IGUSA, J.
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A Desingularization Problem in the Theory of Siegel Modular Functions*

Dedicated to C. L. SIEGEL on his 70th birthday

JUN-ICHI IGUSA

Introduction

We shall try to outline the background of the problem, the problem itself, and our results so that this introduction alone will give a fairly precise idea of what the paper is about. Let $\mathfrak{S}_g$ denote the Siegel upper-half plane of degree g and $\Gamma_g(\lambda)$ the principal congruence group of degree g and of level λ. We shall sometimes exclude the case $g=1$. As a discrete subgroup of $\mathrm{Sp}(g, \mathbf{R})$, the group $\Gamma_g(\lambda)$ operates properly discontinuously on $\mathfrak{S}_g$ from the left and the quotient variety $\Gamma_g(\lambda)\backslash\mathfrak{S}_g$ is non-singular for $\lambda \geqq 3$. Consider, on the other hand, the ring $A(\Gamma_g(\lambda))$ generated by Siegel modular forms belonging to $\Gamma_g(\lambda)$. Then $A(\Gamma_g(\lambda))$ is a positively graded, integrally closed, integral domain of ring finite type over $\mathbf{C}$ and the projective variety $\mathscr{S}(\Gamma_g(\lambda))$ associated with this graded ring is a compactification of $\Gamma_g(\lambda)\backslash\mathfrak{S}_g$ in the sense that this is complex-analytically isomorphic to a Zariski open subset of $\mathscr{S}(\Gamma_g(\lambda))$. Furthermore, the "boundary" $\mathscr{S}(\Gamma_g(\lambda)) - \Gamma_g(\lambda)\backslash\mathfrak{S}_g$ of $\mathscr{S}(\Gamma_g(\lambda))$ is a disjoint union of quasi projective varieties which are the conjugates of the image of $\Gamma_{g_0}(\lambda)\backslash\mathfrak{S}_{g_0}$ under the dual Φ^* of the Siegel operator Φ for $0 \leqq g_0 < g$. These properties of $\mathscr{S}(\Gamma_g(\lambda))$ are obtained by BAILY and $\mathscr{S}(\Gamma_g(\lambda))$ is generally known as the Satake compactification of $\Gamma_g(\lambda)\backslash\mathfrak{S}_g$ (cf. 16, 1, 2).

Now, it was observed by CHRISTIAN that all boundary points of $\mathscr{S}(\Gamma_g(\lambda))$ are singular on $\mathscr{S}(\Gamma_g(\lambda))$ except for the case when $(g, \lambda) = (2, 1)$ (cf. 4, 5). Around the same time, we found that $\mathscr{S}(\Gamma_2(\lambda))$ is not even "almost non-singular" for $\lambda \geqq 3$ although it is a V-manifold, and hence almost non-singular for $\lambda = 1, 2$ (8). Then, it was observed that $\mathscr{S}(\Gamma_g(\lambda))$ is not almost non-singular at any boundary point except for the case when $(g, \lambda) = (2, 1), (2, 2)$. In fact, this was derived from a general theorem of partial desingularization of compactifications of SATAKE's type (10). Since algebraic geometry can most effectively be applied if the variety is non-singular, a problem arises as to whether $\mathscr{S}(\Gamma_g(\lambda))$ admits a natural desingularization or not. This does not mean to desingularize each $\mathscr{S}(\Gamma_g(\lambda))$ separately but it means to find a desingularization functor $\mathscr{D}$ and a natural transformation from $\mathscr{D}$ to $\mathscr{S}$. Furthermore, it is expected that, if we denote by $\mathscr{F}_{g-1}(\lambda)$ the proper transform of $\Phi^*\mathscr{S}(\Gamma_{g-1}(\lambda))$ under the morphism

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$\mathcal{D}(\Gamma_g(\lambda)) \to \mathcal{S}(\Gamma_g(\lambda))$, the restriction of this morphism to $\mathcal{F}_{g-1}(\lambda)$ splits into two morphisms

$$\mathcal{F}_{g-1}(\lambda) \to \mathcal{D}(\Gamma_{g-1}(\lambda)) \to \mathcal{S}(\Gamma_{g-1}(\lambda))$$

such that the fibers of the first morphism are auto-dual abelian varieties of dimension $g-1$ with level λ structures and their "limits", and the second morphism is the desingularization of degree $g-1$. Moreover, if we decompose g as $g_0 + g_1$, the fiber over the image point in $\mathcal{S}(\Gamma_g(\lambda))$ of a point t in $\mathfrak{S}_{g_0}$ is expected to be an extension of the g_1-fold product of the abelian variety with $(t\,1_{g_0})$ as its period matrix by a reducible variety each irreducible component of which is a compactification of the group variety $(\mathbf{C}^*)^{(\frac{1}{2})g_1(g_1-1)}$. With this problem in mind, we investigated the monoidal transform $\mathcal{M}(\Gamma_g(\lambda))$ of $\mathcal{S}(\Gamma_g(\lambda))$ for $\lambda \geqq 3$ along its boundary, i.e., along its singular locus. We have found that the monoidal transformation $\mathcal{M}(\Gamma_g(\lambda)) \to (\Gamma_g(\lambda))$ and the desingularization $\mathcal{D}(\Gamma_g(\lambda)) \to \mathcal{S}(\Gamma_g(\lambda))$ so-to-speak coincide on the image of $\Gamma_{g_0}(\lambda)\backslash\mathfrak{S}_{g_0}$ by Φ^* and on its conjugates for $g_1 \leqq 3$, but not for $g_1 = 4$. In particular $\mathcal{M}$ gives the desingularization functor $\mathcal{D}$ for $g \leqq 3$. This is our *main result.* It seems very likely that closer examination of the non-singular projective varieties $\mathcal{M}(\Gamma_g(\lambda))$ for $g \leqq 3$ will not only enrich our knowledge in algebraic geometry but also in the theory of Siegel modular functions.

It is with pleasure that we mention that the first version of our work was done at Bures-sur-Yvette and was announced at Göttingen in the spring of 1964. Also, we would like to remark that the present version is very closely connected with a paper written by SIEGEL in 1955 (20). Indeed, we shall see that properties of the monoidal transformation are rooted in a certain theory of reduction of positive, non-degenerate matrices, which we shall discuss in the first section of this paper.

1. Fundamental and central cones

We shall use $\mathbf{Z}$ and $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$ to denote the ring of rational integers and the field of rational, real, complex numbers. Also, we shall use $\mathbf{R}_+$ to denote the set of non-negative real numbers. Let g denote a positive integer. We shall use $\mathfrak{Y}$, or more precisely $\mathfrak{Y}_g$, to denote the set of symmetric matrices of degree g with coefficients in $\mathbf{R}$. Then $\mathfrak{Y}$ forms a vector space over $\mathbf{R}$ of dimension $N = (\frac{1}{2})g(g+1)$. If we use "tr" to denote the trace function, and also "det" to denote the determinant, we can identify $\mathfrak{Y}$ with its dual space by $y \to \mathrm{tr}(y\ \ )$. An element y of $\mathfrak{Y}$ is called positive if, for every column vector x with g coefficients in $\mathbf{R}$, the quadratic form $x \to {}^t x y x$ is $\mathbf{R}_+$-valued. The point y is called non-degenerate if the quadratic form ${}^t x y x$ is non-degenerate. This is the case if and only if we have $\det(y) \neq 0$. Also, we say that y is a half-integer matrix or a half-integer point if the quadratic form ${}^t x y x$ is $\mathbf{Z}$-valued on the lattice of integer vectors. This is the case if and only if diagonal coefficients and the twice of other coefficients are integers. We shall denote by $\mathfrak{Y}_+$ the set of all positive elements of $\mathfrak{Y}$. This set forms a non-degenerate, closed convex cone such that its interior points are precisely positive, non-degenerate matrices. Now, we know that the group $GL(g, \mathbf{R})$ operates on $\mathfrak{Y}$ keeping $\mathfrak{Y}_+$ stable as

$$y \to uy{}^t u.$$

Therefore $GL(g, \mathbf{R})$ operates on the interior of $\mathfrak{Y}_+$ and there the action is transitive. In particular $GL(g, \mathbf{Z})$ operates on $\mathfrak{Y}_+$, thus introducing an equivalence relation in its interior. We can construct a closed convex cone in $\mathfrak{Y}_+$ containing a representative for every interior point of $\mathfrak{Y}_+$ in the following way.

Suppose that σ is a fixed interior point of $\mathfrak{Y}_+$ and y a point of $\mathfrak{Y}$. If u is a point of the vector space over $\mathbf{R}$ of $g \times g$ matrices with coefficients in $\mathbf{R}$, we get a quadratic form on this vector space as $u \to \mathrm{tr}(\sigma u y{}^t u)$. We see immediately that this quadratic form is positive, non-degenerate if and only if y is an interior point of $\mathfrak{Y}_+$. In this case, the quadratic form attains its minimum on any given closed discrete subset of the vector space. After this remark, we consider the set F_σ of points y of $\mathfrak{Y}_+$ satisfying

$$\mathrm{tr}(\sigma u y{}^t u) - \mathrm{tr}(\sigma y) \geqq 0$$

for every u in $GL(g, \mathbf{Z})$. Clearly F_σ is a closed convex cone in $\mathfrak{Y}_+$. Furthermore, if y_0 is an arbitrary interior point of $\mathfrak{Y}_+$ and if the quadratic form $\mathrm{tr}(\sigma u y_0{}^t u)$ attains its minimum on $GL(g, \mathbf{Z})$ at $u = v$, say, then $y = v y_0{}^t v$ is in F_σ, and y is equivalent to y_0 with respect to $GL(g, \mathbf{Z})$. We call F_σ the *fundamental cone* associated with σ. The fundamental cone is non-degenerate. The following lemma is straightforward:

Lemma 1. *Suppose that u is an element of $GL(g, \mathbf{Z})$. Then u maps an interior point of F_σ to a point of F_σ if and only if it keeps σ invariant in the sense ${}^t u \sigma u = \sigma$.*

We shall denote by $\mathrm{Aut}(\sigma)$ the finite subgroup of $GL(g, \mathbf{Z})$ consisting of those u satisfying ${}^t u \sigma u = \sigma$. The lemma states that the stabilizer of F_σ in $GL(g, \mathbf{Z})$ is precisely $\mathrm{Aut}(\sigma)$.

In the special case when σ is a half-integer matrix, we consider the set C_σ of points y of $\mathfrak{Y}_+$ satisfying

$$\mathrm{tr}(\sigma' y) - \mathrm{tr}(\sigma y) \geqq 0$$

for every half-integer, interior point σ' of $\mathfrak{Y}_+$. Clearly C_σ is a closed convex cone contained in F_σ. We call C_σ the *central cone* associated with σ. We note that the stabilizer of C_σ in $GL(g, \mathbf{Z})$ is $\mathrm{Aut}(\sigma)$ at least when C_σ is non-degenerate. It would be interesting to investigate in general whether F_σ and C_σ are "chambers" in the sense they have only a finite number of "walls". The properties of the fundamental cone was investigated in the case $g \leqq 4$ for a very special σ by SELLING and CHARVE (17, 3). We were led to the considerations of both F_σ and C_σ for the same special σ from an entirely different line of thoughts.[1] We shall summarize their results mixing with our additional results, which we shall use in the later sections.

The particular σ we have mentioned is the following half-integer $g \times g$ matrix

[1] Prof. SIEGEL has kindly suggested to us the following references: G. VORONOÏ, Crelles J., **133** (1908), pp. 97—178; **134** (1908), pp. 198—287; **136** (1909), pp. 67—181. M. KOECHER, Math. Ann., **141** (1960), pp. 384—432; **144** (1961), pp. 175—182.

$$\sigma_0 = \begin{pmatrix} 1 & \frac{1}{2} & \dots & \frac{1}{2} \\ \frac{1}{2} & 1 & \dots & \frac{1}{2} \\ & & \dots & \\ \frac{1}{2} & \frac{1}{2} & \dots & 1 \end{pmatrix}.$$

If x is a column vector with coefficients $x_1, x_2, \dots, x_g$ and if we introduce a column vector X with coefficients $x_1, x_2, \dots, x_g, x_{g+1} = -(x_1 + x_2 + \dots + x_g)$, we have

$${}^t x \sigma_0 x = (\tfrac{1}{2}) {}^t X X = (\tfrac{1}{2}) \sum_{i=1}^{g+1} (x_i)^2 .$$

This shows clearly that σ_0 is positive and non-degenerate. For a closer investigation of the corresponding quadratic form $\operatorname{tr}(\sigma_0 u y {}^t u)$, it is convenient to introduce a new coordinate system in the space $\mathfrak{Y}$.

Suppose that y is a point of $\mathfrak{Y}_g$ with coefficients y_{ij} for $1 \leqq i, j \leqq g$. We define a point Y of $\mathfrak{Y}_{g+1}$ with coefficients y_{ij} for $1 \leqq i, j \leqq g+1$, where the additional $g+1$ coefficients are introduced as

$$\sum_{j=1}^{g+1} y_{ij} = 0 \qquad i = 1, 2, \dots, g+1 .$$

The correspondence $y \to Y$ can be extended by linearity to the vector spaces of matrices with coefficients in $\mathbf{C}$. At any rate, the point Y, hence also y, is uniquely determined by its N coefficients y_{ij} for $1 \leqq i < j \leqq g+1$. We arrange these coefficients lexicographically and call them *normal coordinates* of the point y. Also, we call Y the matrix associated with y. The usefulness of the normal coordinates comes partly from the following identity. Suppose that σ is an arbitrary point of $\mathfrak{Y}$. We shall denote g column vectors contained in an arbitrary $g \times g$ matrix u by $u_1, u_2, \dots, u_g$ and also we put $u_{g+1} = 0$. Then we have

$$\operatorname{tr}(\sigma u y {}^t u) = \sum_{1 \leqq i < j \leqq g+1} (-y_{ij}) {}^t(u_i - u_j) \sigma (u_i - u_j) .$$

We note that the matrix σ_0 is characterized by the condition ${}^t(u_i - u_j)\sigma(u_i - u_j) = 1$ for $1 \leqq i < j \leqq g+1$ when $u = 1_g$. After these remarks, we shall determine $\operatorname{Aut}(\sigma_0)$.

