BEAUMONT, R. A., and R. S. PIERCE
Math. Annalen 153, 21—37 (1964)

Isomorphic Direct Summands of Abelian Groups*

By

R. A. BEAUMONT and R. S. PIERCE in Seattle, Wash./USA

1. Introduction

In a previous paper [1], the authors studied those abelian groups which contain isomorphic proper subgroups. It was found that the non-existence of such a subgroup is somewhat exceptional. In the case of torsion groups of power greater than the continuum, it was even shown that there are always isomorphic proper subgroups which are direct summands. This observation suggests the study of arbitrary abelian groups which have isomorphic proper direct summands. It turns out that these groups enjoy many special properties which make them worthy of consideration. One of these is the following result which is basic in this investigation.

(1.1) **Lemma.** *An abelian group G has an isomorphic proper direct summand if and only if there exist φ, ψ in the endomorphism ring of G such that $\psi\varphi = 1$ and $\varphi\psi \neq 1$.*

Proof. Let φ be a monomorphism of G into itself such that $G = \varphi(G) \oplus H$ with $H \neq 0$. Then the endomorphism ψ of G defined by $\psi = \varphi^{-1}$ on $\varphi(G)$ and $\psi = 0$ on H satisfies $\psi\varphi = 1$ and $\varphi\psi \neq 1$. Conversely, if φ and ψ are endomorphisms of G which satisfy $\psi\varphi = 1$ and $\varphi\psi \neq 1$, then φ is one-to-one, $\operatorname{Ker}\psi \neq 0$, and it is readily verified that $G = \varphi(G) \oplus \operatorname{Ker}\ \psi$.

It follows from Lemma 1.1 that the notion of a group with an isomorphic proper direct summand is self-dual: There is a monomorphism φ of G into itself with $\operatorname{Im}\varphi$ a proper direct summand if and only if there is an epimorphism ψ of G onto itself with $\operatorname{Ker}\psi$ a proper direct summand. Moreover, Lemma 1.1 suggests that the proper object to study is a system $\langle G; \varphi, \psi\rangle$, where G is an abelian group and φ and ψ are endomorphisms such that $\psi\varphi = 1$. We show that the study of such systems is equivalent to the study of modules over a ring Δ which is freely generated over Z by non-commuting indeterminates X and Y subject to the relation $XY = 1$.

Because of the many connections between pure subgroups and direct summands, it is of interest to consider abelian groups with isomorphic proper pure subgroups along with an investigation of groups with isomorphic proper direct summands. To simplify our terminology we call an abelian group G:

an I-*group* if G has an isomorphic proper subgroup;

an IP-*group* if G has an isomorphic proper pure subgroup;

an ID-*group* if G has an isomorphic proper direct summand.

* This work was supported by the National Science Foundation research grant No. G-19915.

Clearly, G is an ID-group implies G is an IP-group implies G is an I-group. Suitable examples show that neither of these implications can be reversed.

Throughout this paper, group means abelian group. The notation and terminology used is a mixture of that in Fuchs [4], Kaplansky [5], and Cartan and Eilenberg [2].

2. Characterization of IP-groups and ID-groups

The problem of characterizing IP-groups and ID-groups can be pursued along the familiar road of reduction to the reduced and divisible cases, and for torsion groups, to the primary case. We first give some simple lemmas.

(2.1) **Lemma.** *If G has a direct summand which is an ID-group (IP-group), then G is an ID-group (IP-group).*

(2.2) **Lemma.** *If G is a direct sum of infinitely many copies of the same non-zero group, then G is an ID-group.*

(2.3) **Lemma.** *Let φ be a monomorphism of G into G and let N be a fully invariant subgroup of G.*

(a) *If $\varphi(G)$ is a direct summand of G and N is not an ID-group, then $\varphi(N) = N$.*

(b) *If $\varphi(G)$ and N are pure subgroups of G and N is not an IP-group, then $\varphi(N) = N$.*

Proof. To prove (a), let $G = \varphi(G) \oplus H$. Clearly, $\varphi(N) \subseteq N \cap \varphi(G)$. By Lemma 1.1, there is an endomorphism ψ of G such that $\psi\varphi$ is the identity on G. If $\varphi(x) \in N$, then $x = \psi\varphi(x) \in \psi(N) \subseteq N$, since N is fully invariant. Thus, $\varphi(N) = \varphi(G) \cap N$. By Lemma 21.1 in [4], it follows that

$$N = (\varphi(G) \cap N) \oplus (H \cap N) = \varphi(N) \oplus (H \cap N) .$$

Since N is not an ID-group, $N = \varphi(N)$.

To prove (b), it is sufficient to show that $\varphi(N)$ is pure in N. Assume that $\varphi(x) = my$, where $x, y \in N$ and m is a positive integer. Since $\varphi(G)$ is pure in G, there exists $z \in G$ such that

$$\varphi(x) = m\varphi(z) = \varphi(mz) .$$

Thus, $x = mz$, and since N is pure in G, there exists $w \in N$ such that $x = mw$. Therefore,

$$\varphi(x) = \varphi(mw) = m\varphi(w) ,$$

where $\varphi(w) \in \varphi(N)$.

(2.4) **Lemma.** *Let N be a fully invariant subgroup of G.*

(a) *If N and G/N are not ID-groups, then G is not an ID-group.*

(b) *If N is a pure subgroup of G and N and G/N are not IP-groups, then G is not an IP-group.*

Proof. To prove (a), assume that φ is a monomorphism of G into G such that $G = \varphi(G) \oplus H$. Since N is not an ID-group, it follows from Lemma 2.3 (a) that $N = \varphi(N) \subseteq \varphi(G)$. Therefore,

$$G/N = \varphi(G)/N \oplus (H + N)/N, \quad \text{and} \quad \varphi(G)/N = \varphi(G)/\varphi(N) \cong G/N .$$

Since G/N is not an ID-group, $H + N = N = \varphi(N) \subseteq \varphi(G)$. Thus, $H = 0$, and G is not an ID-group. The proof of (b) follows similarly from Lemma 2.3 (b).

(2.5) **Lemma.** *Let $G = \sum \oplus N_i$ be the direct sum of any number of fully invariant subgroups. Then G is an ID-group (IP-group) if and only if some N_i is an ID-group (IP-group).*

Proof. By Lemmas 2.1 and 2.3.

(2.6) **Theorem.** *Let G_T be the torsion subgroup of G. If G_T and G/G_T are not ID-groups (IP-groups), then G is not an ID-group (IP-group). Moreover, G_T is an ID-group (IP-group) if and only if some primary component of G_T is an ID-group (IP-group).*

Proof. By Lemmas 2.4 and 2.5.

(2.7) **Corollary.** *A group G of finite reduced rank is not an IP-group.*

Proof. Since G has finite rank, both G_T and G/G_T have finite rank. By [1, Theorem 1], G_T is not an I-group. Suppose that φ is a monomorphism of G/G_T into G/G_T such that $\varphi(G/G_T)$ is a pure subgroup of G/G_T. Then $(G/G_T)/\varphi(G/G_T)$ is torsion free. However, since $\text{rank}(\varphi(G/G_T)) = \text{rank}(G/G_T) < \infty$, it follows that $(G/G_T)/\varphi(G/G_T)$ is a torsion group. Thus, $\varphi(G/G_T) = G/G_T$, so that G/G_T is not an IP-group. By Theorem 2.6, G is not an IP-group.

(2.8) **Theorem.** *Let $G = K \oplus D$, where K is reduced and D is divisible. Then G is an ID-group (IP-group) if and only if either K is an ID-group (IP-group) or D is an ID-group (IP-group). Moreover, D is an ID-group (IP-group) if and only if D has infinite torsion free rank, or infinite p-rank for some prime p.*

Proof. The first statement follows from Lemmas 2.1 and 2.4. If D is an IP-group, then either D/D_T or some primary component of D has infinite rank by Theorem 2.6 and Corollary 2.7. Conversely, if D has infinite torsion free rank or infinite p-rank, then D has a direct summand which is either a sum of infinitely many copies of Q or a sum of infinitely many copies of $Z(p^\infty)$. Hence D is an ID-group by Lemmas 2.1 and 2.2.

