)\,\omega(\zeta, z)}{\Phi(\zeta, z)^m}$ is uniformly bounded for $z \in \Omega$.

Proof. We need only show this locally near $\partial\Omega$. Take $\zeta^0 \in \partial\Omega$. After changing the coordinates system linearly, we can assume without loss of generality that there exists an open neighborhood $\tilde{U}$ of ζ^0 in $\tilde{\Omega}$ such that $\frac{\partial\varrho}{\partial\bar{\zeta}_j}(\zeta) \neq 0$ for $\zeta \in \tilde{U}$ and $1 \leqq j \leqq n$. We can also assume that $H(\zeta, z)$ is defined on $\tilde{U} \times \tilde{U}$ and $\Phi(\zeta, z) = H(\zeta, z)\, F(\zeta, z)$ on $\tilde{U} \times \tilde{U}$.

Fix $1 \leqq v \leqq n$. By (3.2) we can find C^1 functions $\phi_i(\zeta), \phi_{ij}(\zeta)$ on $\tilde{U}$ such that

$$\phi_i(\zeta) = \frac{\partial\varrho}{\partial\zeta_i}(\zeta), \quad \phi_{ij} = \frac{\partial^2\varrho}{\partial\zeta_i\partial\zeta_j}(\zeta) \quad \text{for} \quad \zeta \in \tilde{U} \cap \partial\Omega,$$

$$\frac{\partial\phi_i}{\partial\bar{\zeta}_v}(\zeta) = \alpha_i(\zeta)\, \varrho(\zeta)^{2(m-n)-l+2},$$

$$\frac{\partial\phi_{ij}}{\partial\bar{\zeta}_v}(\zeta) = \alpha_{ij}(\zeta)\, \varrho(\zeta)^{2(m-n)-l+1} \quad \text{on} \quad \tilde{U} \tag{V}$$

for some C^0 functions $\alpha_i(\zeta), \alpha_{ij}(\zeta)$ on $\tilde{U}$. Also, by (3.2) we can find C^1 functions $a_v(\zeta, z)$ on $\tilde{U} \times \tilde{\Omega}$ such that $\tilde{a}_v(\zeta, z) = a_v(\zeta, z)\, H(\zeta, z)^{-m}$ on $(\tilde{U} \cap \partial\Omega) \times \tilde{\Omega}$ and

$$\frac{\partial\tilde{a}_v}{\partial\bar{\zeta}_v}(\zeta, z) = \gamma_v(\zeta, z)\, \varrho(\zeta)^{2(m-n)-l+1} \quad \text{on} \quad \tilde{U} \times \tilde{\Omega} \tag{VI}$$

for some C^0 function $\gamma_v(\zeta, z)$ on $\tilde{U} \times \tilde{\Omega}$. Let

$$\tilde{F}(\zeta, z) = \sum_{i=1}^{n} \phi_i(\zeta)(z_i - \zeta_i) + \frac{1}{2} \sum_{i,j=1}^{n} \phi_{ij}(\zeta)(z_i - \zeta_i)(z_j - \zeta_j).$$

By (3.1) there exist $c_v > 0$ and a relatively compact open neighborhood U_v of ζ^0 in $\tilde{U}$ such that

$$|\tilde{F}(\zeta, z)| \geqq c_v|\zeta - z|^2 \quad \text{for} \quad \zeta, z \in U_v \quad \text{and} \quad \varrho(z) \leqq 0 \leqq \varrho(\zeta). \tag{VII}$$

Let

$$R_v(\zeta, z) = g(\zeta, z)\left(\frac{\frac{\partial\tilde{a}_v}{\partial\bar{\zeta}_v}(\zeta, z)}{\tilde{F}(\zeta, z)^m} - m\frac{\tilde{a}_v(\zeta, z)\frac{\partial\tilde{F}}{\partial\bar{\zeta}_v}(\zeta, z)}{\tilde{F}(\zeta, z)^{m+1}}\right).$$

It follows from (V), (VI) and (VII) that

$$|R_v(\zeta, z)| \leqq \text{const}\, |\zeta - z|^{-2n+1} \quad \text{for} \quad \zeta \in U_v - \Omega^- \quad \text{and} \quad z \in U_v \cap \Omega. \tag{VIII}$$