For each n, we shall denote by π_n the symmetric group of permutations of the first n positive integers $1, 2, \dots, n$. Then we have a representation $\pi_{g+1} \to \to GL(g, \mathbf{Z})$ defined in the following way. Suppose that $p: i \to i'$ is an element of π_{g+1}. Then, to a column vector X with coefficients $x_1, x_2, \dots, x_{g+1}$, we associate another column vector X' whose i'-th coefficient is x_i. Clearly, if X is contained in the hyperplane $x_1 + x_2 + \dots + x_{g+1} = 0$, so is X'. If we drop the $(g+1)$-th coefficients from X, X', we get column vectors x, x' and an element $u(p)$ of $GL(g, \mathbf{Z})$ satisfying $x' = u(p)x$. The correspondence $p \to u(p)$ defines the representation in question. Because this is a monomorphism, we sometimes identify π_{g+1} with its image in $GL(g, \mathbf{Z})$.

Lemma 2. *We have* $\operatorname{Aut}(\sigma_0) = \pi_{g+1} \cup -\pi_{g+1}$.

Proof. We shall denote the right hand side by $\pm \pi_{g+1}$. If u is taken from $\pm \pi_{g+1}$, we have

$$\begin{aligned} {}^t(ux)\sigma_0(ux) &= {}^t x' \sigma_0 x' = (\tfrac{1}{2}) {}^t X' X' \\ &= (\tfrac{1}{2}) {}^t X X = {}^t x \sigma_0 x \end{aligned}$$

for every x, hence ${}^t u\sigma_0 u=\sigma_0$. Conversely, suppose that u is taken from $\mathrm{Aut}(\sigma_0)$. Then we have ${}^t(u_i-u_j)\sigma_0(u_i-u_j)=1$ for $1\leqq i<j\leqq g+1$. In particular, we have ${}^t u_i\sigma_0 u_i=1$ for $i=1,2,\ldots,g$. Consequently, each u_i either has only one non-zero coefficient which is ± 1, or has only two non-zero coefficients which are 1 and -1. Since we have $\det(u)\neq 0$, not all u_i are of the second type. By multiplying an element of the subgroup $\pm\pi_g$ of $\pm\pi_{g+1}$ to u from the right, we can assume that the coefficients of u_g are $0,\ldots,0,0,1$. Then, by multiplying an element of the subgroup π_{g-1} of π_{g+1} to u from the left, we can assume that the coefficients of u_{g-1} are either $0,\ldots,0,1,0$ or $0,\ldots,0,-1,1$. In fact, we have only to examine the condition ${}^t(u_i-u_g)\sigma_0(u_i-u_g)=1$ for $i=g-1$. In the second case, we change u to $-u(g,g+1)u$, and we will get back to the first case. Then, by multiplying an element of the subgroup π_{g-2} of π_{g+1} to u from the left, we can change u into 1_g. This completes the proof.

We shall next determine the central cone C_σ for $\sigma=\sigma_0$. Using the identification of $\mathfrak{Y}$ to its dual space, we define a half-integer matrix σ_{ij} of $\mathfrak{Y}$ as

$$\mathrm{tr}(\sigma_{ij}y)=\mathrm{tr}(\sigma_0 y)-y_{ij}$$

for $1\leqq i<j\leqq g+1$. Since we have

$${}^t x\sigma_{ij}x=(\tfrac{1}{2})\Big(\sum_{k\neq i,j}(x_k)^2+(x_i-x_j)^2\Big),$$

in which k runs over the indices $1,2,\ldots,g+1$ excluding i and j, clearly σ_{ij} is positive and non-degenerate. Furthermore, they are conjugate to each other with respect to π_{g+1}. Also, we shall denote by e_{ij} the point of $\mathfrak{Y}$ such that its normal coordinates are all zero except at (i,j) where it is -1. We shall prove the following lemma:

Lemma 3. *The central cone C for $\sigma=\sigma_0$ is given by*

$$C=\sum_{1\leqq i<j\leqq g+1}\mathbf{R}_+ e_{ij}.$$

In other words C consists of points y of $\mathfrak{Y}$ such that the normal coordinates of $-y$ are all in $\mathbf{R}_+$.

Proof. Suppose that y is a point of $\mathfrak{Y}$ with the property $y_{ij}\leqq 0$ for $1\leqq i<j\leqq g+1$. Let σ denote an arbitrary half-integer, positive, non-degenerate matrix. Then, for any integer vector x different from zero, we have ${}^t x\sigma x\geqq 1$. Hence, for $u=1_g$ we have

$$\begin{aligned}\mathrm{tr}(\sigma y)&=\sum_{1\leqq i<j\leqq g+1}(-y_{ij})\,{}^t(u_i-u_j)\sigma(u_i-u_j)\\ &\geqq\sum_{1\leqq i<j\leqq g+1}(-y_{ij})=\mathrm{tr}(\sigma_0 y).\end{aligned}$$

This shows that y is in C. Conversely, suppose that y is taken from C. Then, we have $\mathrm{tr}(\sigma_{ij}y)-\mathrm{tr}(\sigma_0 y)=-y_{ij}\geqq 0$ for $1\leqq i<j\leqq g+1$. This already proves the lemma.

We shall now compare F_σ and C_σ for $\sigma=\sigma_0$. For this purpose, we shall introduce a point $e_{ij,k_1k_2k_3}$ of $\mathfrak{Y}$ whose normal coordinates are zero except at (ij) where it is 1 and at (ik_p), (jk_p) for $p=1,2,3$ where they are -1. We are assuming that the five indices i, j, k_1, k_2, k_3 are distinct. Therefore these points exist for $g \geqq 4$ and they are conjugate to each other with respect to π_{g+1}. We note that $e_{ij,k_1k_2k_3}$ is not in C because one of the normal coordinates is strictly positive.

Lemma 4. *The fundamental cone F for $\sigma = \sigma_0$ contains $e_{ij,k_1k_2k_3}$ for $g \geqq 4$, hence it is strictly larger than the central cone C. However, in the case $g \leqq 3$, we have $F = C$.*

Proof. We shall prove the first part. We have only to show that $y = e_{1g+1,234}$ is in F for $g \geqq 4$. The non-zero coefficients among y_{ij} for $1 \leqq i \leqq j \leqq g$ are 2, 2, 2, 2 at (11), (22), (33), (44) and -1, -1, -1 at (12), (13), (14). Let v denote a $g \times g$ matrix whose non-zero coefficients are 1, 2, 2, 2 at (11), (22), (33), (44) and -1, -1, -1 at (12), (13), (14) all multiplied by the square root of $\frac{1}{2}$. Then we have $y = v{}^tv$. Consequently F contains y if and only if

$$\operatorname{tr}(\sigma_0 u y {}^tu) = (\tfrac{1}{2}) \sum_{i=1}^{4} {}^t(2u_i - u_1)\sigma_0(2u_i - u_1)$$

is at least equal to $\operatorname{tr}(\sigma_0 y) = 5$. We shall separate seven cases.

(1) ${}^tu_1 \sigma_0 u_1 = 1$. In this case, we have ${}^t(2u_i - u_1)\sigma_0(2u_i - u_1) = 1, 3, \ldots$, and the value 1 is taken only by $i = 1$; hence $\operatorname{tr}(\sigma_0 u y {}^tu) \geqq (\frac{1}{2})(1+3+3+3) = 5$.

(2) ${}^tu_1 \sigma_0 u_1 = 2$. In this case, we have ${}^t(2u_i - u_1)\sigma_0(2u_i - u_1) = 2, 4, \ldots$, and the value 2 is taken at most by three values of i; hence $\operatorname{tr}(\sigma_0 u y {}^tu) \geqq$ $\geqq (\frac{1}{2})(2+2+2+4) = 5$.

(3) ${}^tu_1 \sigma_0 u_1 = 3$. In this case, we have ${}^t(2u_i - u_1)\sigma_0(2u_i - u_1) = 1, 3, \ldots$, and the value 1 is taken at most by one i; hence $\operatorname{tr}(\sigma_0 u y {}^tu) \geqq (\frac{1}{2})(3+1+3+3) = 5$.

(4) ${}^tu_1 \sigma_0 u_1 = 4$. In this case, we have ${}^t(2u_i - u_1)\sigma_0(2u_i - u_1) = 2, 4, \ldots$, hence $\operatorname{tr}(\sigma_0 u y {}^tu) \geqq (\frac{1}{2})(4+2+2+2) = 5$.

(5) ${}^tu_1 \sigma_0 u_1 = 5$. In this case, we have ${}^t(2u_i - u_1)\sigma_0(2u_i - u_1) = 1, 3, \ldots$, and the value 1 is taken at most by one i; hence $\operatorname{tr}(\sigma_0 u y {}^tu) \geqq (\frac{1}{2})(5+1+3+3) = 6$.

(6) ${}^tu_1 \sigma_0 u_1 = 6$. In this case, we have ${}^t(2u_i - u_1)\sigma_0(2u_i - u_1) = 2, 4, \ldots$, hence $\operatorname{tr}(\sigma_0 u y {}^tu) \geqq (\frac{1}{2})(6+2+2+2) = 6$.

(7) ${}^tu_1 \sigma_0 u_1 \geqq 7$. In this case, we simply have $\operatorname{tr}(\sigma_0 u y {}^tu) \geqq (\frac{1}{2})(7+1+1+1) = 5$.

We shall illustrate the argument in detail. Case (2) is most complicated. Since σ_0 is a half-integer, positive, non-degenerate matrix, we have ${}^t(2u_i - u_1)\sigma_0(2u_i - u_1) \geqq 1$ and also ${}^t(2u_i - u_1)\sigma_0(2u_i - u_1) \equiv {}^tu_1 \sigma_0 u_1 \mod 2$. Therefore, in the case when ${}^tu_1 \sigma_0 u_1 = 2$, the possible values are 2, 4, ... On the other hand, by multiplying an element of π_{g+1} to u from the left, we can assume that the coefficients of u_1 are 1, 1, -1, -1, 0, ..., 0. Then we see that an integer column vector x with coefficients $x_1, x_2, \ldots, x_g$ satisfies ${}^t(2x - u_1)\sigma_0(2x - u_1) = 2$ if and only if $x_1, x_2 = 1$ or 0, $x_3, x_4 = -1$ or 0, $x_5 = \cdots = x_g = 0$ and

$$x_{g+1} = -(x_1 + x_2 + \cdots + x_g) = 0\,.$$

Therefore, there exist five solution vectors among which only three are linearly independent.

We shall prove the second part. In general, consider an equation of the form

$$\operatorname{tr}(\sigma_0 u y {}^tu) - \operatorname{tr}(\sigma_0 y) = -n_{ij} y_{ij}\,,$$

in which y is now a variable point of $\mathfrak{Y}$. If we exclude the already discussed case $n_{ij}=0$, we have $n_{ij}\geqq 1$. Also, because of the conjugacy property, we have only to consider the case $(ij)=(12)$. Then, a solution u exists in $GL(g,\mathbf{Z})$ if and only if we have $g\leqq 3$. Moreover, in the case $g=2$, we have $n_{12}=2$ and u is in the coset

$$\mathrm{Aut}(\sigma_0)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

In the case $g=3$, we have $n_{12}=1$ and u is in the coset

$$\mathrm{Aut}(\sigma_0)\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 1 & 1\end{pmatrix}.$$

At any rate, for $g\leqq 3$, a point y of F satisfies $-y_{ij}\geqq 0$ for $l\leqq i<j\leqq g+1$, hence it is in C. This completes the proof.

The matrices we have introduced are sufficient to describe F for $g=4$ (but not for $g\geqq 5$). In fact, we have the following situation:

Lemma 5. *The fundamental cone F for $g=4$ is given by*

$$F=\sum_{1\leqq i<j\leqq 5}\mathbf{R}_+ e_{ij}+\sum_{1\leqq i<j\leqq 5}\mathbf{R}_+ e_{ij,k_1k_2k_3},$$

in which $e_{ij,k_1k_2k_3}$ depends only on (i,j).

Proof. Consider an equation of the form

$$\mathrm{tr}(\sigma_0 u y {}^t u)-\mathrm{tr}(\sigma_0 y)=-n_{i_1j_1}y_{i_1j_1}-n_{i_2j_2}y_{i_2j_2},$$

in which y is a variable point of $\mathfrak{Y}$. We exclude the known case where $n_{i_1j_1}=n_{i_2j_2}=0$. Then the equation has a solution u in $GL(4,\mathbf{Z})$ if and only if the right hand side is of the form $-y_{ij}-y_{ik}$ with distinct i,j,k. In the same way, we can analyze an equation such that $\mathrm{tr}(\sigma_0 u y {}^t u)-\mathrm{tr}(\sigma_0 y)$ consists of three terms. In particular, an equation of the form

$$\mathrm{tr}(\sigma_0 u y {}^t u)-\mathrm{tr}(\sigma_0 y)=-y_{i_1i_2}-y_{i_2i_3}-y_{i_3i_4},$$

in which i_1, i_2, i_3, i_4 are distinct, has a solution u in $GL(4,\mathbf{Z})$. In both cases, we can write down the solutions explicitly. Suppose that y is a point of F not in C. Then, one of its normal coordinates, say y_{ab}, is strictly positive. Then, for $k\neq a,b$, we have $-y_{ab}-y_{pk}\geqq 0$ for $p=a,b$. If y_{ab} is the only coordinate which is strictly positive, the point y is in the chamber

$$\sum_{(ij)\neq(ab)}\mathbf{R}_+ e_{ij}+\mathbf{R}_+ e_{ab,k_1k_2k_3}.$$

On the other hand, if there is another coordinate which is strictly positive, it is of the form y_{cd} with distinct a,b,c,d. Moreover, if k is the remaining index among 1, 2, 3, 4, 5, we have $-y_{ab}-y_{pk}\geqq 0$ for $p=a,b$ and $-y_{cd}-y_{pk}\geqq 0$ for $p=c,d$. Also we have $-y_{ab}-y_{cd}-y_{ij}\geqq 0$ for $i=a,b$ and $j=c,d$. Therefore, the point y is in the chamber

$$\sum_{(ij)\neq(ab),(cd)} \mathbf{R}_+ e_{ij} + \mathbf{R}_+ e_{ab,cdk} + \mathbf{R}_+ e_{cd,abk}.$$

Moreover, there is no other possibilities for the point y. We have thus shown that F can be decomposed into three types of "simple chambers" with C at the "center". At any rate, the lemma is proved already.