(2.9) **Theorem.** *A reduced p-group G is an ID-group if and only if $f_G(n)$ (the n-th Ulm invariant of G) is infinite for some non-negative integer n. If this condition is satisfied, then $G = K \oplus C$, where C is a non-zero bounded ID-group.*

Proof. Let G be an ID-group, say $G = G_1 \oplus H$, where $G \cong G_1$ and $H \neq 0$. Then

$$f_G(n) = f_{G_1}(n) + f_H(n) = f_G(n) + f_H(n)$$

for $n = 0, 1, 2, \ldots$. Thus either $f_G(n) = \infty$ or $f_H(n) = 0$. However, $H \neq 0$ and H reduced implies $f_H(n) \neq 0$ for some n. Therefore, $f_G(n) = \infty$ for some n. Conversely, suppose that $f_G(n)$ is infinite. Let B be a basic subgroup of G. Then $B = B_1 \oplus B_2 \oplus \ldots$, where B_{i+1} is a direct sum of $f_G(i)$ copies of $Z(p^{i+1})$. In particular, $C = B_{n+1}$ is a non-zero bounded ID-group by 2.2. Since B_{n+1} is pure in B, and B is pure in G, it follows that $C = B_{n+1}$ is a direct summand of G. Therefore, by 2.1, G is an ID-group.

(2.10) **Corollary.** *If G_T is an ID-group, then G is an ID-group.*

Proof. By Theorems 2.6, 2.8, and 2.9, if G_T is an ID-group, then G_T has a non-zero direct summand C which is an ID-group and which is either divisible or a group of bounded order. In either case C is a direct summand of G, since G_T is pure in G. Thus, G is an ID-group by 2.1.

(2.11) **Corollary.** *If G is a reduced p-group such that $|G| > 2^{\aleph_0}$, then G is an ID-group.*

Proof. If $|G| > 2^{\aleph_0}$, then $|B| > \aleph_0$, where B is a basic subgroup of G. Hence for some n, $|B_{n+1}| > \aleph_0$. Therefore, $f_G(n) > \aleph_0$, so that G is an ID-group by Theorem 2.9.

Remark. If G is a countable reduced p-group, then it follows from A, page 135 in [4], that G has a direct summand H which is an unbounded direct sum of cyclic groups. Therefore [4, Lemma 31.1], H has a proper basic subgroup B such that $H \cong B$. Thus, H is an IP-group, and by Lemma 2.1, G is an IP-group. These remarks and Theorem 2.9 show that the group $B = \sum_{n<\omega} \oplus Z(p^n)$ is an IP-group but not an ID-group. The torsion completion of B, $\overline{B} = \sum^*_{n<\omega} \oplus Z(p^n)$, is an example of a group with cardinality $2^{\aleph_0}$ which is an I-group but not an IP-group. CRAWLEY [3] has given an example of a subgroup of $\overline{B}$ containing B which is not an I-group.

The results on torsion free ID-groups are sparse. The following general result provides some information in this case.

(2.12) **Theorem.** *Let G be a reduced group such that G/pG is finite for all primes p. Then G is not an ID-group.*

Proof. Suppose that φ is a monomorphism of G into G such that $G = \varphi(G) \oplus H$. We have

$$G/pG \cong \varphi(G)/\varphi(pG) = \varphi(G)/p\varphi(G) = \varphi(G)/pG \cap \varphi(G) \cong (pG + \varphi(G))/pG\,.$$

Since G/pG is finite, it follows that $pG + \varphi(G) = G$. Therefore,

$$G/\varphi(G) = (pG + \varphi(G))/\varphi(G) = p(G/\varphi(G))$$

for all p. Consequently, $G/\varphi(G) \cong H$ is divisible. Since G is reduced, it follows that $H = 0$. Thus, G is not an ID-group.

Remark. The converse of Theorem 2.12 does not hold for either torsion or torsion free groups. The p-group $B = \sum_{n<\omega} \oplus Z(p^n)$ has $|B/pB| = \aleph_0$ and B is not an ID-group. Let V_ω be a rational vector space of countably infinite dimension with basis $e_1, e_2, \ldots, e_n, \ldots$. Then the subgroup G of V_ω generated by e_k/p_k^m and $(e_i + e_j)/2$ for all i, j, k and all positive integers m, such that $i < j$ and $p_1, p_2, \ldots, p_n, \ldots$ are the odd primes in their natural order, has $|G/p_nG| = \aleph_0$ for all n. Moreover, G is an indecomposable torsion free group [4, page 151], so that G is not an ID-group.

(2.13) **Corollary.** *Let $p_1, p_2, \ldots, p_n, \ldots$ be an enumeration of the primes and let $G = \sum_n \oplus G_n$, where each G_n is a direct sum of a finite number of copies of the p_n-adic integers. Then G is not an IP-group.*

Proof. Since each element of G_n has finite p_n-height and is p_m-divisible for $m \neq n$, it follows that G_n is a fully invariant subgroup of G for all n. Hence by Lemma 2.5, it is sufficient to show that each G_n is not an IP-group. Since G_n is complete in the p_n-adic topology, it follows that an isomorphic proper pure subgroup of G_n is a direct summand [5, Theorem 23]. That is, if G_n is an

IP-group, then G_n is an ID-group. However, G_n is not an ID-group by Theorem 2.12, since $G_n = p_m G_n$ for $m \neq n$ and $G_n/p_n G_n$ is a finite direct sum of cyclic groups of order p_n.

3. ID-systems

It was noted in Section 1 that for the classification of ID-groups, it is convenient to consider not just the groups themselves, but also the endomorphisms which determine the isomorphic direct summand.

(3.1) **Definition.** An ID-*system* is a triple $\langle G; \varphi, \psi\rangle$, where G is an abelian group and φ and ψ are endomorphisms of G such that $\psi\varphi = 1$. Two ID-systems $\langle G; \varphi, \psi\rangle$ and $\langle G'; \varphi', \psi'\rangle$ are *isomorphic* if there is a group isomorphism θ of G onto G' such that $\theta\varphi = \varphi'\theta$ and $\theta\psi = \psi'\theta$.

(3.2) *Examples.* (a) Let G be any group. Let φ be an automorphism of G. Let $\psi = \varphi^{-1}$. Then $\langle G; \varphi, \psi\rangle$ is an ID-system. In this case G need not be an ID-group.

(b) Let H be any group. Denote by P_H the complete direct sum of countably many copies of H. Let σ and τ be the *right* and *left shift endomorphisms* of P_H defined by

$$\sigma((x_1, x_2, x_3, \ldots)) = (0, x_1, x_2, \ldots)\,,$$

$$\tau((x_1, x_2, x_3, \ldots)) = (x_2, x_3, \ldots)\,.$$

Then $\langle P_H; \sigma, \tau\rangle$ is an ID-system. If S_H denotes the direct sum of countably many copies of H, then S_H can be considered as a subgroup of P_H. Moreover, $\sigma(S_H) \subseteq S_H$ and $\tau(S_H) \subseteq S_H$. Hence $\langle S_H; \sigma', \tau'\rangle$ is an ID-system, where σ' and τ' are the restrictions of σ and τ to S_H. More generally, a subgroup T of P_H with $S_H \subseteq T$ determines an ID-system provided that $\sigma(T) \subseteq T$ and $\tau(T) \subseteq T$. We call such a T a *total shift invariant* (t. s. i.) *subgroup* of P_H.

If $H \neq 0$, then any t. s. i. subgroup of P_H is an ID-group. We will see presently that every ID-group is obtained by an extension process from groups of the type given in 3.2.

The study of ID-systems is equivalent to the study of modules over a certain ring. This important observation makes it possible to apply the methods of homological algebra to the theory of ID-systems.

(3.3) **Definition.** The ID-*ring* Δ is the residue class ring

$$Z\{X, Y\}/(XY - 1)\,,$$

where $Z\{X, Y\}$ is the polynomial ring with identity in non-commuting indeterminates X, Y with integral coefficients, and $(XY - 1)$ is the ideal of $Z\{X, Y\}$ generated by $XY - 1$. Let ξ and η denote the residue classes of X and Y respectively in Δ.