Let B_v be a relatively compact open neighborhood of ζ^0 in U_v such that ∂B_v is C^1 and the normal vector of ∂B_v and $\partial\Omega$ are independent at every point of $\partial B_\nu \cap \partial\Omega$. For $z \in B_\nu \cap \Omega$, by applying Stoke's theorem to

$$d_\zeta\left(\frac{g(\zeta,z)\,\tilde{a}_\nu(\zeta,z)}{\tilde{F}(\zeta,z)^m}\,d\bar{\zeta}_1 \wedge \cdots \wedge \widehat{d\bar{\zeta}_\nu} \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n\right)$$
$$= (-1)^{\nu-1} R_\nu(\zeta,z)\,d\bar{\zeta} \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n$$

on $B_\nu - \Omega^-$, we obtain

$$\int_{\zeta \in B_\nu \cap \partial\Omega} \frac{g(\zeta,z)\,a_\nu(\zeta,z)}{\Phi(\zeta,z)^m}\,d\bar{\zeta}_1 \wedge \cdots \wedge \widehat{d\bar{\zeta}_\nu} \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_1 \wedge \cdots \wedge d\zeta n$$
$$= -\int_{\zeta \in \partial B_\nu - \Omega} \frac{g(\zeta,z)\,\tilde{a}_\nu(\zeta,z)}{\tilde{F}(\zeta,z)^m}\,d\bar{\zeta}_1 \wedge \cdots \wedge \widehat{d\bar{\zeta}_\nu} \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n$$
$$+ \int_{\zeta \in B_\nu - \Omega^-} (-1)^{\nu-1} R_\nu(\zeta,z)\,d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n.$$

It follows from (VIII) that, if U is a relatively compact open neighborhood of ζ^0 in $\bigcap_{\nu=1}^{n} B_\nu$, then

$$\int_{\zeta \in \partial\Omega} \frac{g(\zeta,z)\,\omega(\zeta,z)}{\Phi(\zeta,z)^m}$$

is uniformly bounded for $z \in U \cap \Omega$. Q.E.D.

3.4. Proposition. *Let $k \in \mathbb{N}$. If $N \geq k+3$, then $D_z^\alpha T_\Omega(\bar{\partial}_\zeta(\bar{\zeta}-\bar{z})^\gamma)$ is uniformly bounded on Ω for $\alpha, \gamma \in \mathbb{N}^n$ with $|\alpha| \leq k$, and the bound can be chosen to be independent of small perturbations of Ω (where $T_\Omega(\bar{\partial}_\zeta(\bar{\zeta}-\bar{z})^\gamma)$ means*

$$\left.\int_{\zeta \in \partial\Omega} \bar{\partial}_\zeta(\bar{\zeta}-\bar{z})^\gamma \wedge K(\zeta,z) - \int_{\zeta \in \Omega} \bar{\partial}_\zeta(\bar{\zeta}-\bar{z})^\gamma \wedge L(\zeta,z)\right).$$

Proof. Using $\dfrac{\partial}{\partial z_j} = \left(\dfrac{\partial}{\partial z_j} + \dfrac{\partial}{\partial \zeta_j}\right) - \dfrac{\partial}{\partial \zeta_j}$ and applying Stoke's theorem as in (2.2) and using (2.4), we conclude from (3.3) that $D_z^\alpha \int_{\zeta \in \partial\Omega} (\bar{\zeta}-\bar{z})^\gamma C(\zeta,z)$ is uniformly bounded on Ω for $|\alpha| \leq k$ if $N \geq k+3$. The arguments used in obtaining the bound clearly imply that the bound can be chosen to be independent of small perturbations of Ω. The proposition now follows from

$$T_\Omega(\bar{\partial}_\zeta(\bar{\zeta}-\bar{z})^\gamma) = -\int_{\zeta \in \partial\Omega} (\bar{\zeta}-\bar{z})^\gamma C(\zeta,z)$$

which is a consequence of (II) when we substitute $u(\zeta)$ by $(\bar{\zeta}-\bar{z}')^\gamma$ and let $z' = z$. Q.E.D.