We note that, in the case $g=4$, the matrices $e_{ij,k_1k_2k_3}$ and the matrices $2\sigma_{i'j'}$ are conjugate with respect to $GL(4,\mathbf{Z})$. This can be proved by minimizing the quadratic form $u\to\mathrm{tr}(\sigma_0 u\sigma_{ij}{}^t u)$ on $GL(4,\mathbf{Z})$. In particular $e_{ij,k_1k_2k_3}$ are interior points of $\mathfrak{Y}_+$ for $g=4$. We can show that $\mathbf{R}_+ e_{ij,k_1k_2k_3}$ are "edges" of the fundamental cone F not only for $g=4$ but also for all $g\geqq 4$.

Now, suppose that y is a point of the central cone C satisfying $y_{ii}>0$ for $i=1,2,\ldots,g+1$. Then, we can introduce an equivalence relation in the set of $g+1$ indices. We say that i is equivalent to j if there exists a sequence of indices $k_0=i$, $k_1,\ldots,k_n=j$ all satisfying $1\leqq k\leqq g+1$ such that $-y_{k_pk_{p+1}}>0$ for $p=0,1,\ldots,n-1$. We shall show that the number of equivalence classes is equal to $g+1-\mathrm{rank}(y)$. We consider the matrix Y associated with y, i.e., the matrix of degree $g+1$ with coefficients y_{ij} for $1\leqq i,j\leqq g+1$. We apply a permutation to the $g+1$ indices so that indices of each equivalence class become consecutive. If there are n equivalence classes, we then have a splitting of Y of the following form

$$Y=\begin{pmatrix} Y_1 & & & \\ & Y_2 & & \\ & & \ddots & \\ & & & Y_n\end{pmatrix}.$$

Therefore we have

$$\begin{aligned}\mathrm{rank}(y)=\mathrm{rank}(Y)&=\sum_{i=1}^{n}\mathrm{rank}(Y_i)\\ &\leqq \sum_{i=1}^{n}(\deg(Y_i)-1)\\ &=g+1-n.\end{aligned}$$

Consequently, our statement will be proved if we can show that we have $\mathrm{rank}(Y)=g$ provided $g+1$ indices are equivalent to each other. Therefore, it is sufficient to show that y is positive, non-degenerate when $g+1$ indices are equivalent to each other. Consider the quadratic form $u\to\mathrm{tr}(\sigma u y{}^t u)$ for any positive, non-degenerate σ. If we have $\mathrm{tr}(\sigma u y{}^t u)=0$, we get ${}^t(u_i-u_j)\sigma(u_i-u_j)=0$, hence $u_i=u_j$ whenever $-y_{ij}>0$. Since all indices are equivalent to each other, we get $u_1=u_2=\cdots=u_{g+1}=0$, and this proves the assertion.

We shall introduce a notation. If Γ is a group of integer matrices, for any positive integer λ, we shall denote by $\Gamma(\lambda)$ the kernel of the homomorphism $\Gamma\to\Gamma \bmod \lambda$. We shall consider $GL(g,\mathbf{Z})(\lambda)$ in the next lemma:

Lemma 6. *Suppose that y is an interior point of $\mathfrak{Y}_+$ contained in C. Let σ denote either σ_0 or σ_{ij} for $1 \leqq i < j \leqq g+1$. Then we have*

$$\operatorname{tr}(\sigma u y {}^t u) - \operatorname{tr}(\sigma y) \geqq 0$$

for every u in $GL(g, \mathbf{Z})(\lambda)$ provided $\lambda \geqq 3$. Moreover, the equality sign holds if and only if $u = 1_g$.

Proof. For a moment, we shall denote the column vector u_i in the special case $u = 1_g$ by e_i for $i = 1, 2, \ldots, g+1$. Suppose that x is an integer vector satisfying $x \equiv e_i - e_j \bmod \lambda$. Then we have ${}^t x \sigma x \geqq 1$ and the equality sign holds if and only if $x = e_i - e_j$. There is one exception, and that is the case when $\sigma = \sigma_{ij}$ for the same i, j as above. In this case, we have ${}^t x \sigma x \geqq 2$ and the equality sign holds if and only if $x = e_i - e_j$. The verification is straightforward. Therefore, we have

$$\begin{aligned} \operatorname{tr}(\sigma u y {}^t u) &= \sum_{1 \leqq i < j \leqq g+1} (-y_{ij}) {}^t(u_i - u_j) \sigma (u_i - u_j) \\ &\geqq \sum_{1 \leqq i < j \leqq g+1} (-y_{ij}) {}^t(e_i - e_j) \sigma (e_i - e_j) \\ &= \operatorname{tr}(\sigma y). \end{aligned}$$

The equality sign holds if and only if $u_i - u_j = e_i - e_j$, i.e., $u_i - e_i = u_j - e_j$ whenever $-y_{ij} > 0$. Since we are assuming that y is positive, non-degenerate, the $g+1$ indices are equivalent to each other; hence $u_1 - e_1 = u_2 - e_2 = \cdots = u_{g+1} - e_{g+1} = 0$. This proves the assertion.

We shall consider a sequence of points in $\mathfrak{Y}$. Suppose that y is a "typical term" of the sequence. We say that the sequence tends to ∞ if $y - y'$ eventually becomes positive for any given y'. We shall express this fact by $y \to \infty$. If we have $y \to \infty$ and if u is an arbitrary element of $GL(g, \mathbf{R})$, we also have $u y {}^t u \to \infty$. Moreover, if we consider a submatrix Y_1 of y consisting of y_{ij} for $i, j = k_1, k_2, \ldots, k_d$, say, then we also have $Y_1 \to \infty$ (in the space $\mathfrak{Y}_d$). Finally, in the case $g = 1$, we have $y \to \infty$ if and only if $y \to +\infty$ on $\mathbf{R}$. Combining them together, we see that $y \to \infty$ implies $y_{ii} \to +\infty$ for $i = 1, 2, \ldots, g+1$. After these remarks, we shall prove the following lemma:

Lemma 7. *Suppose that a sequence of points is given in $\mathfrak{Y}$ with the property $y \to \infty$ and with the normal coordinates of y bounded above. Define a point y^0 of C by the condition*

$$(y^0)_{ij} = \begin{cases} 0 & y_{ij} \text{ bounded} \\ -1 & \text{otherwise} \end{cases}$$

for $1 \leqq i < j \leqq g+1$. Then y^0 is an interior point of $\mathfrak{Y}_+$.

Proof. Since $y \to \infty$, we have $y_{ii} \to +\infty$, hence $y_{ij} \to -\infty$ for some j. This implies $(y^0)_{ij} = -1$, hence $(y^0)_{ii} \geqq 1$ for $i = 1, 2, \ldots, g+1$. Suppose, now, that y^0 is not an interior point of $\mathfrak{Y}_+$. Then, we have $\operatorname{rank}(y^0) \leqq g-1$. Therefore, by applying a permutation to the $g+1$ indices, we can assume that the matrix Y^0 associated with y^0 splits into n parts, say. Accordingly, if we write

$$Y = \begin{pmatrix} Y_1 & * & \ldots & * \\ * & Y_2 & \ldots & * \\ & \ldots & \ldots & \\ * & * & \ldots & Y_n \end{pmatrix}.$$

the *-parts have bounded coefficients. Moreover Y_1 is a part of y, hence $Y_1 \to \infty$. Therefore, if we put $\deg(Y_1) = d$ and if we consider the matrix associated with Y_1, its $(d+1,\ d+1)$-coefficient has to tend to $+\infty$. On the other hand, it is bounded. This is a contradiction. The lemma is thus proved.

2. The analytic local ring $\mathcal{O}$

We shall fix our terminology first. Suppose that X is an analytic space. In other words X is a ringed space which locally looks like a piece of complex-analytic subvariety of a finite dimensional vector space over $\mathbf{C}$. The analytic space X is called normal if all of its local rings are integrally closed, integral domains. On the other hand, suppose that X is a quasi projective variety over $\mathbf{C}$. In other words X is isomorphic to a locally closed subvariety of a complex projective space, i.e., a projective space over $\mathbf{C}$. Then X carries a unique structure of an analytic space, which is normal in the analytic sense if and only if it is normal in the algebraic sense. In order to distinguish these two structures, we shall, if there is an ambiguity, use "Hausdorff topology" and "analytic local rings" etc. vis-à-vis "Zariski topology" and "algebraic local rings" etc. We refer to SERRE [18] for basic results concerning this subject. We say that a normal analytic space is *almost non-singular* if it admits a non-singular covering locally at every point. The *V-manifold* in the sense of SATAKE [15] is almost non-singular (the converse of which seems to be unknown).

We shall also recall basic facts about the *Satake compactifications* (cf. 1, 2). Consider the vector space of symmetric matrices of degree g with coefficients in $\mathbf{C}$. If τ is a point of this vector space over $\mathbf{C}$, we shall denote its real and imaginary parts by $\mathrm{Re}(\tau)$ and $\mathrm{Im}(\tau)$. They are points of the vector space $\mathfrak{Y}$ over $\mathbf{R}$. If $\mathrm{Im}(\tau)$ is in the interior of $\mathfrak{Y}_+$, we say that τ is a point of the *Siegel upper-half plane* $\mathfrak{S}_g$ of degree g. Clearly $\mathfrak{S}_g$ is an open convex cone. We know that the group of complex-analytic automorphisms of $\mathfrak{S}_g$ is given by $\mathrm{Sp}(g, \mathbf{R})/\pm 1_{2g}$. Moreover, if

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an element of $\mathrm{Sp}(g, \mathbf{R})$, the complex-analytic automorphism of $\mathfrak{S}_g$ determined by $\pm M$ is $\tau \to M \cdot \tau = (a\tau + b)(c\tau + d)^{-1}$. If Γ is a discrete subgroup of $\mathrm{Sp}(g, \mathbf{R})$, it operates properly discontinuously on $\mathfrak{S}_g$. Hence the quotient variety $\Gamma \backslash \mathfrak{S}_g$ is defined, and it is a V-manifold. If Γ operates without fixed points, the quotient variety is even non-singular. In the case when Γ is commensurable with $\mathrm{Sp}(g, \mathbf{Z})$, we consider, for every integer k, the vector space $A(\Gamma)_k$ over $\mathbf{C}$ of *Siegel modular forms* of weight k, i.e., holomorphic functions ψ on $\mathfrak{S}_g$ satisfying $\psi(M \cdot \tau) = \det(c\tau + d)^k \psi(\tau)$ for every M in Γ. These vector spaces generate a positively graded ring

$$A(\Gamma) = \bigoplus_{k \geqq 0} A(\Gamma)_k,$$

which is integrally closed and of finite type over $A(\Gamma)_0 = \mathbf{C}$. The projective variety $\mathcal{S}(\Gamma)$ associated with $A(\Gamma)$ contains a Zariski open set which is complex-analytically isomorphic to $\Gamma\backslash\mathfrak{S}_g$. Furthermore, the boundary $\mathcal{S}(\Gamma)-\Gamma\backslash\mathfrak{S}_g$ is a disjoint union of a finite number of quasi projective varieties each complex-analytically isomorphic to quotient varieties of the form $\Gamma_0\backslash\mathfrak{S}_{g_0}$ for $0\leqq g_0<g$. We shall explain $\mathcal{S}(\Gamma)$ more in detail in the case when Γ is the *principal congruence group* $\Gamma_g(\lambda)=\mathrm{Sp}(g,\mathbf{Z})(\lambda)$ of level λ.

We write g in the form $g=g_0+g_1$ for $0\leqq g_0<g$. Accordingly, write a typical point τ of $\mathfrak{S}_g$ as

$$\tau=\begin{pmatrix} t & z \\ {}^t z & w \end{pmatrix},$$

or simply as $\tau=(t,z,w)$, in which z, for instance, is a $g_0\times g_1$ matrix. Then, for every ψ in $A(\Gamma_g(\lambda))$, we define $\Phi\psi$ as

$$(\Phi\psi)(t)=\lim_{\mathrm{Im}(w)\to\infty}\psi(\tau).$$

This limit always exists and it depends only on t, hence $\Phi\psi$ is well defined. Moreover Φ gives a weight-preserving ring homomorphism $A(\Gamma_g(\lambda))\to A(\Gamma_g(\lambda))$, which is almost surjective in the sense it is surjective for all high weights. Consequently, we get a complex-analytic embedding $\Phi^*:\mathcal{S}(\Gamma_{g_0}(\lambda))\to\mathcal{S}(\Gamma_g(\lambda))$, and the image of $\Gamma_{g_0}(\lambda)\backslash\mathfrak{S}_{g_0}$, which we simply call the image of $\mathfrak{S}_{g_0}$, is a quasi projective subvariety of $\mathcal{S}(\Gamma_g(\lambda))$. Now $\mathcal{S}(\Gamma_g(\lambda))$ admits $\mathrm{Sp}(g,\mathbf{Z}/\lambda\mathbf{Z})$ as a group of (complex-analytic or algebraic) automorphisms. This group transforms the image of $\mathfrak{S}_{g_0}$ to its conjugates. The fact is that these varieties considered for $g_0=0,1,\ldots,g-1$ are mutually disjoint and their union is the entire boundary $\mathcal{S}(\Gamma_g(\lambda))-\Gamma_g(\lambda)\backslash\mathfrak{S}_g$. Because of this, as long as local properties of $\mathcal{S}(\Gamma_g(\lambda))$ at the boundary points are concerned, we have only to investigate $\mathcal{S}(\Gamma_g(\lambda))$ at those points lying on the image of $\mathfrak{S}_{g_0}$ by Φ^* for some g_0.