(3.4) **Lemma.** *Every element of Δ can be expressed uniquely in the form*

$$\alpha = P(\xi, \eta) = \sum_{i,j \geq 0} n_{ij}\eta^i\xi^j,\ n_{ij} \in Z\,.$$

Hence Δ is a free Z-module.

(3.5) **Theorem.** *There is a one-to-one correspondence between ID-systems and Δ-modules. If $\langle G; \varphi, \psi\rangle$ is an ID-system, then the corresponding Δ-module is* *the group G with the module operation defined by*

$$P(\xi, \eta) \cdot x = P(\psi, \varphi)(x).$$

Two ID-systems are isomorphic if and only if the corresponding Δ-modules are isomorphic.

The proof of this theorem is routine.

If Λ is any ring, M is a Λ-module, and $\alpha \in \Lambda$, then the set of all elements of infinite α-height in M is

$$\alpha^\omega M = \bigcap_{n<\omega} \alpha^n M.$$

Clearly, $\alpha^\omega M$ is a subgroup of M, but in general, if Λ is not commutative, then $\alpha^\omega M$ is not a submodule of M. However, for the ID-ring Δ, we can prove the following result.

(3.6) **Lemma.** *Let G be a Δ-module. Then*

(1) *$\eta^\omega G$ is a submodule of G; for $x \in \eta^\omega G$, $\eta \xi x = x$.*

(2) *$G/\eta^\omega G$ is a Δ-module without non-zero elements of infinite η-height.*

Proof. An element x belongs to $\eta^\omega G$ if and only if there exist $x_1, x_2, x_3, \ldots$ in G such that $x = \eta x_1 = \eta^2 x_2 = \eta^3 x_3 = \cdots$. If this condition is satisfied, then $\eta x = \eta(\eta x_1) = \eta^2(\eta x_2) = \eta^3(\eta x_3) = \cdots$ and

$$\xi x = \xi \eta x_1 = \xi \eta^2 x_2 = \xi \eta^3 x_3 = \cdots$$
$$= x_1 = \eta x_2 = \eta^2 x_3 = \cdots.$$

Thus, $\eta x \in \eta^\omega G$ and $\xi x \in \eta^\omega G$. Therefore $\eta^\omega G$ is a submodule of G. Moreover, for $x \in \eta^\omega G$, $\eta \xi x = \eta \xi \eta x_1 = \eta x_1 = x$. Finally, by a standard argument, if $x + \eta^\omega G$ has infinite η-height in $G/\eta^\omega G$, then $x \in \eta^\omega G$, which implies (2).

This lemma shows that $\eta^\omega G$ is a submodule of G on which ξ and η act as inverse automorphisms. We will call a module of this kind an *automorphic module* (or automorphic Δ-module). The automorphic modules are exactly the Δ-modules corresponding to ID-systems of the type defined in 3.2 (a).

If T is a Δ-module corresponding to a t. s. i. subgroup of a product P_H (see 3.2 (b)), then we will call T a *shift module.*

(3.7) **Theorem.** *If G is a Δ-module, then G is isomorphic to a shift module if and only if G has no elements of infinite η-height.*

Proof. Suppose that G is isomorphic to the shift module T. To show that G has no elements of infinite η-height, it suffices to prove that T has no non-zero elements of infinite η-height. However, this is clear because

$$\eta^k(x_1, x_2, \ldots) = (\overbrace{0, 0, \ldots, 0}^{k}, x_1, x_2, \ldots).$$

Conversely, assume that G has no non-zero elements of infinite η-height. Let

$$H = \{x \in G \mid \xi x = 0\}.$$

Then G decomposes as a group,

$$G = H \oplus \eta(G) = H \oplus \eta(H) \oplus \eta^2(G) = \cdots = H \oplus \eta(H) \oplus \cdots \oplus \eta^n(H) \oplus \eta^{n+1}(G).$$

For $x \in G$, map

$$\mu: x \to (x_1, x_2, x_3, \ldots, x_n, \ldots) \in P_H,$$

where

$$x = x_1 + \eta x_2 + \cdots + \eta^n x_n + \eta^{n+1} y_n,$$

$x_1, x_2, \ldots, x_n \in H$, $y_n \in G$. It is clear that μ is a well-defined group homomorphism of G into P_H. Moreover, if $\mu(x) = (x_1, x_2, x_3, \ldots)$, then $\mu(\xi x) = (x_2, x_3, \ldots) = \tau(\mu(x))$, and $\mu(\eta x) = (0, x_1, x_2, x_3, \ldots) = \sigma(\mu(x))$. Thus, the image T of μ is a subgroup of P_H such that $\sigma(T) \subseteqq T$ and $\tau(T) \subseteqq T$. Moreover, if $x_1, x_2, \ldots, x_n$ are in H, then

$$\mu(x_1 + \eta x_2 + \cdots + \eta^n x_n) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots),$$

so that $S_H \subseteqq T$. That is, T is a t. s. i. subgroup of P_H. Since $\mu\xi = \tau\mu$ and $\mu\eta = \sigma\mu$, it follows that μ is a Δ-homomorphism onto the Δ-module determined by T. Moreover, the kernel of μ is $\eta^\omega G = 0$, so that μ is a Δ-isomorphism.

It is perhaps worthwhile to interpret 3.6 and 3.7 as statements about ID-systems. The result is the following theorem.

(3.8) **Theorem.** *Let $\langle G; \varphi, \psi \rangle$ be an ID-system. Let $K = \bigcap_{n<\omega} \varphi^n(G) = \varphi^\omega G$, $H = \operatorname{Ker} \psi$, and $\alpha = \varphi|_K$. Then α is an automorphism of K and there is a t. s. i. subgroup T of P_H and an epimorphism μ such that the following diagrams are row exact and commutative:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & G & \xrightarrow{\mu} & T & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\sigma} & & \\ 0 & \longrightarrow & K & \longrightarrow & G & \xrightarrow{\mu} & T & \longrightarrow & 0, \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & G & \xrightarrow{\mu} & T & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha^{-1}} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\tau} & & \\ 0 & \longrightarrow & K & \longrightarrow & G & \xrightarrow{\mu} & T & \longrightarrow & 0. \end{array}$$

4. Extensions of ID-systems

The results of the previous section show that every Δ-module is an extension of an automorphic module by a shift module. We now wish to classify the different extensions of a fixed automorphic module K by a fixed shift module T. Since the module structure (though not the group structure) of automorphic and shift modules can be considered to be known, this program is essentially equivalent to that of classifying the module structure of all Δ-modules. It is obvious that the appropriate classifying structure is the group $\operatorname{Ext}^1_\Delta(T, K)$. The main result of this section is a theorem which relates $\operatorname{Ext}^1_\Delta(T, K)$ to $\operatorname{Ext}_Z(T, K)$.

If T and K are any Δ-modules, then the group $\operatorname{Hom}_Z(T, K)$ carries the structure of a left and right Δ-module with the operations

$$(\alpha\chi)(x) = \alpha \cdot \chi(x), \quad (\chi\beta)(x) = \chi(\beta \cdot x),$$

where $\chi \in \operatorname{Hom}_Z(T, K)$ and $\alpha, \beta \in \Delta$. Obviously,

$$(\alpha\chi)\beta = \alpha(\chi\beta).$$

The same remark applies to the derived functor Ext_Z of Hom_Z: $\operatorname{Ext}_Z(T, K)$ is a two sided Δ-module, and

$$(\alpha\mathfrak{A})\beta = \alpha(\mathfrak{A}\beta)$$

for all $\alpha, \beta \in \Delta$ and $\mathfrak{A} \in \operatorname{Ext}_Z(T, K)$.

(4.1) **Lemma.** *Let T and K be Δ-modules, where K is an automorphic module. Then the inclusion mapping*

$$i_1 : \mathrm{Hom}_\Delta(T, K) \to \mathrm{Hom}_Z(T, K)$$

sends $\mathrm{Hom}(T, K)$ *onto* $\{\chi \in \mathrm{Hom}_Z(T, K) \mid \xi\chi = \chi\xi\}$.

Proof. Clearly, if $\chi \in \mathrm{Hom}_\Delta(T, K)$, then $\chi \in \mathrm{Hom}_Z(T, K)$ and $\xi\chi = \chi\xi$. Conversely, if $\chi \in \mathrm{Hom}_Z(T, K)$ and $\xi\chi = \chi\xi$, then since K is an automorphic module, $\chi\eta = \eta\xi\chi\eta = \eta\chi\xi\eta = \eta\chi$. Therefore, $\chi \in \mathrm{Hom}_\Delta(T, K)$.