3.5. It follows from (3.4) that $\dfrac{\partial}{\partial z_j} T_\Omega(d\bar{\zeta}_i)$ is $\frac{1}{2}$-Hölder bounded on Ω and the bound can be chosen to be independent of small perturbations of Ω if $N \geq 5$. Hence the Main Theorem is proved for $k=1$.

§ 4. The Case $k>1$

4.1. For $l \in \mathbb{N}$ and $g(\zeta) \in C^l(\Omega)$ define

$$g^{(l)}(\zeta, z) = g(\zeta) - \sum_{|\beta|+|\beta'| \leq l} \frac{1}{\beta!} \frac{1}{(\beta')!} (D^\beta \bar{D}^{\beta'} g)(z)(\zeta - z)^\beta (\bar\zeta - \bar z)^{\beta'}.$$

For $g \in C^0(\Omega)$ define $g^{(-1)}(\zeta, z) = g(\zeta)$. The following identities hold:

$$\frac{\partial}{\partial \zeta_j} g^{(l)}(\zeta, z) = \left(\frac{\partial g}{\partial \zeta_j}\right)^{(l-1)} (\zeta, z),$$

$$\frac{\partial}{\partial \bar\zeta_j} g^{(l)}(\zeta, z) = \left(\frac{\partial g}{\partial \bar\zeta_j}\right)^{(l-1)} (\zeta, z),$$

$$\left(\frac{\partial}{\partial \zeta_j} + \frac{\partial}{\partial z_j}\right) g^{(l)}(\zeta, z) = \left(\frac{\partial g}{\partial \zeta_j}\right)^{(l)} (\zeta, z).$$

Suppose $f(\zeta) \in C^k(\Omega)$ and let $g_{z'}(\zeta) = f^{(k)}(\zeta, z')$. Then

$$(g_{z'})^{(l)}(\zeta, z) = f^{(k)}(\zeta, z) \quad \text{for} \quad z' = z \quad \text{and} \quad l \leq k.$$

4.2. Proposition. *If* $|\alpha| = k > 0$ *and* $f = \sum_{i=1}^{n} f_i d\bar z_i$ *is a* C^∞ $\bar\partial$*-closed* $(0, 1)$*-form on an open neighborhood of* Ω^-, *then*

$$\begin{aligned} D_z^\alpha T_\Omega(f) = & \sum_{\substack{|\beta|+|\beta'|<k \\ 1 \leq i \leq n}} a^\alpha_{\beta\beta' i}(z)(D^\beta \bar D^{\beta'} f_i)(z) \\ & + \sum_{\substack{|\beta|+|\beta'| \leq k \\ 1 \leq i \leq n}} \int_{\zeta \in \partial\Omega} (D^\beta \bar D^{\beta'} f_i)^{(k-1-|\beta|-|\beta'|)}(\zeta, z) K^\alpha_{\beta\beta' i}(\zeta, z) \\ & + \sum_{\substack{|\beta|+|\beta'| \leq k \\ 1 \leq i \leq n}} \int_{\zeta \in \Omega} (D^\beta \bar D^{\beta'} f_i)^{(k-1-|\beta|-|\beta'|)}(\zeta, z) L^\alpha_{\beta\beta' i}(\zeta, z) \end{aligned} \qquad (\mathrm{IX})^\alpha_\Omega$$

where.

i) $a^\alpha_{\beta\beta' i}(z)$ *has uniformly bounded derivatives of order* $\leq N-k-3$ *and the bounds can be chosen to be independent of small perturbations of* Ω,

ii) $K^\alpha_{\beta\beta' i}(\zeta, z)$ *is an* $(n, n-1)$*-form in* ζ *whose coefficients are obtained by applying to the coefficients of* $K(\zeta, z)$ *the two operations* $\frac{\partial}{\partial \zeta_l} + \frac{\partial}{\partial z_l}, \frac{\partial}{\partial \bar\zeta_l}$ $(1 \leq l \leq n)$ *a total number of* $k - |\beta| - |\beta'|$ *times and then taking linear combinations with coefficients in* $C^{N-1}(\tilde\Omega)$,

iii) $L^\alpha_{\beta\beta' i}(\zeta, z)$ *is an* (n, n)*-form in* ζ *whose coefficients are obtained by applying to the coefficients of* $L(\zeta, z)$ *the operation* $\frac{\partial}{\partial \zeta_l}$ $(1 \leq l \leq n)$ $k - |\beta| - |\beta'|$ *times and then taking linear combinations with coefficients in* $\mathbb{C}$.