Suppose that t_0 is a point of $\mathfrak{S}_{g_0}$ and consider its image in $\mathcal{S}(\Gamma_g(\lambda))$. In the following, we shall describe the analytic local ring $\mathcal{O}$ of $\mathcal{S}(\Gamma_g(\lambda))$ at the image point of t_0 more or less explicitly using *Fourier-Jacobi series* by PYATETSKI-SHAPIRO ([14], cf. also [10]). We shall denote by $\mathfrak{Z}$ the vector space of $g_0\times g_1$ matrices with coefficients in $\mathbf{C}$ and, for t in $\mathfrak{S}_{g_0}$ and z,z' in $\mathfrak{Z}$, we put

$$L_t(z,z')=(\tfrac{1}{4})({}^t z\,\mathrm{Im}(t)^{-1}(\bar{z}'-z')+{}^t(\bar{z}'-z')\,\mathrm{Im}(t)^{-1}z).$$

Then $L_t(z,z')$ is in the vector space of symmetric matrices of degree g_1 with coefficients in $\mathbf{C}$. Moreover $L_t(z,z')$ is "quasi-hermitian" in the sense it is $\mathbf{C}$-linear in z, $\mathbf{R}$-linear in z' and

$$(1/2(-1)^{\frac{1}{2}})(L_t(z,z')-L_t(z',z))=(\tfrac{1}{4})\,\mathrm{Im}({}^t z\,\mathrm{Im}(t)^{-1}\,\bar{z}'+{}^t\bar{z}'\,\mathrm{Im}(t)^{-1}\,z)$$

is real, i.e., $\mathfrak{Y}_{g_1}$-valued. We also note that, because of

$$\mathrm{Re}(L_t(z,z))={}^t\mathrm{Im}(z)\,\mathrm{Im}(t)^{-1}\,\mathrm{Im}(z)$$

for every t in $\mathfrak{S}_{g_0}$, the point τ is contained in $\mathfrak{S}_g$ if and only if $\mathrm{Im}(w)-\mathrm{Re}(L_t(z,z))$ is positive, non-degenerate, i.e., in the interior of $\mathfrak{Y}_+$. We take an open neighborhood V of t_0 in $\mathfrak{S}_{g_0}$ and an element r of $\mathfrak{Y}$, and define an open subset $S(V,r)$ of $\mathfrak{S}_g$ as the set of points (t,z,w) such that t is in V and $\mathrm{Im}(w)-\mathrm{Re}(L_t(z,z))-r$ is in the interior of $\mathfrak{Y}_+$. Then we have $S(\mathfrak{S}_{g_0},0)=\mathfrak{S}_g$. What is more important is that, if we take V sufficiently small and r sufficiently large, the elements M of $\mathrm{Sp}(g, \mathbf{Z})$ with the property $M \cdot S(V, r) \cap S(V, r) \neq \emptyset$ are of the following form

$$M = \begin{pmatrix} a_{11} & 0 & b_{11} & * \\ * & a_{22} & * & * \\ c_{11} & 0 & d_{11} & * \\ 0 & 0 & 0 & * \end{pmatrix}.$$

We shall denote by P the subgroup of $\mathrm{Sp}(g, \mathbf{R})$ consisting of elements of this form, and by $P_{\mathbf{Z}}$ the subgroup of P of integer matrices. We note that, if M is an element of P and if we denote by M_0 the element of $\mathrm{Sp}(g_0, \mathbf{R})$ defined by a_{11}, b_{11}, c_{11}, d_{11} and put $u = a_{22}$, we have

$$M \cdot S(V, r) = S(M_0 \cdot V, ur^tu).$$

Now, suppose that f is a holomorphic function on $S(V, r)$ with the property $f(M \cdot \tau) = f(\tau)$ for all M in $P_{\mathbf{Z}}(\lambda)$ for which $M_0 = l_{2g_0}$ and $u = 1_{g_1}$. Then it admits a convergent Fourier expansion

$$f(t, z, w) = \sum_{\sigma} \theta_\sigma(t, z)\, e((1/\lambda)\operatorname{tr}(\sigma w)),$$

in which $e(\ ) = \exp(2\pi(-1)^{\frac{1}{2}}\ )$ and σ runs over half-integer matrices. The convergence means "normal convergence" in every compact subset (of $S(V, r)$). Furthermore $(t, z) \to \theta_\sigma(t, z)$ defines a holomorphic function on $V \times \mathfrak{Z}$ with the property

$$(\theta) \qquad \theta_\sigma(t, z + tm' + m'') = \theta_\sigma(t, z)\, e\big(-(1/\lambda)\operatorname{tr}(\sigma(2^tm'z + {}^tm'tm'))\big)$$

for all $g_0 \times g_1$ integer matrices m', m'' satisfying $m', m'' \equiv 0 \bmod \lambda$. Therefore, for a fixed t, it is a *theta-function* with

$$(2/\lambda)\operatorname{tr}\big(\sigma({}^tz \operatorname{Im}(t)^{-1}\bar{z})\big)$$

as its *Riemann form* (21). Since the real part of a Riemann form is positive, we see that σ is positive. This so-called Köcher effect subsists except for the case when $g_0 = 0$, $g_1 = 1$ and, in this case, we make the well-known modification. On the other hand, if we evaluate the imaginary part of the Riemann form at $(tm' + m'', tn' + n'')$, we get

$$(2/\lambda)\operatorname{tr}\big(\sigma({}^tm'n'' - {}^tm''n')\big).$$

We know that the reduced Pfaffian of this $\mathbf{Z}$-valued alternating form defined on the lattice of points $tm' + m''$ in $\mathfrak{Z}$ with $m' \equiv m'' \equiv 0 \bmod \lambda$ gives the dimension of the vector space of theta-functions $\theta_\sigma(t, z)$ (for a fixed t). We note that this is independent of t and, in the case when σ is non-degenerate, it is given by $(\det(2\sigma)\lambda^{g_1})^{g_0}$.

Now, if f is invariant by all M in $P_{\mathbf{Z}}(\lambda)$ for which we only assume $M_0 = 1_{2g_0}$, we get

$$\theta_{{}^tu\sigma u}(t, z) = \theta_\sigma(t, z^tu)$$

for all u in $GL(g_1, \mathbf{Z})$. Therefore, if we put

$$H_\sigma(t, z, w) = \sum_u{}' \theta_\sigma(t, z^tu)\, e((1/\lambda)\operatorname{tr}(\sigma uw^tu)),$$

we have

$$f(t, z, w) = \sum_{\sigma}' H_\sigma(t, z, w).$$

The primes in the summations indicate that in the first one it is taken over distinct ${}^t u \sigma u$ for u in $GL(g_1, \mathbf{Z})(\lambda)$ and in the second one it is taken over inequivalent, half-integer, positive matrices σ, the equivalence being taken with respect to the same group.

Conversely, suppose that σ is a given half-integer, positive matrix. Then a holomorphic function $\theta_\sigma(t, z)$ in $V \times \mathfrak{Z}$ satisfying the functional equation (θ) can be constructed by theta-series. Hence we can define $H_\sigma(t, z, w)$ by $\theta_\sigma(t, z)$ formally as above. The problem is whether the series for $H_\sigma(t, z, w)$ actually defines a holomorphic function in $S(V, r)$, for some r, invariant by all M in $P_{\mathbf{Z}}(\lambda)$ for which $M_0 = 1_{2g_0}$. This is just a problem of convergence.

Lemma 8. *Suppose that r is positive and non-degenerate. Then, for (t, z) in a compact subset of $V \times \mathfrak{Z}$ and for w such that $\mathrm{Im}(w) - \mathrm{Re}(L_t(z, z)) - r$ is positive, the series*

$$H_\sigma(t, z, w) = \sum_{u}' \theta_\sigma(t, z\,{}^t u)\, e((1/\lambda)\,\mathrm{tr}(\sigma u w\,{}^t u))$$

is dominated by a series of the form

$$\text{const.} \sum_{u}' \exp(-\mu\,\mathrm{tr}({}^t u \sigma u)),$$

in which the summations are taken over distinct ${}^t u \sigma u$ for u in $GL(g_1, \mathbf{Z})(\lambda)$ and $\mu > 0$.

Proof. If we let all coefficients of m', m'' vary from 0 to λ, the corresponding point $tm' + m''$ describes a compact subset of $\mathfrak{Z}$. We write $z\,{}^t u$ in the form $z_0 + tm' + m''$ with z_0 in this compact set and with m', m'' satisfying $m', m'' \equiv 0 \bmod \lambda$. Then, observing that the exponential factor in the functional equation (θ) is $e(-((-1)^{\frac{1}{2}}/\lambda)\,\mathrm{tr}(\sigma L_t(2z + tm' + m'', tm' + m'')))$, we have

$$\begin{aligned} &|\theta_\sigma(t, z\,{}^t u)\, e((1/\lambda)\,\mathrm{tr}(\sigma u w\,{}^t u))| \\ &\quad = |\theta_\sigma(t, z_0)|\, \exp(-(2\pi/\lambda)\,\mathrm{tr}(\sigma\,\mathrm{Re}(L_t(z_0, z_0)))) \times \\ &\qquad \times \exp(-(2\pi/\lambda)\,\mathrm{tr}(\sigma u(\mathrm{Im}(w) - \mathrm{Re}(L_t(z, z)))\,{}^t u)) \leqq \\ &\quad \leqq \text{const.} \exp(-(2\pi/\lambda)\,\mathrm{tr}(\sigma u r\,{}^t u)) \leqq \\ &\quad \leqq \text{const.} \exp(-\mu\,\mathrm{tr}({}^t u \sigma u)) \end{aligned}$$

for $\mu = (2\pi/\lambda)$-times the smallest eigen-value of r, and the summation of $\exp(-\mu\,\mathrm{tr}(\sigma'))$ over all half-integer, positive matrices σ' is convergent (cf. 19). This proves the lemma.

We can, now, give a description of the analytic local ring $\mathcal{O}$ of $\mathfrak{S}(\Gamma_g(\lambda))$ at the image point of t_0. We observe that the stabilizer of t_0 in $\Gamma_{g_0}(\lambda)$ is a finite group. If we take an element M of $P_{\mathbf{Z}}(\lambda)$ such that M_0 is in this finite group and if we write down the invariance condition for the corresponding transformation

$$(t, z, w) \to (M_0 \cdot t, {}^t(c_{11} t + d_{11})^{-1} z, w - {}^t z (c_{11} t + d_{11})^{-1} c_{11} z),$$

we get

$$(\theta') \qquad \theta_\sigma(M_0 \cdot t, {}^t(c_{11} t + d_{11})^{-1} z) = \theta_\sigma(t, z)\, e((1/\lambda)\operatorname{tr}(\sigma {}^t z (c_{11} t + d_{11})^{-1} c_{11} z)).$$

Conversely, if we use such $\theta_\sigma(t, z)$ for the construction of $H_\sigma(t, z, w)$, it is invariant by all elements M of $P_{\mathbf{Z}}(\lambda)$ for which $M_0 \cdot t_0 = t_0$. We shall summarize our results in the following way:

Theorem 1. *Suppose that t_0 is an arbitrary point of $\mathfrak{S}_{g_0}$. Then the analytic local ring $\mathcal{O}$ of $\mathcal{S}(\Gamma_g(\lambda))$ at the image point of t_0 consists of convergent series of the form*

$$f(t, z, w) = \sum_\sigma{}' H_\sigma(t, z, w),$$

in wich

$$H_\sigma(t, z, w) = \sum_u{}' \theta_\sigma(t, z {}^t u)\, e((1/\lambda)\operatorname{tr}(\sigma u w {}^t u)).$$

The conditions for $\theta_\sigma(t, z)$ are that it is holomorphic in $V \times \mathfrak{Z}$ for some unspecified open neighborhood V of t_0 and satisfies the functional equations (θ) *and* (θ').