It is convenient to identify the elements of the groups $\mathrm{Ext}^1_\Delta(T, K)$ and $\mathrm{Ext}_Z(T, K)$ with equivalence classes of short exact sequences

$$0 \to K \to G \to T \to 0\,,$$

considered respectively as Δ-modules and Z-modules. Two such sequences

$$0 \to K \to G \to T \to 0 \quad \text{and} \quad 0 \to K \to G' \to T \to 0$$

are equivalent if there is an isomorphism k of G onto G' (considered as Δ-modules and Z-modules in the respective cases) such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \to & K & \to & G & \to & T & \to & 0 \\ & & \| & & \downarrow{\scriptstyle k} & & \| & & \\ 0 & \to & K & \to & G' & \to & T & \to & 0\,. \end{array}$$

Thus, if two sequences of Δ-modules are equivalent, then they are equivalent as Z-modules. It follows that the mapping i_2, which associates with each equivalence class $\mathfrak{A}$ of sequences of Δ-modules the equivalence class $\mathfrak{A}'$ of sequences of Z-modules containing $\mathfrak{A}$, is a well-defined mapping of $\mathrm{Ext}^1_\Delta(T, K)$ into $\mathrm{Ext}_Z(T, K)$. Because of the way in which the addition operation (Baer composition) is defined in $\mathrm{Ext}^1_\Delta(T, K)$ and $\mathrm{Ext}_Z(T, K)$ (see [2], page 290), it is clear that i_2 is a group homomorphism. We will call i_2 the *reduction homomorphism.*

It should be remarked that each short exact sequence of abelian groups in the class $\mathfrak{A}' = i_2\mathfrak{A}$ can be regarded as a sequence of Δ-modules. For if

$$0 \longrightarrow K \xrightarrow{f} G' \xrightarrow{g} T \longrightarrow 0$$

is in $i_2\mathfrak{A}$, then there is a sequence of Δ-modules in $\mathfrak{A}$,

$$0 \longrightarrow K \xrightarrow{f'} G \xrightarrow{g'} T \longrightarrow 0\,,$$

such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{f'} & G & \xrightarrow{g'} & T & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle k} & & \| & & \\ 0 & \longrightarrow & K & \xrightarrow{f} & G' & \xrightarrow{g} & T & \longrightarrow & 0 \end{array}$$

commutes. Defining $\alpha \cdot x = k(\alpha \cdot k^{-1}(x))$ for $\alpha \in \Delta$ and $x \in G'$ makes G' into a Δ-module in such a way that f and g are Δ-homomorphisms.

(4.2) **Lemma.** *Let K and T be Δ-modules, where K is an automorphic module. Then the reduction homomorphism*

$$i_2 : \mathrm{Ext}^1_\Delta(T, K) \to \mathrm{Ext}_Z(T, K)$$

sends $\mathrm{Ext}^1_\Delta(T, K)$ *onto* $\{\mathfrak{A} \in \mathrm{Ext}_Z(T, K) \mid \xi\mathfrak{A} = \mathfrak{A}\xi\}$. *The kernel of* i_2 *consists of those equivalence classes of sequences of* Δ*-modules which split as sequences of abelian groups.*

Proof. If $\mathfrak{A}$ is in the image of i_2, then the class $\mathfrak{A}$ contains the sequence

$$0 \longrightarrow K \xrightarrow{f} G \xrightarrow{g} T \longrightarrow 0,$$

where G is a Δ-module and f and g are Δ-homomorphisms. Since K is an automorphic module, the following diagram is commutative:

$$\begin{array}{rcccccccc}
\mathfrak{A}\xi : 0 & \longrightarrow & K & \xrightarrow{f\eta} & G & \xrightarrow{g} & T & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\xi} & & \\
\mathfrak{A} : 0 & \longrightarrow & K & \xrightarrow{f} & G & \xrightarrow{g} & T & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\xi} & & \| & & \| & & \\
\xi\mathfrak{A} : 0 & \longrightarrow & K & \xrightarrow{f\eta} & G & \xrightarrow{g} & T & \longrightarrow & 0 .
\end{array}$$

Therefore $\xi\mathfrak{A} = \mathfrak{A}\xi$.

Conversely, suppose that $\xi\mathfrak{A} = \mathfrak{A}\xi$. It follows as in the proof of Lemma 4.1 that $\eta\mathfrak{A} = \mathfrak{A}\eta$. Let

$$0 \longrightarrow K \xrightarrow{f} G \xrightarrow{g} T \longrightarrow 0$$

be a sequence belonging to the class $\mathfrak{A} \in \mathrm{Ext}_Z(T, K)$. Then there exist endomorphisms ψ and φ of G such that the following diagrams commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{f} & G & \xrightarrow{g} & T & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\xi} & & \\
0 & \longrightarrow & K & \xrightarrow{f} & G & \xrightarrow{g} & T & \longrightarrow & 0 ,
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{f} & G & \xrightarrow{g} & T & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\eta} & & \\
0 & \longrightarrow & K & \xrightarrow{f} & G & \xrightarrow{g} & T & \longrightarrow & 0 .
\end{array}$$

By the commutativity of these diagrams, $(\psi\varphi - 1)(G) \subseteq \mathrm{Ker}\, g = \mathrm{Im}\, f$ and $(\psi\varphi - 1)(\mathrm{Im}\, f) = 0$. Consequently,

$$\psi(2\varphi - \varphi\psi\varphi) - 1 = -(\psi\varphi - 1)^2 = 0 .$$

Therefore G can be made into a Δ-module by defining

$$\xi \cdot x = \psi(x), \quad \eta \cdot x = (2\varphi - \varphi\psi\varphi)(x) .$$

With the module operations so defined, it is easily verified that f and g are Δ-homomorphisms. Thus,

$$\mathfrak{A} : 0 \longrightarrow K \xrightarrow{f} G \xrightarrow{g} T \longrightarrow 0$$

is in the image of i_2.

The final statement of the lemma is obvious.

(4.3) **Lemma.** *Let K and T be Δ-modules, where K is an automorphic module. Then there is a homomorphism*

$$d_1 : \mathrm{Hom}_Z(T, K) \to \mathrm{Ext}^1_\Delta(T, K)$$

such that (i) $\mathrm{Im}\, d_1 = \mathrm{Ker}\, i_2$, *and* (ii) $\mathrm{Ker}\, d_1 = \{\xi\chi - \chi\xi \mid \chi \in \mathrm{Hom}_Z(T, K)\}$.

Proof. Let θ be in $\mathrm{Hom}_Z(T, K)$. Define the Δ-module G_θ as follows: as a group, G_θ is the (external) direct sum $K \oplus T$; the module operations on G are defined by

$$\xi \cdot (k, t) = (\xi \cdot k + \theta(t), \xi \cdot t), \quad \eta \cdot (k, t) = (\eta \cdot k - \eta \cdot \theta(\eta \cdot t), \eta \cdot t) .$$