Proof. The case $k=1$ follows from (2.3), (2.4), and (3.5). For the general case we use induction on k. Suppose $(\mathrm{IX})^{\alpha}_{\Omega}$ is true for $|\alpha|=k$. Fix $1\leqq \nu\leqq n$ and let

$$\tilde{\alpha}=(\alpha_1,\ldots,\alpha_{\nu-1},\alpha_\nu+1,\alpha_{\nu+1},\ldots,\alpha_n).$$

We are going to derive $(\mathrm{IX})^{\tilde{\alpha}}_{\Omega}$.

For $\gamma\in\mathbb{N}^n$ with $|\gamma|>0$, let J_γ be the set of all (α,i) with $\alpha\in\mathbb{N}^n$ and $1\leqq i\leqq n$ such that $D^\alpha D_i=D^\gamma$. Since $\bar{\partial}f=0$, we have $\bar{D}^\alpha f_i=\bar{D}^\beta f_j$ for $(\alpha,i),(\beta,j)\in J_\gamma$. For $\gamma\in\mathbb{N}^n$ with $|\gamma|>0$, define $f_\gamma(z)=(\bar{D}^\alpha f_i)(z)$ for $(\alpha,i)\in J_\gamma$. This definition is independent of the choise of (α,i). From the definitions of $f_i^{(k)}$ and f_γ we obtain

$$\begin{aligned}\sum_{i=1}^{n} f_i^{(k)}(\zeta,z)\,d\bar{\zeta}_i &= f(\zeta)-\sum_{|\beta|+|\gamma|\leqq k+1}\frac{1}{\beta!}(\zeta-z)^\beta\sum_{(\beta',i)\in J_\gamma}\frac{1}{(\beta')!}\\ &\qquad\cdot(D^\beta\bar{D}^{\beta'}f_i)(z)(\bar{\zeta}-\bar{z})^{\beta'}\,d\bar{\zeta}_i\\ &= f(\zeta)-\sum_{|\beta|+|\gamma|\leqq k+1}\frac{1}{\beta!}\frac{1}{\gamma!}(\zeta-z)^\beta(D^\beta f_\gamma)(z)\,\bar{\partial}_\zeta(\bar{\zeta}-\bar{z})^\gamma.\end{aligned}$$

Substituting $f_i(\zeta)$ in $(\mathrm{IX})^{\alpha}_{\Omega}$ by $f_i^{(k)}(\zeta,z')$, using the above expression for $\sum_{i=1}^{n} f_i^{(k)}(\zeta,z')\,d\bar{\zeta}_i$, and then setting $z'=z$, we conclude from (4.1) that

$$\begin{aligned}D^\alpha_z T_\Omega(f) &= \sum_{|\beta|+|\gamma|\leqq k+1}\frac{1}{\beta!}\frac{1}{\gamma!}(D^\beta f_\gamma)(z)\,D^\alpha_z T_\Omega(\bar{\partial}_\zeta(\bar{\zeta}-\bar{z})^\gamma)\\ &\quad+\sum_{\substack{|\beta|+|\beta'|\leqq k\\ 1\leqq i\leqq n}} a^\alpha_{\beta\beta' i}(z)\,D^\beta\bar{D}^{\beta'}f_i(z)\\ &\quad+\sum_{\substack{|\beta|+|\beta'|\leqq k\\ 1\leqq i\leqq n}}\int_{\zeta\in\partial\Omega}(D^\beta\bar{D}^{\beta'}f_i)^{(k-|\beta|-|\beta'|)}(\zeta,z)\,K^\alpha_{\beta\beta' i}(\zeta,z)\\ &\quad+\sum_{\substack{|\beta|+|\beta'|\leqq k\\ 1\leqq i\leqq n}}\int_{\zeta\in\Omega}(D^\beta\bar{D}^{\beta'}f_i)^{(k-|\beta|-|\beta'|)}(\zeta,z)\,L^\alpha_{\beta\beta' i}(\zeta,z).\end{aligned}\qquad (\mathrm{X})_f$$