The analytic local ring $\mathcal{O}$ describes the variety $\mathcal{S}(\Gamma_g(\lambda))$ in some (Hausdorff) neighborhood of the image point of t_0. In order to examine the boundary $\mathcal{S}(\Gamma_g(\lambda)) - \Gamma_g(\lambda)\backslash\mathfrak{S}_g$ in this neighborhood, we shall determine the ideal $\mathcal{J}$ in $\mathcal{O}$ associated with this closed subset of $\mathcal{S}(\Gamma_g(\lambda))$. For this purpose, we take an arbitrary element f of $\mathcal{O}$ and try to examine its restriction to the image of $\mathcal{S}(\Gamma_{g-1}(\lambda))$ by Φ^*. In other words, we examine the limit of $f(t, z, w)$ when $\operatorname{Im}(w_{g_1 g_1}) \to +\infty$. We shall use the original Fourier expansion

$$f(t, z, w) = \sum_\sigma \theta_\sigma(t, z)\, e((1/\lambda)\operatorname{tr}(\sigma w)).$$

Since the convergence is normal for (t, z) in any given compact subset of $V \times \mathfrak{Z}$ and for w such that $\operatorname{Im}(w) - \operatorname{Re}(L_t(z, z)) - r$ is positive for some r, the summation and the limit process are commutative. We see that the terms which survive are only those for which σ is of the following form

$$\sigma = \begin{pmatrix} \sigma^* & 0 \\ 0 & 0 \end{pmatrix}.$$

In this case, if we denote by w^* the symmetric matrix of degree $g_1 - 1$ obtained from w by deleting its g_1-th line and column vectors, we have $\operatorname{tr}(\sigma w) = \operatorname{tr}(\sigma^* w^*)$. Incidentally, also in this case, if we denote the g_1-th column vector in z by z^{**} and put $z = (z^* \, z^{**})$, then $\theta_\sigma(t, z)$ is periodic in z^{**} with $(t\, 1_{g_0})$ as its period matrix. Therefore $\theta_\sigma(t, z)$ is independent of z^{**} and it can be written as $\theta_{\sigma^*}(t, z^*)$. In this way, the said limit of $f(t, z, w)$ takes the following form

$$f^*(t, z^*, w^*) = \sum_{\sigma^*} \theta_{\sigma^*}(t, z^*)\, e((1/\lambda)\operatorname{tr}(\sigma^* w^*)).$$

In particular, we have $f^* = 0$ if and only if $\theta_{\sigma^*}(t, z^*) = 0$ for all σ^*, i.e., if and only if $\theta_\sigma(t, z) = 0$ whenever σ splits as above. On the other hand, we observe that the ideal $\mathcal{J}$ is stable by the subgroup of $P_{\mathbf{Z}}$ consisting of elements M for which $M_0 \cdot t_0 = t_0$. Hence, the ideal $\mathcal{J}$ is certainly stable under the transformation $f(t, z, w) \to f(t, z^t v, v w^t v)$ for v in $GL(g_1, \mathbf{Z})$. Therefore, if f is an element of $\mathscr{I}$, we get $\theta_\sigma(t, z^t v) = 0$, i.e., $\theta_\sigma(t, z) = 0$ whenever ${}^t v \sigma v$ splits as before. Since v is an arbitrary element of $GL(g_1, \mathbf{Z})$, this simply means that σ is degenerate. Conversely, suppose that $\theta_\sigma(t, z) = 0$ whenever σ is degenerate. Then, this property is preserved under every automorphism of $\mathcal{O}$ coming from an element M in $P_{\mathbf{Z}}$ for which $M_0 \cdot t_0 = t_0$. Therefore, the restriction of f vanishes not only on the image of $\mathscr{S}(\Gamma_{g-1}(\lambda))$ but also on all of its conjugates passing through the image point of t_0. Then f vanishes along the boundary in the neighborhood of the image point of t_0, hence f is contained in $\mathscr{I}$. We have thus obtained the following result:

Supplement. *The ideal $\mathscr{I}$ in $\mathcal{O}$ associated with the boundary $\mathscr{S}(\Gamma_g(\lambda)) - \Gamma_g(\lambda) \backslash \mathfrak{S}_g$ consists of those convergent series in $H_\sigma(t, z, w)$ for which σ is positive and non-degenerate.*

Theorem 1 and this supplement provide all that is necessary to investigate the blowing up of $\mathcal{O}$ with respect to $\mathscr{I}$. We note that, in the case when σ is positive, non-degenerate and if we take $\lambda \geqq 3$, the only element u of $GL(g_1, \mathbf{Z})(\lambda)$ for which ${}^t u \sigma u = \sigma$ holds is 1_{g_1}. In fact, the set of such elements forms a finite group and we know, on the other hand, that 1_{g_1} is the only element of $GL(g_1, \mathbf{Z})(\lambda)$ with a finite order for $\lambda \geqq 3$. Consequently, if σ is positive and non-degenerate, the series for H_σ is extended over all elements u in $GL(g_1, \mathbf{Z})(\lambda)$ provided $\lambda \geqq 3$. Also, in the case when $\lambda \geqq 3$, the functional equation (θ') for $\theta_\sigma(t, z)$ disappears because the stabilizer of t_0 in $\Gamma_{g_0}(\lambda)$ consists of 1_{g_0} only.

3. The monoidal transform $\mathscr{M}(\Gamma_g(\lambda))$

We shall consider the monoidal transformation $\mathscr{M}(\Gamma_g(\lambda)) \to \mathscr{S}(\Gamma_g(\lambda))$ for $\lambda \geqq 3$ along its boundary. We shall recall the definition of a more general process of "blowing up". Suppose that A is a noetherian integral domain and let I denote an ideal in A. Then we can introduce the following graded ring

$$B = \bigoplus_{n \geqq 0} I^n \qquad (I^0 = A).$$

We see that B is also an integral domain and it is of finite type over A; hence B is noetherian. Algebraically speaking, the graded ring B is the blowing up of A with respect to I. A geometric interpretation which we shall use is the following one. We consider the set of all maximal ideals in A and denote it by spec(A). This can be converted into a ringed space. Similarly, we consider the set of all graded maximal ideals in B not containing

$$B_+ = \bigoplus_{n > 0} I^n,$$

and denote it by proj(B). Again, this can be converted into a ringed space. In the case when $f_1, f_2, \ldots, f_p$ form an A-module base of I, proj(B) can be covered by the open subsets spec$(A[f_1/f_i, f_2/f_i, \ldots, f_p/f_i])$ for $f_i \neq 0$. If y is a point of proj(B), then $x = A \cap y$ is a point of spec(A). The correspondence $y \to x$ gives a morphism proj$(B) \to$ spec(A), which is called the *blowing up* of spec(A) with respect to the ideal I. The blowing up is surjective and it is an isomorphism outside the zeros of I. If I is an intersection of prime ideals, it can be considered as the ideal of a closed subset of spec(A). In this case, the blowing up proj(B)$\to$spec(A) is called the *monoidal transformation* along this closed subset and proj(B) is called a *monoidal transform* of spec(A). In general, if we wish to investigate the blowing up locally, we have only to take the corresponding local ring O and consider the blowing up of spec(O) with respect to OI. In the case when A is an integral domain of finite type over $\mathbf{C}$, we can also use analytic local rings for this purpose. At any rate, since the processes of blowing up and localization are compatible, we can define the blowing up globally using a sheaf of ideals. We refer to GROTHENDIECK [6] and also to [7, 23] for basic results concerning this process.

Now, as we have said in the beginning, we shall denote by $\mathscr{M}(\Gamma_g(\lambda))$ the monoidal transform of $\mathscr{S}(\Gamma_g(\lambda))$ for $\lambda \geqq 3$ along its singular locus $\mathscr{S}(\Gamma_g(\lambda)) - \Gamma_g(\lambda)\backslash\mathfrak{S}_g$. If we denote by $I(\Gamma_g(\lambda))_k$ the vector subspace over $\mathbf{C}$ of $A(\Gamma_g(\lambda))_k$ consisting of *cusp forms* and if we consider the graded ideal

$$I(\Gamma_g(\lambda)) = \bigoplus_{k \geqq 0} I(\Gamma_g(\lambda))_k$$

in $A(\Gamma_g(\lambda))$, this ideal, made inhomogeneous at the image point of t_0, generates an ideal I in the algebraic local ring O of $\mathscr{S}(\Gamma_g(\lambda))$ at the image point of t_0. The ideal I is an intersection of prime ideals (corresponding to the conjugates of the image of $\mathscr{S}(\Gamma_{g-1}(\lambda))$ passing through the image point of t_0), and so is $\mathcal{O}I$. This implies $\mathcal{O}I = \mathscr{I}$. Therefore, we can investigate the monoidal transformation using the blowing up of spec($\mathcal{O}$) with respect to $\mathscr{I}$. We observe that $Sp(g, \mathbf{Z}/\lambda\mathbf{Z})$ operates on $A(\Gamma_g(\lambda))$ as a group of weight-preserving automorphisms keeping $I(\Gamma_g(\lambda))$ stable. Therefore $Sp(g, \mathbf{Z}/\lambda\mathbf{Z})$ operates not only on $\mathscr{S}(\Gamma_g(\lambda))$ but also on $\mathscr{M}(\Gamma_g(\lambda))$ as a group of automorphisms. We shall examine the singular locus of $\mathscr{M}(\Gamma_g(\lambda))$.

Lemma 9. *Let σ denote either σ_0 or σ_{ij} for $1 \leqq i < j \leqq g_1 + 1$. Then, for (t, z) in a compact subset of $V \times \mathfrak{Z}$ and for* Im$(w) \to \infty$ *with the normal coordinates of* Im(w) *bounded above, the series*

$$H_\sigma(t, z, w)\, e(-(1/\lambda)\,\mathrm{tr}(\sigma w)) = \sum_u \theta_\sigma(t, z\,{}^t u)\, e((1/\lambda)\,\mathrm{tr}(({}^t u \sigma u - \sigma) w))$$

will eventually be dominated by a series of the form

$$\text{const.} \sum_u \exp(-\mu\, \mathrm{tr}({}^t u \sigma u)),$$

in which u runs over $GL(g_1, \mathbf{Z})(\lambda)$ for $\lambda \geqq 3$ and $\mu > 0$. Moreover $H_\sigma(t, z, w) \times e(-(1/\lambda)\,\mathrm{tr}(\sigma w))$ converges to $\theta_\sigma(t, z)$ for Im$(w) \to \infty$ *with the normal coordinates of* Im(w) *bounded above, and the convergence is uniform when (t, z) is restricted to the compact set.*

Proof. We put Im$(w) = y$. We shall first show that there exists a positive, non-degenerate matrix r satisfying $y_{ij} \leqq r_{ij}$ for $1 \leqq i < j \leqq g_1 + 1$, provided y is taken sufficiently large. Since $y \to \infty$ with $y_{ij} \leqq \beta$, say, we can associate a positive, non-degenerate matrix y^0 with $(y^0)_{ij} = 0, \ -1$ for $1 \leqq i < j \leqq g_1 + 1$

to this sequence as in Lemma 7. We shall denote by r_0 the symmetric matrix defined by $(r_0)_{ij}=\beta$ for $1\leqq i<j\leqq g_1+1$. Put $r=\alpha y^0+r_0$ with α in $\mathbf{R}$. Then r becomes positive and non-degenerate for α sufficiently large. Moreover, according as $y_{ij}\to-\infty$ or remains bounded, we have $r_{ij}=-\alpha+\beta$ or β. Therefore r satisfies our requirements. We shall next show that, for (t,z) in the given compact subset of $V\times\mathfrak{Z}$, we have

$$|\theta_\sigma(t,z{}^tu)|\leqq\text{const.}\exp(\gamma\,\mathrm{tr}({}^tu\sigma u))$$

for some γ in $\mathbf{R}$. In fact, we take a positive, non-degenerate matrix r_1 sufficiently large so that $r_1-\mathrm{Re}(L_t(z,z))-r$ remains positive for all (t,z) in the compact set. Then, we have only to take $w=(-1)^{\frac{1}{2}}r_1$ in Lemma 8 and put $\gamma=(2\pi/\lambda)$-times the largest eigenvalue of r_1-r. Finally, we observe that $y-\delta y^0-r$ will eventually be contained in the central cone C for any δ in $\mathbf{R}$, and it tends to ∞ with y. In fact, according as $y_{ij}\to-\infty$ or remains bounded, we have $(y-\delta y^0-r)_{ij}=y_{ij}-r_{ij}+\delta$ or $y_{ij}-r_{ij}$. Therefore, using Lemma 6 partially, we get $\mathrm{tr}(({}^tu\sigma u-\sigma)(y-\delta y^0-r))\geqq 0$ for every u in $GL(g_1,\mathbf{Z})(\lambda)$, provided y is taken sufficiently large. Consequently, for (t,z) in the compact set, we eventually have the following estimation

$$\begin{aligned}|\theta_\sigma(t,z{}^tu)\,e((1/\lambda)\,\mathrm{tr}(({}^tu\sigma u-\sigma)w))|&\leqq\\ &\leqq\text{const.}\exp(\gamma\,\mathrm{tr}({}^tu\sigma u))\times\\ &\quad\times\exp(-(2\pi/\lambda)\,\mathrm{tr}(({}^tu\sigma u-\sigma)(\delta y^0+r))).\end{aligned}$$

Since δ is arbitrary in $\mathbf{R}$, if we take it sufficiently large, the matrix $(2\pi/\lambda)\times\times(\delta y^0+r)-\gamma 1_{g_1}$ becomes positive and non-degenerate. Therefore, if we denote its smallest eigen-value by μ, we have

$$\begin{aligned}&|\theta_\sigma(t,z{}^tu)\,e((1/\lambda)\,\mathrm{tr}(({}^tu\sigma u-\sigma)w))|\leqq\\ &\qquad\leqq\text{const.}\exp((2\pi/\lambda)\,\mathrm{tr}(\sigma(\delta y^0+r)))\exp(-\mu\,\mathrm{tr}({}^tu\sigma u)).\end{aligned}$$

Clearly, the first exponential factor can be included in the constant factor. This proves the first part. As for the second part, because of what we have shown, it is enough to prove that, each $\theta_\sigma(t,z{}^tu)\,e((1/\lambda)\,\mathrm{tr}(({}^tu\sigma u-\sigma)w))$ for $u\neq 1_{g_1}$ tends uniformly to zero for $\mathrm{Im}(w)=y\to\infty$. If we use the estimation we have obtained above, we get

$$\begin{aligned}|\theta_\sigma(t,z{}^tu)\,e((1/\lambda)\,\mathrm{tr}(({}^tu\sigma u-\sigma)w))|&\leqq\\ &\leqq\text{const.}\exp(-\mu\,\mathrm{tr}({}^tu\sigma u))\times\\ &\quad\times\exp(-(2\pi/\lambda)\,\mathrm{tr}(({}^tu\sigma u-\sigma)(y-\delta y^0-r))).\end{aligned}$$

Therefore, applying Lemma 6 now in full, we see that this tends to zero for $y\to\infty$. This completes the proof.

In the same way, but this time without using Lemma 6, we can show that, if σ is an arbitrary half-integer, positive, non-degenerate matrix, the series for $H_\sigma(t,z,w)\,e(-(1/\lambda)\,\mathrm{tr}(\sigma_0 w))$ has the same kind of dominant series.