It is easy to verify that with these definitions, $\xi\eta = 1$. Moreover, with

$$f_\theta : K \to G_\theta, \quad g_\theta : G_\theta \to T$$

defined by $f_\theta(k) = (k, 0)$, $g_\theta(k, t) = t$, respectively, f_θ and g_θ are Δ-homomorphisms, and

$$0 \longrightarrow K \xrightarrow{f_\theta} G_\theta \xrightarrow{g_\theta} T \longrightarrow 0$$

is a short exact sequence of Δ-modules. Let $\mathfrak{A}_\theta$ denote the class of this sequence. Define

$$d_1 : \theta \to \mathfrak{A}_\theta .$$

It can be checked that $d_1(\theta + \varkappa) = \mathfrak{A}_\theta + \mathfrak{A}_\varkappa$, where $\theta, \varkappa \in \mathrm{Hom}_Z(T, K)$ and $\mathfrak{A}_\theta + \mathfrak{A}_\varkappa$ is obtained by the Baer composition. Thus, d_1 is a homomorphism of $\mathrm{Hom}_Z(T, K)$ into $\mathrm{Ext}^1_\Delta(T, K)$. It is evident from the the definition of G_θ that the sequence

$$0 \longrightarrow K \xrightarrow{f_\theta} G_\theta \xrightarrow{g_\theta} T \longrightarrow 0$$

splits as a sequence of abelian groups. Hence $\mathrm{Im}\, d_1 \subseteq \mathrm{Ker}\, i_2$. On the other hand, suppose that $i_2\mathfrak{A} = 0$. Then (by the remarks preceding Lemma 4.2) $\mathfrak{A}$ contains a sequence

$$0 \longrightarrow K \xrightarrow{f} K \oplus T \xrightarrow{g} T \longrightarrow 0 ,$$

where $f(k) = (k, 0)$, $g((k, t)) = t$, $K \oplus T$ is a Δ-module, and f and g are Δ-homomorphisms. For $\alpha \in \Delta$, let $\alpha \cdot (0, t) = (\theta_\alpha(t), t_\alpha)$. Then $\theta_\alpha \in \mathrm{Hom}_Z(T, K)$. Since $g(\alpha \cdot (0, t)) = \alpha \cdot g((0, t)) = \alpha \cdot t$, it follows that $t_\alpha = \alpha \cdot t$. Moreover, $\alpha \cdot (k, 0) = \alpha \cdot f(k) = f(\alpha \cdot k) = (\alpha \cdot k, 0)$. Therefore,

$$\alpha \cdot (k, t) = \alpha \cdot (k, 0) + \alpha \cdot (0, t) = (\alpha \cdot k + \theta_\alpha(t), \alpha \cdot t) .$$

Since $\xi\eta = 1$, it follows that

$$(k, t) = \xi\eta \cdot (k, t) = \xi \cdot (\eta \cdot k + \theta_\eta(t), \eta \cdot t) = (k + \xi \cdot \theta_\eta(t) + \theta_\xi(\eta \cdot t), t) .$$

Therefore $\xi\theta_\eta = -\theta_\xi\eta$, and since K is an automorphic module, $\theta_\eta = \eta\xi\theta_\eta = -\eta\theta_\xi\eta$. This proves that the sequence under consideration is in the class $\mathfrak{A}_{\theta_\xi}$. Hence $\mathfrak{A} = d_1(\theta_\xi)$, completing the proof of (i).

Suppose that $d_1(\theta) = 0$ for $\theta \in \mathrm{Hom}_Z(T, K)$. Then the sequence

$$0 \longrightarrow K \xrightarrow{f_\theta} G_\theta \xrightarrow{g_\theta} T \longrightarrow 0$$

splits. Hence, there is a Δ-homomorphism $h : T \to G_\theta$ such that $g_\theta h$ is the identity on T. Evidently there exists $\chi \in \mathrm{Hom}_Z(T, K)$ such that $h(t)$

$= (-\chi(t), t)$. From the identity $\xi h(t) = h(\xi \cdot t)$, it follows easily that $\theta = \xi\chi - \chi\xi$. Conversely, suppose that $\theta = \xi\chi - \chi\xi$, where $\chi \in \mathrm{Hom}_Z(T, K)$. Define $h: T \to G_\theta$ by $h(t) = (-\chi(t), t)$. Then $g_\theta h$ is the identity on T, and it can be checked that h is a Δ-homomorphism. Therefore, the Δ-module sequence

$$0 \longrightarrow K \xrightarrow{f_\theta} G_\theta \xrightarrow{g_\theta} T \longrightarrow 0$$

splits, so that $d_1(\theta) = \mathfrak{A}_\theta = 0$. This completes the proof of Lemma 4.3.

(4.4) **Theorem.** *Let K and T be Δ-modules, where K is an automorphic module. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{Hom}_\Delta(T,K) \xrightarrow{i_1} \mathrm{Hom}_Z(T,K) \xrightarrow{c_1} \mathrm{Hom}_Z(T,K) \xrightarrow{d_1} \mathrm{Ext}^1_\Delta(T,K)$$
$$\xrightarrow{i_2} \mathrm{Ext}_Z(T,K) \xrightarrow{c_2} \mathrm{Ext}_Z(T,K) \xrightarrow{d_2} \mathrm{Ext}^2_\Delta(T,K) \longrightarrow 0\,,$$

where i_1 is the inclusion homomorphism, i_2 is the reduction homomorphism, $c_1: \chi \to \xi\chi - \chi\xi$, $\chi \in \mathrm{Hom}_Z(T, K)$, $c_2: \mathfrak{A} \to \xi\mathfrak{A} - \mathfrak{A}\xi$, $\mathfrak{A} \in \mathrm{Ext}_Z(T, K)$, d_1 is defined as in 4.3, and d_2 is a homomorphism derived from d_1.

Proof. It follows from 4.1, 4.2, and 4.3 that this sequence is exact up to $\mathrm{Ext}_Z(T, K) \xrightarrow{c_2} \mathrm{Ext}_Z(T, K)$. To complete the sequence, let

$$0 \to S \to F \to T \to 0$$

be a short exact sequence of Δ-modules with F Δ-free. Then since Δ is Z-free by 3.4, it follows that F is Z-free, and consequently S is also Z-free. Thus, using the results alrcady established, together with standard results of homological algebra, we have the following diagram with exact rows and columns:

$$\begin{array}{ccccccc}
\mathrm{Hom}_Z(F,K) & \xrightarrow{c_1} & \mathrm{Hom}_Z(F,K) & \xrightarrow{d_1} & \mathrm{Ext}^1_\Delta(F,K) = 0 & & \\
\downarrow r & & \downarrow r & & & & \\
\mathrm{Hom}_Z(S,K) & \xrightarrow{c_1} & \mathrm{Hom}_Z(S,K) & \xrightarrow{d_1} & \mathrm{Ext}^1_\Delta(S,K) & \xrightarrow{i_2} & \mathrm{Ext}_Z(S,K) = 0 \\
\downarrow \partial & & \downarrow \partial & & & & \\
\mathrm{Ext}_Z(T,K) & \xrightarrow{c_2} & \mathrm{Ext}_Z(T,K) & & & & \\
\downarrow & & \downarrow & & & & \\
\mathrm{Ext}_Z(F,K) & \longrightarrow & \mathrm{Ext}_Z(F,K) & & & & \\
\| & & \| & & & & \\
0 & & 0\,, & & & &
\end{array}$$

where r is the restriction homomorphism $\chi \to \chi|_S$ and ∂ is the connecting homomorphism. Obviously $rc_1 = c_1 r$. Moreover, since $\partial(\chi\xi) = (\partial\chi)\xi$ and $\partial(\xi\chi) = \xi(\partial\chi)$, it follows that $\partial c_1 = c_2\partial$. Therefore, the diagram commutes. From the commutivity and exactness, it follows easily that

$$\begin{aligned}\partial^{-1}(\mathrm{Im}\, c_2) &= \mathrm{Im}\, r + \mathrm{Im}\, c_1 = \mathrm{Im}\, rc_1 + \mathrm{Im}\, c_1 = \mathrm{Im}\, c_1 r + \mathrm{Im}\, c_1 \\ &= \mathrm{Im}\, c_1 = \mathrm{Ker}\, d_1\,.\end{aligned}$$

Thus, there is a homomorphism e of $\mathrm{Ext}_Z(T, K)$ onto $\mathrm{Ext}^1_\Delta(S. K)$ such that $\mathrm{Ker}\, e = \mathrm{Im}\, c_2$. Finally, from the exactness of the sequence

$$0 = \mathrm{Ext}^1_\Delta(F,K) \to \mathrm{Ext}^1_\Delta(S,K) \to \mathrm{Ext}^2_\Delta(T,K) \to \mathrm{Ext}^2_\Delta(F,K) = 0\,,$$

it follows that there is an epimorphism

$$d_2 : \mathrm{Ext}_Z(T, K) \to \mathrm{Ext}^1_\Delta(T, K)$$

with $\mathrm{Ker}\, d_2 = \mathrm{Ker}\, e = \mathrm{Im}\, c_2$. This completes the proof of the theorem.

(4.5) **Corollary.** *Let K and T be Δ-modules, where K is an automorphic module. Then* $\mathrm{Ext}^n_\Delta(T, K) = 0$ *for* $n > 2$.