We apply $\dfrac{\partial}{\partial z_\nu}$ to both sides of $(\mathrm{X})_f$ and we use the following two identities

$$\begin{aligned}&\frac{\partial}{\partial z_\nu}\int_{\zeta\in\partial\Omega}(D^\beta\bar{D}^{\beta'}f_i)^{(k-|\beta|-|\beta'|)}(\zeta,z)\,K^\alpha_{\beta\beta' i}(\zeta,z)\\ &\quad=\int_{\zeta\in\partial D}\left(\frac{\partial}{\partial z_\nu}+\frac{\partial}{\partial\zeta_\nu}\right)((D^\beta\bar{D}^{\beta'}f_i)^{(k-|\beta|-|\beta'|)}(\zeta,z)\,K^\alpha_{\beta\beta' i}(\zeta,z))\\ &\qquad-\int_{\zeta\in\partial D}\frac{\partial}{\partial\zeta_\nu}((D^\beta\bar{D}^{\beta'}f_i)^{(k-|\beta|-|\beta'|)}(\zeta,z)\,K^\alpha_{\beta\beta' i}(\zeta,z))\end{aligned}$$

$$\frac{\partial}{\partial z_\nu} \int_{\zeta\in\Omega} (D^\beta \bar D^{\beta'} f_i)^{(k-|\beta|-|\beta'|)}(\zeta, z)\, L^\alpha_{\beta\beta' i}(\zeta, z)$$
$$= \int_{\zeta\in\Omega} \left(\frac{\partial}{\partial z_\nu} + \frac{\partial}{\partial \zeta_\nu}\right)((D^\beta \bar D^{\beta'} f_i)^{(k-|\beta|-|\beta'|)}(\zeta, z)\, L^\alpha_{\beta\beta' i}(\zeta, z))$$
$$- \int_{\zeta\in\Omega} \frac{\partial}{\partial \zeta_\nu}((D^\beta \bar D^{\beta'} f_i)^{(k-|\beta|-|\beta'|)}(\zeta, z)\, L^\alpha_{\beta\beta' i}(\zeta, z)).$$

Now we transform

$$\int_{\zeta\in\partial\Omega} \frac{\partial}{\partial \zeta_\nu}((D^\beta \bar D^{\beta'} f_i)^{(k-|\beta|-|\beta'|)}(\zeta, z)\, K^\alpha_{\beta\beta' i}(\zeta, z))$$

by Stoke's theorem as in (2.2) and compare the final result with $(\mathrm{X})_{D_\nu f}$. The proposition now follows from (3.4) and (2.4). Q.E.D.

4.3. Proof of the Main Theorem

Let $\Omega_m = \left\{\varrho < -\frac{1}{m}\right\}$. Assume $l \in \mathbb{N}$ with $l \leqq k$ and $f \in C^\infty_{(0,1)}(\Omega)$ with $\|f\|_{\Omega,l} < \infty$. Using $(\mathrm{IX})^\alpha_{\Omega_m}$ and employing the standard method of estimation as mentioned in (2.3), we conclude that, for m sufficiently large,

$$\|D^\alpha_z T_{\Omega_m}(f)\|_{\Omega_m, \frac{1}{2}} \leqq C\|f\|_{\Omega_m, l}$$

for $|\alpha| \leqq l$. Since

$$\sup_m \|T_{\Omega_m}(f)\|_{\Omega_m, 0} < \infty$$

and $\bar\partial T_{\Omega_m}(f) = f$ on Ω_m, there exist $u \in C^\infty(\Omega)$ and a subsequence $\{m_\nu\}$ such that the derivatives of $T_{\Omega_{m_\nu}}(f)$ converge uniformly on compact subsets of Ω to the derivatives of u. Then $\bar\partial u = f$ on Ω and $\|u\|_{\Omega, l+\frac{1}{2}} \leqq C\|f\|_{\Omega, l}$. Q.E.D.

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