Now, take an arbitrary point ω of $\mathcal{M}(\Gamma_g(\lambda))$. We shall examine the analytic local ring of $\mathcal{M}(\Gamma_g(\lambda))$ at ω. Since the monoidal transformation $\mathcal{M}(\Gamma_g(\lambda))\to\to\mathcal{S}(\Gamma_g(\lambda))$ is an isomorphism over $\Gamma_g(\lambda)\backslash\mathfrak{S}_g$ and since this is non-singular, we have only to consider the case when the projection of ω is on the boundary. We apply an automorphism to $\mathcal{M}(\Gamma_g(\lambda))$ so that the projection of ω is the image point of a point t_0 of $\mathfrak{S}_{g_0}$ for some $g_0 < g$. We then take a sequence of points in $\Gamma_g(\lambda)\backslash\mathfrak{S}_g$ which converges to ω. We are regarding $\Gamma_g(\lambda)\backslash\mathfrak{S}_g$ not only as an open subset in $\mathscr{S}(\Gamma_g(\lambda))$ but also as an open subset in $\mathcal{M}(\Gamma_g(\lambda))$. Then we take representatives in $\mathfrak{S}_g$ of these points and thus obtain a sequence of points with (t, z, w), say, as a typical term. By taking a subsequence if necessary, we can assume that (t, z) converges to (t_0, z_0), say, and that $\mathrm{Im}(w)\to\infty$. We can also assume that $\mathrm{Re}(w)$ converges to some point of $\mathbf{R}^N$ for $N=(\frac{1}{2})g_1(g_1+1)$ and that some of the normal coordinates of $\mathrm{Im}(w)$ are convergent whenever they are bounded. Again, by applying an automorphism to $\mathcal{M}(\Gamma_g(\lambda))$ coming from a transformation of $\mathfrak{S}_g$ of the form $(t, z, w)\to(t, z^t v, v w^t v)$ with v in $GL(g_1, \mathbf{Z})$ and taking a subsequence if necessary, we can assume that $\mathrm{Im}(w)\to\infty$ in the fundamental cone F. Although we have replaced ω by one of its conjugates, we have not made any restriction on the nature of the point ω. Now we make a rather strong assumption that we have $\mathrm{Im}(w)\to\infty$ with the normal coordinates of $\mathrm{Im}(w)$ bounded above. According to Lemma 4, this is not a restriction in the case when $g_1 \leqq 3$. In fact, we could assume that we have $\mathrm{Im}(w)\to\infty$ in the central cone C. At any rate, *in the case when* $\mathrm{Im}(w)\to\infty$ *with the normal coordinates of* $\mathrm{Im}(w)$ *bounded above, the analytic local ring of* $\mathcal{M}(\Gamma_g(\lambda))$ *at* ω *is regular.* In fact, we can give a set of local parameters explicitly. For this purpose, we shall denote the coefficients of the matrix w by w_{ij} for $1\leqq i, j\leqq g_1$ and introduce w_{ij} for $1\leqq i, j\leqq g_1+1$ as before. Furthermore, we put

$$\xi_{ij}=e((1/\lambda)(-w_{ij})) \qquad 1\leqq i<j\leqq g_1+1\,.$$

Then, there exists a point ξ_0 in $\mathbf{C}^N$ for $N=(\frac{1}{2})\,g_1(g_1+1)$ such that $\xi\to\xi_0$ when $\mathrm{Im}(w)\to\infty$ with the normal coordinates of $\mathrm{Im}(w)$ bounded above. We shall show that *the analytic local ring of* $\mathcal{M}(\Gamma_g(\lambda))$ *at* ω *contains local parameters of* $\mathfrak{S}_{g_0}$ *at* t_0, *local parameters of* $\mathfrak{Z}$ *at* z_0 *and local parameters of* $\mathbf{C}^N$ *at* ξ_0, *and they form a set of local parameters in the analytic local ring.*

We shall denote the said local parameters symbolically by $t-t_0$, $z-z_0$ and $\xi-\xi_0$. Then, for any half-integer, positive matrix σ, the corresponding $H_\sigma(t, z, w)$ can be considered as a convergent power-series in $t-t_0$, $z-z_0$ and in ξ provided $\mathrm{Im}(w)$ is very large with the normal coordinates of $\mathrm{Im}(w)$ bounded above. On the other hand, if σ is also non-degenerate, the Riemann form of $\theta_\sigma(t, z)$ is λ-time another Riemann form. Since we are assuming that $\lambda\geqq 3$, by an important theorem in the theory of theta-functions (cf. 21), the vector space of theta-functions $\theta_\sigma(t_0, z)$ gives rise to a projective embedding of the complex torus, which is the quotient variety of $\mathfrak{Z}$ by the lattice of points $t_0 m' + m''$ with $m', m''\equiv 0 \bmod \lambda$. Therefore, using this theorem only partially, we can find $\dim(\mathfrak{Z})+1$ theta-functions $\theta_{\sigma_0, i}(t, z)$ for $i=0, 1, \ldots, g_0 g_1$ such that $\theta_{\sigma_0}(t_0, z_0)=\theta_{\sigma_0,0}(t_0, z_0)\neq 0$ and such that the Jacobian at z_0, say $J(t_0, z_0)$, of the complex-analytic mapping

$$z\to((\theta_{\sigma_0,i}/\theta_{\sigma_0})(t_0, z))_{i=1,\ldots,g_0 g_1}$$

is different from zero. We also choose one $\theta_{\sigma_{ij}}(t,z)$ for each σ_{ij} such that $\theta_{\sigma_{ij}}(t_0,z_0)\neq 0$. We can assume that $\theta_{\sigma_0,i}(t,z)$ and $\theta_{\sigma_{ij}}(t,z)$ are all holomorphic in $V\times\mathfrak{Z}$ for some V. Then, if we denote by ζ the product of all ξ_{ij}, we get

$$H_{\sigma_0}(t,z,w)=\theta_{\sigma_0}(t,z)\,\zeta(1+\cdots)$$
$$H_{\sigma_{ij}}(t,z,w)=\theta_{\sigma_{ij}}(t,z)\,\zeta\,\xi_{ij}(1+\cdots),$$

in which the unwritten parts are convergent power-series in $t-t_0$, $z-z_0$ and in $(\xi_{ij})^\lambda$ such that they vanish term-by-term at $\xi=\xi_0$. This follows from Lemma 9. Similarly, if we denote by $H_{\sigma_0,i}(t,z,w)$ the series for $\theta_{\sigma_0,i}(t,z)$, we get

$$H_{\sigma_0,i}(t,z,w)=\zeta(\theta_{\sigma_0,i}(t,z)+\cdots),$$

in which the unwritten part is a convergent power-series in $t-t_0$, $z-z_0$ and in $(\xi_{ij})^\lambda$ such that it vanishes term-by-term at $\xi=\xi_0$. Therefore, its derivative with respect to any ξ_{ij} vanishes also term-by-term at $\xi=\xi_0$. On the other hand, the remark following Lemma 9 shows that, if σ is an arbitrary half-integer, positive, non-degenerate matrix, the quotient $(H_\sigma/H_{\sigma_0})(t,z,w)$ is a convergent power-series in $t-t_0$, $z-z_0$, ξ. Therefore, the analytic local ring of $\mathcal{M}(\Gamma_g(\lambda))$ at ω is contained in the ring of convergent power-series in $t-t_0$, $z-z_0$, $\xi-\xi_0$ with coefficients in $\mathbf{C}$. On the other hand, the analytic local ring $\mathcal{O}$ always contains the ring of convergent power-series in $t-t_0$. In fact, any holomorphic function in V can be considered as $H_0(t,z,w)$. Also, the Jacobian at (t_0,z_0,ξ_0) of the system of functions

$$(H_{\sigma_0,i}/H_{\sigma_0})(t,z,w) \qquad i=1,\ldots,g_0g_1$$
$$(H_{\sigma_{ij}}/H_{\sigma_0})(t,z,w) \qquad 1\leqq i<j\leqq g_1+1$$

with respect to the coefficients of z and ξ is equal to

$$J(t_0,z_0)\cdot\prod_{1\leqq i<j\leqq g_1+1}(\theta_{\sigma_{ij}}/\theta_{\sigma_0})(t_0,z_0).$$

Since this is different from zero by assumption, we can conclude that $z-z_0$ and $\xi-\xi_0$ are contained in the analytic local ring of $\mathcal{M}(\Gamma_g(\lambda))$ at ω. This proves the assertion.

What we have shown implies that the projection to $\mathcal{S}(\Gamma_g(\lambda))$ of the singular locus of $\mathcal{M}(\Gamma_g(\lambda))$ is contained in the union of all conjugates of the image of $\mathcal{S}(\Gamma_{g_0}(\lambda))$ for $g_0=g-4$. More precise information are contained in the following statement:

Theorem 2. *The projection of the singular locus of $\mathcal{M}(\Gamma_g(\lambda))$ to $\mathcal{S}(\Gamma_g(\lambda))$ is precisely the union of all conjugates of the image of $\mathcal{S}(\Gamma_{g_0}(\lambda))$ for $g_0=g-4$. The situation does not improve even if we apply a monoidal transformation to $\mathcal{M}(\Gamma_g(\lambda))$ along its singular locus. In particular $\mathcal{M}(\Gamma_g(\lambda))$ is non-singular for $g\leqq 3$.*

Let (t_0,z_0) denote an arbitrary point of $\mathfrak{S}_{g_0}\times\mathfrak{Z}$. We know by Lemma 5 that the fundamental cone F of degree $g_1=4$ can be decomposed into three types of simple chambers with C at the center. In order to obtain a singular point of $\mathcal{M}(\Gamma_g(\lambda))$ lying over the image point of t_0, we have to take a sequence such that $\mathrm{Im}(w)$ stays outside the central cone C. Consider the following neighboring chamber:

$$C_{12}=\sum_{(ij)\neq(12)}\mathbf{R}_+e_{ij}+\mathbf{R}_+\varepsilon_{12},$$

in which $\varepsilon_{12} = e_{12,345}$. We take a sequence in C_{12} which tends to ∞ without approaching to any one of its walls. We can simply take μw_0 with

$$w_0 = (-1)^{\frac{1}{2}} \left(\varepsilon_{12} + \sum_{(ij) \neq (12)} e_{ij}\right)$$

for $\mu \to +\infty$. We shall denote by ω the point of $\mathcal{M}(\Gamma_g(\lambda))$ which is determined by the sequence $(t_0, z_0, \mu w_0)$. Also, we shall denote by $\mathcal{O}_1$ the analytic local ring of $\mathcal{M}(\Gamma_g(\lambda))$ at the point ω. In order to examine $\mathcal{O}_1$, we shall introduce new parameters η_{ij} for $1 \leqq i < j \leqq 5$ such that we have an identity of the form

$$e((1/\lambda)\operatorname{tr}(\sigma w)) = \eta_{12}^{\operatorname{tr}(\varepsilon_{12}\sigma)} \left(\prod_{(ij) \neq (12)} \eta_{ij}^{\operatorname{tr}(e_{ij}\sigma)}\right)$$

for every half-integer matrix σ. This is the way suggested by SIEGEL [20] to introduce local parameters. We note that the parameters ζ_{ij} could have been introduced in the same way by

$$e((1/\lambda)\operatorname{tr}(\sigma w)) = \prod_{1 \leqq i < j \leqq g_1 + 1} \zeta_{ij}^{\operatorname{tr}(e_{ij}\sigma)}.$$

In both cases, the matrix σ is determined uniquely by the corresponding monomial. (In fact, we have only to solve systems of linear equations with unimodular coefficient matrices.) The new parameters are related to the old parameters as follows

$$\eta_{ij} = \begin{cases} 1/\zeta_{12} & (ij) = (12) \\ \zeta_{12}\zeta_{ij} & i = 1, 2;\ j = 3, 4, 5 \\ \zeta_{ij} & (ij) = (34),\ (35),\ (45). \end{cases}$$

Moreover, if $\operatorname{Im}(w)$ is in C_{12}, we have $|\eta_{ij}| \leqq 1$ for $1 \leqq i < j \leqq 5$. We shall denote all of the η_{ij} symbolically by η. In the case when $w = \mu w_0$, we have $\eta \to 0$ for $\mu \to +\infty$.

Lemma 10. *Let σ denote an arbitrary half-integer, positive matrix of degree four. Then we have $\operatorname{tr}(\sigma\varepsilon_{ij}) \geqq 2$ except for the case when $\sigma = 0$. Moreover, if σ is non-degenerate, we have*

$$\operatorname{tr}(\sigma\varepsilon_{ij}) \geqq \operatorname{tr}(\sigma_{ij}\varepsilon_{ij}) = 4;$$

the equality sign holds if and only if we have $\sigma = \sigma_{ij}$.

The lemma implies that C_{12} is a part of the central cone associated with σ_{12}. At any rate, the proof we know requires some case-by-case examination similar to the proof of Lemma 4. We shall leave the proof as an exercise to the reader.