Proof. Let $0 \to S \to F \to W \to 0$ be an exact sequence of Δ-modules, with F Δ-free. Then S is Z-free and

$$0 = \mathrm{Ext}_Z(S, K) \xrightarrow{d_2} \mathrm{Ext}^2_\Delta(S, K) \longrightarrow 0$$

is exact. Hence $\mathrm{Ext}^2_\Delta(S, K) = 0$. Since $\mathrm{Ext}^3_\Delta(W, K) \cong \mathrm{Ext}^2_\Delta(S, K)$, it follows that $\mathrm{Ext}^3_\Delta(W, K) = 0$ for all Δ-modules W. Therefore, $\mathrm{Ext}^n_\Delta(T, K) = 0$ for $n > 2$.

The following result exhibits the functorial nature of the exact sequence of Theorem 4.4. The proof is omitted since it is lengthy and contains no surprises.

(4.6) **Theorem.** *Let T, T', K, and K' be Δ-modules, where K and K' are automorphic modules. Denote the exact sequence of Theorem 4.4 by $D(T, K)$. Let $\lambda : T' \to T$ and $\mu : K \to K'$ be Δ-homomorphisms. Then there exist translations*

$$\lambda^* : D(T, K) \to D(T', K) \quad \textit{and} \quad \mu_* : D(T, K) \to D(T, K'),$$

where the component maps of λ^* and μ_* are the induced maps of λ and μ respectively.

5. Extensions of trivial ID-systems

Answers to interesting questions about ID-groups can be found by determining the conditions under which the reduction homomorphism

$$i_2 : \mathrm{Ext}^1_\Delta(T, K) \to \mathrm{Ext}_Z(T, K)$$

is zero, one-to-one, or onto. Note that $i_2 = 0$ if and only if each ID-group which is a module extension of K by T has K as a (group) direct summand. The homomorphism i_2 is one-to-one if and only if (roughly speaking) each ID-group which is a module extension of K by T has an essentially unique ID-structure. Finally, i_2 is onto if and only if every group which is an extension of K by T is an ID-group.

We examine these problems in the particular case where the automorphic module K is *trivial*, that is, K is simply an abelian group on which Δ acts by $\xi x = \eta x = x$. Such a trivial Δ-module corresponds to an ID-system $\langle K; \varphi, \psi\rangle$, where $\varphi = \psi$ is the identity mapping. The results of Section 4 can be put in a form which is convenient for computations when K is a trivial Δ-module.

(5.1) **Theorem.** *Let K and T be Δ-modules, where K is a trivial Δ-module. Let $\varrho : T \to T$ be defined by $\varrho(t) = t - \xi t$. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{Hom}_Z(T/\varrho(T), K) \xrightarrow{p^*} \mathrm{Hom}_Z(T, K) \xrightarrow{\varrho_1^*} \mathrm{Hom}_Z(T, K) \xrightarrow{d_1} \mathrm{Ext}^1_\Delta(T, K)$$
$$\xrightarrow{i_2} \mathrm{Ext}_Z(T, K) \xrightarrow{\varrho_2^*} \mathrm{Ext}_Z(T, K) \xrightarrow{i^*} \mathrm{Ext}_Z(\mathrm{Ker}\, \varrho, K) \longrightarrow 0,$$

where p is the natural projection of T onto $T/\varrho(T)$, i is the inclusion mapping of $\operatorname{Ker}\varrho$ *into T, and p^*, ϱ_i^*, and i^* are the mappings induced by p, ϱ, and i.*

Proof. The group homomorphism ϱ can be factored, $\varrho = j\nu$, to obtain two short exact sequences

$$0 \longrightarrow \operatorname{Ker}\varrho \xrightarrow{i} T \xrightarrow{\nu} \varrho(T) \longrightarrow 0\,,$$

$$0 \longrightarrow \varrho(T) \xrightarrow{j} T \xrightarrow{p} T/\varrho(T) \longrightarrow 0\,.$$

From these we obtain the exact sequences

$$0 \longrightarrow \operatorname{Hom}_Z(\varrho(T), K) \xrightarrow{\nu_1^*} \operatorname{Hom}_Z(T, K)\,,$$

$$\operatorname{Ext}_Z(\varrho(T), K) \xrightarrow{\nu_2^*} \operatorname{Ext}_Z(T, K) \xrightarrow{i^*} \operatorname{Ext}_Z(\operatorname{Ker}\varrho, K) \longrightarrow 0\,,$$

$$0 \longrightarrow \operatorname{Hom}_Z(T/\varrho(T), K) \xrightarrow{p^*} \operatorname{Hom}_Z(T, K) \xrightarrow{j_1^*} \operatorname{Hom}_Z(\varrho(T), K)\,,$$

$$\operatorname{Ext}_Z(T, K) \xrightarrow{j_2^*} \operatorname{Ext}_Z(\varrho(T), K) \longrightarrow 0\,.$$

These sequences, the sequence of Theorem 4.4, and the fact that $\nu_i^* j_i^* = \varrho_i^* = c_i$, yield the required result.

(5.2) **Corollary.** *Let T and K be Δ-modules, where K is a trivial Δ-module. Then*

$$\operatorname{Ext}^2_\Delta(T, K) \cong \operatorname{Ext}_Z(\operatorname{Ker}\varrho, K)$$

and

$$\operatorname{Hom}_\Delta(T, K) \cong \operatorname{Hom}_Z(T/\varrho(T), K)\,.$$

(5.3) **Lemma.** *Let U and V be abelian groups and $\lambda : U \to V$ a homomorphism inducing $\lambda^* : \operatorname{Ext}_Z(V, K) \to \operatorname{Ext}_Z(U, K)$. Then $\lambda^* = 0$ if and only if for every short exact sequence*

$$0 \longrightarrow K \longrightarrow G \xrightarrow{p} V \longrightarrow 0\,,$$

there is a homomorphism $\mu : U \to G$ such that $\lambda = p\mu$.

Proof. Let $\mathfrak{A}$ be the class of $0 \to K \to G \xrightarrow{p} V \to 0$. Then $\mathfrak{A}\lambda^* = 0$ if and only if there is a homomorphism $\nu : K \oplus U \to G$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
\lambda^*\mathfrak{A} : 0 & \longrightarrow & K & \longrightarrow & K \oplus U & \longrightarrow & U & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\nu} & & \downarrow{\scriptstyle\lambda} & & \\
\mathfrak{A} : 0 & \longrightarrow & K & \longrightarrow & G & \xrightarrow{p} & V & \longrightarrow & 0\,.
\end{array}$$

The lemma follows from this observation.

Throughout the remainder of this paper ϱ is the group homomorphism of T into T defined by $\varrho(t) = t - \xi t$.

(5.4) **Theorem.** *The reduction homomorphism i_2 is zero for all trivial Δ-modules K if and only if* $\operatorname{Ker}\varrho$ *is a direct summand of T and there is a subgroup L of $\varrho(T)$ such that T/L is free.*

Proof. By Theorem 5.1, $i_2 = 0$ if and only if

$$\varrho_2^* : \operatorname{Ext}_Z(T, K) \to \operatorname{Ext}_Z(T, K)$$

is one-to-one. The short exact sequences obtained by factoring $\varrho = j\nu$ (as in the proof of Theorem 5.1) yield the following exact sequences:

$$\mathrm{Hom}_Z(T, K) \xrightarrow{i^*} \mathrm{Hom}_Z(\mathrm{Ker}\,\varrho, K) \longrightarrow \mathrm{Ext}_Z(\varrho(T), K) \xrightarrow{\nu^*} \mathrm{Ext}_Z(T, K)\,,$$

$$\mathrm{Ext}_Z(T/\varrho(T), K) \xrightarrow{p^*} \mathrm{Ext}_Z(T, K) \xrightarrow{j^*} \mathrm{Ext}_Z(\varrho(T), K)\,,$$

where $\varrho_2^* = \nu^* j^*$. Since j^* is onto, it follows that ϱ_2^* is one-to-one if and only if both j^* and ν^* are one-to-one. This latter condition holds if and only if $p^* = 0$ and i^* is onto.