As a consequence, we see that, if σ is a half-integer, positive matrix satisfying $\operatorname{tr}(\sigma\varepsilon_{12}) = 2$, we have

$$H_\sigma(t, z, w) = (\eta_{12})^2 \eta_{i_1 j_1} \eta_{i_2 j_2} \eta_{i_3 j_3}\text{-times a convergent power-series in } t - t_0, z - z_0, \eta$$

for some indices $i_1, j_1, i_2, j_2, i_3, j_3$. On the other hand, if σ is non-degenerate and if we denote the product of the ten η_{ij} by ζ, we have

$$H_\sigma(t, z, w) = (\eta_{12})^3 \zeta\text{-times a convergent power-series in } t - t_0, z - z_0, \eta\,.$$

Moreover, there exists one and only one σ for which $H_\sigma(t, z, w)$ is not divisible by $(\eta_{12})^5$, and it is σ_{12}. Furthermore, we have

$$H_{\sigma_{12}}(t, z, w) = \theta_{\sigma_{12}}(t, z)\,(\eta_{12})^3\zeta(1+\cdots),$$

in which the unwritten part is a convergent power-series in $t-t_0$, $z-z_0$ and in $(\eta_{ij})^\lambda$ such that it vanishes term-by-term at $\eta=0$. We are assuming that we have $\theta_{\sigma_{12}}(t_0, z_0) \neq 0$. Therefore, in order to obtain functions belonging to $\mathcal{O}_1$, we have to adjoin quotients of $H_\sigma(t, z, w)$ by $H_{\sigma_{12}}(t, z, w)$ to the analytic local ring $\mathcal{O}$. All these quotients, except for the one corresponding to $\sigma=\sigma_{12}$, are divisible by η_{12}. Furthermore, there exists, in each case, a non-degenerate matrix σ such that the expansion of $H_\sigma(t, z, w)$ takes the following form

$$\theta_\sigma(t, z)\text{-times}\begin{cases}(\eta_{12})^4\,\zeta(1+\cdots) & \\ (\eta_{12})^4\,\eta_{ij}\,\zeta(1+\cdots) & (ij)\neq(12)\\ (\eta_{12})^4\,\eta_{23}\eta_{34}\,\zeta(1+\cdots) & \\ (\eta_{12})^5\,\eta_{ij}\,\zeta(1+\cdots) & i=1,2;\ j=3,4,5\,.\end{cases}$$

We are assuming that we have $\theta_\sigma(t_0, z_0) \neq 0$. It is important to observe that there is no σ which corresponds to $(\eta_{12})^5\zeta$. The σ is assumed to be a half-integer, positive, non-degenerate matrix. We shall show that the analytic local ring $\mathcal{O}_1$ is not regular. We shall denote by Ω the ring of convergent power-series in $t-t_0, z-z_0, \eta$. Then $\mathcal{O}_1$ is contained in Ω. Moreover, we can show, as before, that $\mathcal{O}_1$ contains the ring of convergent power-series in $t-t_0, z-z_0$. Let $\mathfrak{m}_1$ denote the maximal ideal of $\mathcal{O}_1$. Then $\mathfrak{m}_1$ is contained in $\eta_{12}\Omega$, and $\mathfrak{m}_1$ contains $t-t_0, z-z_0$ and

$$\eta_{12}, \eta_{12}\eta_{ij}(ij)\neq(12), \eta_{12}\eta_{23}\eta_{34}\,.$$

If we consider these elements modulo $(\mathfrak{m}_1)^2$, they are linearly independent over $\mathcal{O}_1/\mathfrak{m}_1 = \mathbf{C}$. Consequently, the dimension of the Zariski tangent space $\mathfrak{m}_1/(\mathfrak{m}_1)^2$ over $\mathcal{O}_1/\mathfrak{m}_1$ is larger than the dimension of the variety. Hence $\mathcal{O}_1$ is not regular (22). This proves the first part of the theorem. We, now, apply a monoidal transformation to $\mathcal{M}(\Gamma_g(\lambda))$ along its singular locus. Let $\mathcal{O}_2$ denote the analytic local ring of the monoidal transform of $\mathcal{M}(\Gamma_g(\lambda))$ at the point which corresponds to $t=t_0, z=z_0, \eta=0$. Also, let $\mathfrak{m}_2$ denote the maximal ideal of $\mathcal{O}_2$. Then $\mathcal{O}_2$ is contained in Ω. Moreover $\mathcal{O}_2$ contains the ring obtained from $\mathcal{O}_1$ by adjoining quotients of elements of $\mathfrak{m}_1$ by η_{12}. In particular $\mathfrak{m}_2$ contains

$$\eta_{ij}\ (ij)\neq(12),\ \eta_{12}\eta_{ij} \qquad i=1,2;\ j=3,4,5\,.$$

If we consider these elements modulo $(\mathfrak{m}_2)^2$, they are linearly independent over $\mathcal{O}_2/\mathfrak{m}_2 = \mathbf{C}$. Therefore $\mathcal{O}_2$ is not regular. This proves the second part of the theorem.

It is possible to blow up $\mathcal{S}(\Gamma_4(\lambda))$ suitably along its singular locus to get a non-singular projective variety and we can give the ideal of blowing up explicitly. However, since we do not know the meaning of this non-singular model and since the mere existence of a non-singular model is generally guaranteed by the theorem of HIRONAKA [7], we shall not discuss it here.

4. Fibers of $\mathcal{M}(\Gamma_g(\lambda)) \to \mathcal{S}(\Gamma_g(\lambda))$

As before, we shall assume that $\lambda \geqq 3$ and denote by $\mathcal{M}(\Gamma_g(\lambda)) \to \mathcal{S}(\Gamma_g(\lambda))$ the monoidal transformation along the boundary. The monoidal transformation is an isomorphism over $\Gamma_g(\lambda)\backslash \mathfrak{S}_g$. We shall determine the fiber of $\mathcal{M}(\Gamma_g(\lambda)) \to \mathcal{S}(\Gamma_g(\lambda))$ over the image point of a point t_0 of $\mathfrak{S}_{g_0}$ for $g_0 \geqq g-2$. This can be done by the argument which precedes Theorem 2.

We shall first consider the case when $g_0 = g-1$. In this case, the analytic local ring $\mathcal{O}$ of $\mathcal{S}(\Gamma_g(\lambda))$ at the image point of t_0 consists of convergent series of the form

$$\sum_{k=0}^{\infty} \theta_k(t, z)\xi^k .$$

in which $\xi = e(w/\lambda)$. Moreover $\theta_k(t, z)$ is holomorphic in $V \times \mathfrak{Z}$ for some open neighborhood V of t_0, which does not depend on k, and satisfies the functional equation

$$\theta_k(t, z + tm' + m'') = \theta_k(t, z)\, e(-(k/\lambda)(2{}^t m' z + {}^t m' t m'))$$

for m', $m'' \equiv 0 \bmod \lambda$. Therefore $z \to \theta_k(t, \lambda z)$ is what is known classically as a "theta-function of order $2k\lambda$". On the other hand, the ideal $\mathcal{I}$ consists of convergent series which start from $k=1$. Therefore, the fiber of $\mathcal{M}(\Gamma_g(\lambda))$ over the image point of t_0 is obtained as the image of

$$z \to (\theta_1(t_0, z)),$$

in which we take a suitable base over $\mathbf{C}$ of the vector space of theta-functions corresponding to $k=1$. For instance, if we denote by $\theta_n(t, z)$ the classical theta-function of characteristic n (cf. 9), the following $(2\lambda)^{g_0}$ theta-functions

$$z \to \theta_{\binom{n'}{0}}(2\lambda t_0, 2z) \qquad 2\lambda n' \equiv 0 \bmod 1$$

form such base. At any rate, the fiber is an abelian variety which is complex-analytically isomorphic to the complex torus

$$T_{g_0}(t_0) = \mathbf{C}^{g_0}/(t_0\, 1_{g_0})(\lambda \mathbf{Z})^{2g_0}$$

with $g_0 = g-1$.

We shall next consider the case when $g_0 = g-2$. We choose a base $(\theta_{\sigma_0}(t_0, z))$ over $\mathbf{C}$ of theta-functions belonging to σ_0. Also, for $(ij) = (12), (13), (23)$, we choose a base $(\theta_{\sigma_{ij}}(t_0, z))$ over $\mathbf{C}$ of theta-functions belonging to σ_{ij}. Then, using the notation of the previous section, a part of the fiber of $\mathcal{M}(\Gamma_g(\lambda)) \to \mathcal{S}(\Gamma_g(\lambda))$ over the image point of t_0 has a projective embedding of the form

$$(z, \xi_0) \to (\theta_{\sigma_0}(t_0, z), \theta_{\sigma_{ij}}(t_0, z)\,(\xi_0)_{ij}).$$

The point ξ_0 is the limit of the point ξ when $\mathrm{Im}(w) \to \infty$ with the normal coordinates of $\mathrm{Im}(w)$ bounded above. Therefore ξ_0 is a point of $\mathbf{C}^3$ such that the product of any two of its coordinates is zero. Consequently, the point ξ_0 is on the union, say Δ, of the three coordinate axes in $\mathbf{C}^3$, and every point of Δ can be obtained in this way. Therefore, the part of the fiber we are talking about is an extension by Δ of an abelian variety, which is complex-analytically isomorphic to $T_{g_0}(t_0)^2$ with $g_0 = g-2$. Moreover, the entire fiber of $\mathcal{M}(\Gamma_g(\lambda)) \to \mathcal{S}(\Gamma_g(\lambda))$ over the image point of t_0 consists of a certain number of its copies pieced together in a suitable manner. These copies are obtained by applying the following transformation $(t_0, z, w) \to (t_0, z\,{}^t v, v w\,{}^t v)$ with v in $GL(2, \mathbf{Z})$. We know that we get the original part if v is contained in $GL(2, \mathbf{Z})(\lambda)$. We shall examine how the original part and its neighboring copies are pieced together. We will have to use certain information about the cone C and its neighboring cones. In order to obtain such information, we shall employ the well-known correspondence between the Cayley model and the Poincaré model.

If y is an interior point of $\mathfrak{Y}_+$, its inverse y^{-1} is also an interior point of $\mathfrak{Y}_+$. We shall denote by ζ the root with positive imaginary part of the associated binary quadratic form. Since we have

$$y^{-1} = (1/\det(y)) \begin{pmatrix} y_2 & -y_{12} \\ -y_{12} & y_1 \end{pmatrix},$$

the root of $y_2\zeta^2 - 2y_{12}\zeta + y_1 = 0$ is given by

$$\zeta = (y_{12} + (-1)^{\frac{1}{2}} \det(y)^{\frac{1}{2}})/y_2 .$$

We say that ζ is obtained from y. It is clear that the correspondence $y \to \zeta$ gives rise to a homeomorphism of the space of rays $\mathbf{R}_+ y$ to the ordinary upper-half plane $\mathfrak{S}_1$.

Lemma 11. *The image of the interior of the central cone C under $y \to \zeta$ is the interior of the non-euclidean triangle defined by $-1 \leq \mathrm{Re}(\zeta) \leq 0$, $|\zeta + \frac{1}{2}| \geq \frac{1}{2}$. Moreover, the set of limit points of ζ for $y \to \infty$ with a finite limit for y_{12} is the union $(-1)^{\frac{1}{2}} \mathbf{R}_+ \cup \infty$.*

Proof. We have $\mathrm{Re}(\zeta) = y_{12}/y_2$, hence $-1 < \mathrm{Re}(\zeta) < 0$. Since $|\zeta + \frac{1}{2}|^2 - (\frac{1}{2})^2 = (y_1 + y_{12})/y_2$, we have $|\zeta + \frac{1}{2}| > \frac{1}{2}$. Conversely, if ζ satisfies these inequalities, the above two equations in y_{12}/y_2 and y_1/y_2 can be solved, and we get an interior point of C. This proves the first part. The second part is also straightforward.

Suppose that v is an element of $GL(2, \mathbf{R})$. Then, for ζ in $\mathfrak{S}_1$, we put $v \cdot \zeta = (v_{11}\zeta + v_{12})(v_{21}\zeta + v_{22})^{-1}$. If v is contained in $SL(2, \mathbf{R}) = \mathrm{Sp}(1, \mathbf{R})$, this definition becomes a special case of the action of $\mathrm{Sp}(g, \mathbf{R})$ on $\mathfrak{S}_g$. At any rate, if ζ is obtained from y, the point obtained from $v y\,{}^t v$ is $v \cdot \zeta$ or its complex conjugate according as $\det(v)$ is positive or negative. Also, if v is contained in $GL(2, \mathbf{Z})$, we shall denote by $v \cdot \xi$ the image point of $v w\,{}^t v$ under $w \to \xi$.

Before we state and prove the next lemma, we shall introduce two finite subgroups of $GL(2, \mathbf{Z})/\pm 1_2$. The first one is $\pm \pi_{g_1+1}$ for $g_1 = 2$, and it consists of

$$\pm 1_2, \pm \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \pm \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}.$$

If $\pm v$ is one of them, as $v \cdot \xi$ we respectively get

$$(\xi_{12}, \xi_{13}, \xi_{23}), \quad (\xi_{13}, \xi_{23}, \xi_{12}), \quad (\xi_{23}, \xi_{12}, \xi_{13})$$
$$(\xi_{12}, \xi_{23}, \xi_{13}), \quad (\xi_{13}, \xi_{12}, \xi_{23}), \quad (\xi_{23}, \xi_{13}, \xi_{12}).$$

The second one is the following four group

$$\pm 1_2, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pm \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If $\pm v$ is one of them, as $v \cdot \xi$ we respectively get

$$\begin{array}{ll} (\xi_{12}, \xi_{13}, \xi_{23}), & (1/\xi_{12}, (\xi_{12})^2 \xi_{23}, (\xi_{12})^2 \xi_{13}) \\ (\xi_{12}, \xi_{23}, \xi_{13}), & (1/\xi_{12}, (\xi_{12})^2 \xi_{13}, (\xi_{12})^2 \xi_{23}). \end{array}$$

At any rate, we shall refer to these groups as D_6 and D_4 with their elements ordered as above.