Suppose first that $p^* = 0$ and i^* is onto for all trivial K. Then, in particular, i^* is onto when K is the group $\mathrm{Ker}\,\varrho$. This implies that $\mathrm{Ker}\,\varrho$ is a direct summand of T. Let G be a free group and $f: G \to T/\varrho(T)$ an epimorphism. Since $p^* = 0$, it follows from 5.3 that there is a homomorphism $g: T \to G$ such that $p = fg$. Thus, $L = \mathrm{Ker}\,g \subseteq \mathrm{Ker}\,p = \varrho(T)$, and $T/L \cong g(T) \subseteq G$ is free.

Conversely, assume that $\mathrm{Ker}\,\varrho$ is a direct summand of T and L is a subgroup of $\varrho(T)$ such that T/L is free. The first condition clearly implies that i^* is onto for all K. The composition of the injections $L \xrightarrow{k} \varrho(T) \xrightarrow{j} T$ is the injection l of L into T. Since T/L is free,

$$0 = \mathrm{Ext}_Z(T/L, K) \longrightarrow \mathrm{Ext}_Z(T, K) \xrightarrow{l^*} \mathrm{Ext}_Z(L, K) \longrightarrow 0$$

is exact, and therefore l^* is an isomorphism for all K. Since $l^* = (jk)^* = k^* j^*$, it follows that j^* is one-to-one, and hence that $p^* = 0$ for all K.

Remark. The proof which we have given for Theorem 5.4 establishes somewhat more than is stated in the theorem. To obtain the conclusion that $\varrho(T)$ contains a subgroup L such that T/L is free, it is only necessary to assume that $i_2 = 0$ for all trivial Δ-modules K which are free as groups. Note that in this case L contains the torsion subgroup of T. In particular, if T is a torsion group, then $\varrho(T) = T$. If T is a torsion free group, then the conclusion that $\mathrm{Ker}\,\varrho$ is a direct summand is obtained if $i_2 = 0$ for all trivial Δ-modules K which are torsion free as groups.

(5.5) **Theorem.** *The reduction homomorphism i_2 is one-to-one for all trivial Δ-modules K if and only if ϱ is one-to-one and $\varrho(T)$ is a direct summand of T.*

Proof. By Theorem 5.1, i_2 is one-to-one if and only if

$$\varrho_1^*: \mathrm{Hom}_Z(T, K) \to \mathrm{Hom}_Z(T, K)$$

is onto. Assume that ϱ_1^* is onto for all K. Let $K = T$. Then there exists $\chi \in \mathrm{Hom}_Z(T, T)$ such that $\varrho_1^*(\chi) = \chi\varrho$ is the identity on T. Therefore ϱ is one-to-one and $\varrho(T)$ is a direct summand of T. Conversely, if ϱ is one-to-one and $\varrho(T)$ is a direct summand of T, then there exists $\chi \in \mathrm{Hom}_Z(T, T)$ such that $\chi\varrho$ is the identity on T. Consequently, $\varrho_1^* \chi^*$ is the identity on $\mathrm{Hom}_Z(T, K)$ for all K. Thus, ϱ_1^* is onto for all K.

(5.6) **Theorem.** *Let T be a shift module. The reduction homomorphism is onto for all trivial Δ-modules K which are free groups if and only if T is a free group.*

Proof. By Theorem 5.1, i_2 is onto if and only if

$$\varrho_2^* : \mathrm{Ext}_Z(T, K) \to \mathrm{Ext}_Z(T, K)$$

is zero. Assume that $\varrho_2^* = 0$ for all free K. Let G be a free group and $f : G \to T$ an epimorphism. Then $\mathrm{Ker} f$ is free, and by 5.3, there is a homomorphism $g : T \to G$ such that $\varrho = fg$. Since $g(T)$ is free, it follows that $T = F_1 \oplus \mathrm{Ker} g$, where $F_1 \cong g(T)$ is free. Note that $\mathrm{Ker} g \subseteq \mathrm{Ker}\varrho$, and that since T is a shift module, it follows that $\varrho(x_1, x_2, x_3, \ldots) = (x_1 - x_2, x_2 - x_3, \ldots)$ is zero if and only if $x_1 = x_2 = x_3 = \cdots$. Let $\lambda : \mathrm{Ker}\varrho \to T$ be defined by $\lambda(x, x, x, \ldots) = (x, 0, 0, \ldots)$, and let π be the projection of $T = F_1 \oplus \mathrm{Ker} g$ onto F_1. If $\pi\lambda(x, x, x, \ldots) = \pi(x, 0, 0, \ldots)$ is zero, then $(x, 0, 0, \ldots) \in \mathrm{Ker}\pi = \mathrm{Ker} g \subseteq \mathrm{Ker}\varrho$, so that $x = 0$. That is, $\pi\lambda$ is a monomorphism of $\mathrm{Ker}\varrho$ into F_1. Thus, $\mathrm{Ker}\varrho$ is free, and consequently $\mathrm{Ker} g$ is free. Hence T is a free group.

Conversely, if T is free, then $\mathrm{Ext}_Z(T, K) = 0$ for all K, and hence $\varrho_2^* = 0$ for all K.

(5.7) *Examples.* (a) Let T be the shift module S_H, where H is any group. Then the homomorphism ϱ has an inverse, namely

$$\varrho^{-1}(x_1, x_2, \ldots, x_n, 0, 0, \ldots) = \left(\sum_{j=1}^{n} x_j, \sum_{j=2}^{n} x_j, \ldots, \sum_{j=n}^{n} x_j, 0, 0, \ldots \right).$$

Hence by 5.4 and 5.5, the reduction homomorphism i_2 is both zero and one-to-one for all trivial Δ-modules K. Thus, $\mathrm{Ext}^1_\Delta(T, K) = 0$. This means that an ID-group which is a Δ-module extension of a trivial Δ-module by S_H is the module direct sum $K \oplus_\Delta S_H$.

(b) Let T be a shift module. Thus, T is a t. s. i. subgroup of P_H for some group H. Define $\delta : P_H \to P_H$ by

$$\delta(x_1, x_2, x_3, \ldots) = \left(\sum_{j=1}^{1} x_j, \sum_{j=1}^{2} x_j, \sum_{j=1}^{3} x_j, \ldots \right).$$

Assume that $\delta(T) \subseteq T$. Then $-\eta\delta$ is a right inverse of ϱ. Consequently, $\mathrm{Ker}\varrho$ is a direct summand of T and $\varrho(T) = T$. Therefore by 5.4, i_2 is zero for all trivial Δ-modules K. Note that $\mathrm{Ker}\varrho \cong H$, since for any $x \in H$,

$$(x, x, x, \ldots) = \delta(x, 0, 0, \ldots) \in \delta(S_H) \subseteq \delta(T) \subseteq T.$$

Therefore, if $H \neq 0$, ϱ is not one-to-one, and it follows from 5.5 that i_2 is not one-to-one for all trivial Δ-modules K. Hence for some trivial Δ-module K, $\mathrm{Ext}^1_\Delta(T, K) \neq 0$. We conclude that there exists a group K and Δ-module extensions of K by T with essentially different ID-structures. Of course, all of these extensions are isomorphic to $K \oplus T$ (as groups).

(c) Let $H = Z/pZ = \{\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{p-1}\}$ be the cyclic group of order p. Define $w = (w_1, w_2, w_3, \ldots)$, where $w_i = \bar{1}$ if i is a perfect square and $w_i = \bar{0}$ otherwise. Let T be the subgroup of P_H generated by S_H, $(\bar{1}, \bar{1}, \bar{1}, \ldots)$, $\xi^j w$ for $j = 0, 1, 2, \ldots$, and $\eta^j w$ for $j = 0, 1, 2, \ldots$. Then T is a shift module. Let δ be the endomorphism of P_H defined in (b). For every natural number n, $\delta(w)$ contains a block of n consecutive zeros followed by a block of n consecutive ones. It is easy to see that no element of T has this property. Hence $\delta(w) \notin T$.

We will prove that there is some trivial Δ-module K for which the reduction homomorphism i_2 is not zero. Assume the contrary. Then $i_2 = 0$ for all trivial K. By 5.4, $\operatorname{Ker}\varrho$ is a direct summand of T and $\varrho(T) = T$ (since T is a torsion group). Therefore, ϱ has a right inverse φ. Consequently, for $x \in T$, $\varrho(\varphi - \eta\delta)(x) = 0$. Hence, $(\varphi - \eta\delta)(x) = (\bar{k}, \bar{k}, \bar{k}, \ldots) \in T$, and $-\eta\delta(x) \in T$. Consequently $\delta(x) = -\xi(-\eta\delta(x)) \in T$, contradicting the fact that $\delta(w) \notin T$.

(d) Let $H = Z$. Let r be any positive real number. Define $T_r \subseteqq P_H$ by

$$T_r = \{x \in P_H \mid |x(n)| \leqq cn^r \text{ for some positive integer } c\}.$$

Clearly, T_r is a pure subgroup of P_H, and $S_H \subseteqq T_r$. Moreover, it is easy to verify that $\xi T_r \subseteqq T_r$ and $\eta T_r \subseteqq T_r$, so that T_r is a shift module. We wish to prove that there is a torsion free group K such that $i_2 : \operatorname{Ext}^1_\Delta(T_r, K) \to \to \operatorname{Ext}_Z(T_r, K)$ is not the zero homomorphism. This can be done, using 5.4 and the remark following this theorem, by showing that there is no subgroup $L \subseteqq \varrho(T_r)$ such that T_r/L is free. Suppose that such an L exists. Using the fact proved in [6, Theorem 3] that $\operatorname{Hom}_Z(T_r, Z)$ is countable, it follows that T_r/L has finite rank. Thus T_r/L is finitely generated. Consequently $T_r/\varrho(T_r)$ is finitely generated. To show that this is impossible, we have only to prove that $T_r/\varrho(T_r)$ is a non-zero divisible group. Let δ be the endomorphism of P_Z defined in (b). Set $\tau = -\eta\delta$. Then $\varrho\tau$ is the identity on P_Z. Moreover $\tau\varrho(T_r) \subseteqq T_r$, since $\tau\varrho(x) = x - u$, where $u(n) = x(1)$ for all n. To prove that $T_r/\varrho(T_r)$ is divisible, let $x \in T_r$ and $k > 1$. It suffices to find v and w in T_r such that $x = kv + \varrho(w)$. Let $y = \tau(x)$. Define $z(n)$ and $w(n)$ by the division algorithm:

$$y(n) = kz(n) + w(n), \quad 0 \leqq w(n) < k.$$

Then $z, w \in P_Z$ and w is bounded, so that $w \in T_r$. Hence

$$x = \varrho\tau(x) = \varrho(y) = \varrho(kz + w) = kv + \varrho(w),$$

where $v = \varrho(z)$. Since $x \in T_r$ and $\varrho(w) \in T_r$, it follows that $kv \in T_r$. Thus, $v \in T_r$ since T_r is pure in P_Z. Hence $T_r/\varrho(T_r)$ is divisible. It remains only to prove that $\varrho(T_r) \neq T_r$. Define $x \in P_Z$ by $x(n) = [n^r]$. Clearly, $x \in T_r$. If $x \in \varrho(T_r)$, then $\tau(x) \in \tau\varrho(T_r) \subseteqq T_r$. However, it is easy to see that $\tau(x)(n)$ is of the order of n^{r+1}. Thus, $\tau(x) \notin T_r$.

The proof given in 5.7 (d) yields a somewhat more precise result: the reduction homomorphism $i_2 : \operatorname{Ext}^1_\Delta(T_r, Z) \to \operatorname{Ext}_Z(T_r, Z)$ is not zero. Indeed, writing $\varrho = j\nu$ with $\nu : T_r \to \varrho(T_r)$ and $j : \varrho(T_r) \to T_r$ (injection), we obtain

$$0 \to \operatorname{Hom}_Z(\varrho(T_r), Z) \to \operatorname{Hom}_Z(T_r, Z)$$

and

$$\operatorname{Hom}_Z(\varrho(T_r), Z) \to \operatorname{Ext}_Z(T_r/\varrho(T_r), Z) \to \operatorname{Ext}_Z(T_r, Z) \xrightarrow{j^*} \operatorname{Ext}_Z(\varrho(T_r), Z).$$

Thus, since $\operatorname{Hom}_Z(T_r, Z)$ is countable (by [6, Theorem 3]) and $\operatorname{Ext}_Z(T_r/\varrho(T_r), Z)$ is uncountable (since $T_r/\varrho(T_r)$ is a non-zero divisible group), it follows that $\operatorname{Ker} j^* \neq 0$. Thus, $\varrho_2^* = \nu^* j^*$ is not one-to-one, and by 5.1, i_2 is not zero.

(5.8) Theorem. *Let H be any group. Then there exists a t. s. i. subgroup $T \subseteqq P_H$, and a group K such that the reduction homomorphism i_2 is not zero.*

Proof. If H is a torsion group, let M denote a cyclic subgroup of H with prime order. If H is not a torsion group, let M be an infinite cyclic subgroup of H. In each case, there is a t. s. i. subgroup T_0 of P_M, and a trivial Δ-module K such that the reduction homomorphism

$$i_2 : \mathrm{Ext}^1_\Delta(T_0, K) \to \mathrm{Ext}_Z(T_0, K)$$

is not zero (by 5.7 (c) and (d)). Then $T_0 + S_H$ is a t. s. i. subgroup of P_H. As Δ-modules, $T/T_0 \cong S_H/T_0 \cap S_H = S_H/S_M \cong S_{H/M}$. Thus, by Theorem 4.6, we have the commuting diagram

$$\begin{array}{ccc} \mathrm{Ext}^1_\Delta(T, K) & \to & \mathrm{Ext}_Z(T, K) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^1_\Delta(T_0, K) & \to & \mathrm{Ext}_Z(T_0, K) \end{array}$$

in which the vertical mappings are induced by the inclusion mapping of T_0 into T. Note that the image of the mapping of $\mathrm{Ext}^1_\Delta(T, K)$ into $\mathrm{Ext}^1_\Delta(T_0, K)$ is the kernel of the mapping of $\mathrm{Ext}^1_\Delta(T_0, K)$ into $\mathrm{Ext}^2_\Delta(S_{H/M}, K)$. By 5.2, $\mathrm{Ext}^2_\Delta(S_{H/M}, K)$ is isomorphic to $\mathrm{Ext}_Z(L, K)$, where L is the kernel of the mapping $s \to s - \xi s$ in $S_{H/M}$. Thus, $L = 0$ and $\mathrm{Ext}^2_\Delta(S_{H/M}, K) = 0$. That is, the mapping $\mathrm{Ext}^1_\Delta(T, K) \to \mathrm{Ext}^1_\Delta(T_0, K)$ is onto. Since the reduction homomorphism $\mathrm{Ext}^1_\Delta(T_0, K) \to \mathrm{Ext}_Z(T_0, K)$ is not zero, it follows that the reduction homomorphism $\mathrm{Ext}^1_\Delta(T, K) \to \mathrm{Ext}_Z(T, K)$ is not zero either.

Remark. If H is not a torsion group, we can let $M = Z$. By the remark following 5.7 (d), the group K can be taken to be Z in this case also.

Theorem 5.8 can be reformulated as a statement concerning the existence of ID-groups.

(5.9) **Corollary.** *Let H be an arbitrary non-zero abelian group. Then there exists an ID-group G and a monomorphism φ of G into itself such that $G/\varphi(G) \cong H$ and $\varphi^\omega G$ is not a direct summand of G.*

References

[1] Beaumont, R. A., and R. S. Pierce: Partly transitive modules and modules with proper isomorphic submodules. Trans. Am. Math. Soc. **91**, 209—219 (1959).

[2] Cartan, H., and S. Eilenberg: Homological algebra. Princeton University Press 1956.

[3] Crawley, P.: An infinite primary abelian group without proper isomorphic subgroups. Bull. Am. Math. Soc. **68**, 463—467 (1962).

[4] Fuchs, L.: Abelian groups. Budapest 1958.

[5] Kaplansky, I.: Infinite abelian groups. Ann Arbor 1954.

[6] Specker, E.: Additive Gruppen von Folgen von ganzen Zahlen. Portugaliae Math. **9**, 129—140 (1950).

(Received January 7, 1963)