Lemma 12. *Suppose that a sequence with a typical term ξ is given in $(\mathbf{C}^*)^3$, i.e., in $\mathbf{C}^3$ minus the three coordinate planes. Then $\xi \to (\eta, 0, 0)$ and $v \cdot \xi \to (\eta', 0, 0)$ with $\eta, \eta' \neq 0, \infty$ are compatible if and only if $\pm v$ is an element of D_4 and if $\eta' = \eta$, $1/\eta$, η, $1/\eta$ respectively. Similarly $\xi \to 0$ and $v \cdot \xi \to (\eta', 0, 0)$ with $\eta' \neq \infty$ are compatible if and only if $\pm v$ is an element of D_6 and if $\eta' = 0$.*

Proof. Since the if-part is clear by our previous observation, we shall prove the only if-part. Suppose that $\xi \to (\eta, 0, 0)$ and $v \cdot \xi \to (\eta', 0, 0)$ with $\eta, \eta' \neq 0, \infty$ are compatible. If we consider the absolute values of the coefficients of ξ and $v \cdot \xi$, we get y, $vy^tv \to \infty$ with finite limites for y_{12} and $(vy^tv)_{12}$. Therefore, passing to the Poincaré model, we get two sequences with typical terms ζ, $v \cdot \zeta$ both converging to points of $(-1)^{\frac{1}{2}} \mathbf{R}_+ \cup \infty$. This follows from Lemma 11. Since we can modify both sequences by the elements of D_4, we can assume from the beginning that v is an element of $SL(2, \mathbf{Z})$. By the same reason, we can assume that they converge to some points of the imaginary axis above or equal to $(-1)^{\frac{1}{2}}$ and compactified by ∞. Then $\pm v$ is the first, or the second element of D_4. This proves the first part. We shall pass to the second part. Again, the sequences $\xi \to 0$ and $v \cdot \xi \to (\eta', 0, 0)$ with $\eta' \neq \infty$ give rise to two sequences with typical terms ζ, $v \cdot \zeta$ both converging to some points of the non-euclidean triangle of Lemma 1. Actually, the first sequence is contained in the triangle although the second sequence may not be. At any rate, since we can modify the first sequence by the elements of D_6, we can assume from the beginning that v is an element of $SL(2, \mathbf{Z})$. By the same reason, we can assume that ζ converges to the part of the triangle defined by $-1 \leqq \mathrm{Re}(\zeta) \leqq 0$, $|\zeta| \geqq 1$, $|\zeta + 1| \geqq 1$. We observe that this is a "fundamental domain" of $SL(2, \mathbf{Z}) = \Gamma_1(1)$ operating on $\mathfrak{S}_1$. Moreover, the following seven transformations are the only ones which will transform this fundamental domain to neighboring fundamental domains

$$\zeta \to \zeta, \zeta - 1, -1 - 1/(1+\zeta), -1 - 1/\zeta, -1/(1+\zeta), -1/\zeta, \zeta + 1.$$

Therefore, the transformation $\zeta \to v \cdot \zeta$ has to be one of them. We note that $\zeta \to \zeta$, $-1 - 1/\zeta$, or $-1/(1+\zeta)$ is precisely the transformation $\zeta \to v \cdot \zeta$, in which $\pm v$ is the first, the second or the third element of D_6. On the other hand, if $\zeta \to v \cdot \zeta$ corresponds to one of the remaining four transformations, we respectively have $(v \cdot \xi)_{23} = 1/\xi_{12} \to \infty$, $(v \cdot \xi)_{23} = 1/\xi_{23} \to \infty$, $(v \cdot \xi)_{12} = 1/\xi_{12} \to \infty$, or $(v \cdot \xi)_{12} = 1/\xi_{23} \to \infty$. Therefore, they are not permissible. This completes the proof.

We shall, now, go back to the problem we had before. We shall denote by (z) the point of the complex torus $T_{g_0}(t_0)^2$ with z as a representative in $\mathfrak{Z}$. Suppose that an element v of $GL(2, \mathbf{Z})$ transforms the original part of the fiber to a neighboring copy. Then there exists a sequence in $(\mathbf{C}^*)^3$ with a typical term ξ such that ξ and $v \cdot \xi$ converge to some points of Δ. Lemma 12 gives a complete list of such $\pm v$. For instance, the second element of D_4 will map the part of the ξ_{12}-axis in the closed unit disc to the part of the ξ_{12}-axis outside the unit disc. In this way, we get the entire complex projective line $P_1(\mathbf{C})$, or the Riemann sphere, along the ξ_{12}-axis. Since D_6 operates transitively on the three coordinate axes, we can conclude that the entire fiber is an extension of the abelian variety complex-analytically isomorphic to $T_{g_0}(t_0)^2$ by a reducible rational variety which is a union of a certain number of $P_1(\mathbf{C})$. Since we know that $D_4 GL(2, \mathbf{Z})(\lambda)$ is the stabilizer of $P_1(\mathbf{C})$ in $GL(2, \mathbf{Z})$ and since $GL(2, \mathbf{Z})$ operates transitively on the set of the projective lines, we see that this number is the corresponding index, i.e., it is

$$(1/4)\ [SL(2, \mathbf{Z}) : SL(2, \mathbf{Z})(\lambda)] = (1/4)\, \lambda^3 \prod_{p \mid \lambda} (1 - p^{-2}).$$

Moreover, the way these projective lines meet is exactly like the three coordinate axes in the ξ-space meet at the origin. Since we know that $D_6 GL(2, \mathbf{Z})(\lambda)$ is the stabilizer of the origin in $GL(2, \mathbf{Z})$ and since $GL(2, \mathbf{Z})$ operates transitively on the set of the origins, we see that the number of such points is the corresponding index, i.e., it is

$$(1/6)\ [SL(2, \mathbf{Z}) : SL(2, \mathbf{Z})(\lambda)] = (1/6)\, \lambda^3 \prod_{p \mid \lambda} (1 - p^{-2}).$$

Now, if we fix not only the image point of t_0 but also a point in the first factor of the product $T_{g_0}(t_0)^2$, the fiber will become an extension of $T_{g_0}(t_0)$ by a reducible rational variety which consists of λ projective lines meeting like edges of an λ-gon. In fact, under this restriction, the group $GL(2, \mathbf{Z})$ from which we take v is replaced by its subgroup defined by $v_{12} = 0$. Since the stabilizer in this subgroup of the ξ_{12}-axis consists of the first and the fourth elements of D_4, we have as many copies of the ξ_{12}-axis as there are elements of the form

$$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} k \bmod \lambda .$$

For each k, the corresponding transformation in $T_{g_0}(t_0)$ is $(z_2) \to (z_2 + k z_1)$, in which $z = (z_1 z_2)$. Furthermore, for $k = 1$, the transformation in the ξ-space is given by

$$(\xi_{12}, \xi_{13}, \xi_{23}) \to (1/\xi_{13}, \xi_{12}(\xi_{13})^2, (\xi_{13})^2\, \xi_{23}).$$

This transforms the ξ_{13}-axis to the ξ_{12}-axis with the coordinate transformation $\eta \to 1/\eta$ accompanied. Therefore, if we take λ copies of the extension of $T_{g_0}(t_0)$ by $P_1(\mathbf{C})$ determined by z_1 and identify $(z_2) \times (0)$ in the k-th copy with $(z_2 + z_1) \times (\infty)$ in the $(k+1)$-th copy for $k \bmod \lambda$, we get the fiber in question. This kind of varieties appeared, at least in the case $g_0 = 0$, in the works of KODAIRA [11] and NÉRON [13].

We can also discuss the case when $g_0 = g-3$. We can show, for instance, that the fiber of $\mathcal{M}(\Gamma_g(\lambda)) \to \mathcal{S}(\Gamma_g(\lambda))$ over the image point of t_0 is an extension of an abelian variety complex-analytically isomorphic to the complex torus $T_{g_0}(t_0)^3$ by a reducible rational variety which is a union of

$$(1/24)\,\lambda^8 \prod_{p|\lambda} (1-p^{-2})(1-p^{-3})$$

copies of the monoidal transform of $P_1(\mathbf{C})^3$ along

$$(0,0,\infty) \cup (0,\infty,0) \cup (\infty,0,0) \cup (\infty,\infty,\infty).$$

Therefore, if ξ_{12} denotes the coordinate of the first factor of the product $P_1(\mathbf{C})^3$, the fiber of this monoidal transform over ξ_{12} is $P_1(\mathbf{C})^2$ in general. However, over $\xi_{12}=0$, the fiber becomes the monoidal transform of $P_1(\mathbf{C})^2$ along $(0,\infty) \cup (\infty, 0)$. This fiber contains six exceptional curves of the first kind meeting like edges of a hexagon. We shall probably examine the combinatorial schema of the entire fiber on some other occasion. We shall only state the results for $g_1 = 1, 2$ in the following theorem:

Theorem 3. *The fiber of $\mathcal{M}(\Gamma_g(\lambda)) \to \mathcal{S}(\Gamma_g(\lambda))$ over the image point of (t_0), in $\mathfrak{S}_{g_0}$ is an abelian variety complex-analytically isomorphic to the complex torus $T_{g_0}(t_0)$, or simply the abelian variety $T_{g_0}(t_0)$, for $g_0 = g-1$, and an extension of the abelian variety $T_{g_0}(t_0)^2$ for $g_0 = g-2$ by a reducible rational variety composed of*

$$(1/4)\,\lambda^3 \prod_{p|\lambda} (1-p^{-2})$$

projective lines $P_1(\mathbf{C})$ meeting three at each one of the

$$(1/6)\,\lambda^3 \prod_{p|\lambda} (1-p^{-2})$$

points just like three coordinate axes in $\mathbf{C}^3$. Moreover, the combinatorial schema of the reducible variety is like edges of a tetrahedron for $\lambda = 3$, a cube for $\lambda = 4$, a dodecahedron for $\lambda = 5$ and of a polyhedral decomposition of the Riemann surface associated with the elliptic modular function field of level λ in

$$(1/2)\,\lambda^2 \prod_{p|\lambda} (1-p^{-2})$$

λ-gons for $\lambda \geqq 3$.

We remark also that, if we consider the irreducible part of $\mathcal{M}(\Gamma_g(\lambda)) \to \to \mathcal{S}(\Gamma_g(\lambda))$ lying over the image of $\mathcal{S}(\Gamma_{g-1}(\lambda))$ by Φ^*, i.e., if we consider the so-called proper transform of $\Phi^* \mathcal{S}(\Gamma_{g-1}(\lambda))$ under the monoidal transformation, we get a fiber system of abelian varieties complex-analytically isomorphic to $T_{g-1}(t)$ for t in $\mathfrak{S}_{g-1}$ and their limit varieties. Furthermore, if we put

$$t = \begin{pmatrix} t_0 & z_1 \\ {}^t z_1 & w_1 \end{pmatrix}$$

with t_0 in $\mathfrak{S}_{g-2}$ and use $\xi_{12}, \xi_{13}, \xi_{23}$ as before for $g_0 = g-2$, we have $\xi_{23} = 0$ and we see that the abelian variety degenerates for $\mathrm{Im}(w_1) \to +\infty$ to the λ-gon with the ξ_{12}-axis and the ξ_{13}-axis as two of its edges. This shows that the faces of the polyhedral decomposition stated in Theorem 3 correspond to distinct conjugates of $\Phi^* \mathscr{S}(\Gamma_{g-1}(\lambda))$ passing through the image point of t_0.

5. The blowing up $\mathscr{M}(\Gamma) \to \mathscr{S}(\Gamma)$ for $g = 2, 3$

We shall consider the special case $g \leqq 3$ (excluding the trivial case $g = 1$). In this case, the monoidal transform $\mathscr{M}(\Gamma_g(\lambda))$ of $\mathscr{S}(\Gamma_g(\lambda))$ is non-singular by Theorem 2. Furthermore, if q is a positive integer, we have a morphism $\mathscr{M}(\Gamma_g(q\lambda)) \to \mathscr{M}(\Gamma_g(\lambda))$, which is a covering, such that the monoidal transformation commutes with this morphism and with the known morphism $\mathscr{S}(\Gamma_g(q\lambda)) \to \mathscr{S}(\Gamma_g(\lambda))$. In fact, if we take the product of the morphisms $\mathscr{M}(\Gamma_g(q\lambda)) \to \mathscr{S}(\Gamma_g(q\lambda))$, $\mathscr{S}(\Gamma_g(q\lambda)) \to \mathscr{S}(\Gamma_g(\lambda))$ and the birational transformation $\mathscr{S}(\Gamma_g(\lambda)) \to \mathscr{M}(\Gamma_g(\lambda))$ in this order, we get a rational transformation $\mathscr{M}(\Gamma_g(q\lambda)) \to \mathscr{M}(\Gamma_g(\lambda))$. Moreover, the information obtained in Section 3 about analytic local rings of the monoidal transforms show clearly that this rational transformation is a complex-analytic morphism, and hence it is an algebraic morphism. We shall show that this morphism is a covering. We know that every point of $\mathscr{M}(\Gamma_g(\lambda))$ is conjugate to a point ω such that the analytic local ring $\mathcal{O}$ of $\mathscr{M}(\Gamma_g(\lambda))$ at ω is the ring of convergent power-series in $t - t_0$, $z - z_0$, $\xi - \xi_0$ with coefficients in $\mathbf{C}$. Moreover, the analytic local ring of $\mathscr{M}(\Gamma_g(q\lambda))$ at any point lying over ω is obtained by adjoining $(\xi_{ij})^{1/q}$ to $\mathcal{O}$. Therefore, if we put $N = (\frac{1}{2})\, g_1(g_1 + 1)$, the morphism $\mathscr{M}(\Gamma_g(q\lambda)) \to \mathscr{M}(\Gamma_g(\lambda))$ is a covering, and it is locally abelian of type $(q, q, \ldots, q)$ such that the branch locus locally looks like the union of a certain number of coordinate hyperplanes in $\mathbf{C}^N$. We note that this type of ramifications is the simplest we can expect.

We shall now define a normal projective variety $\mathscr{M}(\Gamma)$ for every subgroup Γ of $\mathrm{Sp}(g, \mathbf{R})$ in the commensurability class of $\mathrm{Sp}(g, \mathbf{Z})$. We take a subgroup Γ' of $\Gamma \cap \Gamma_g(1)$ which is normal of finite index in Γ. This is certainly possible. On the other hand, we know that Γ' contains $\Gamma_g(\lambda)$ for some large λ, say $\lambda \geqq 3$. This is a special case of a recent result by MENNICKE and independently by BASS-LAZARD-SERRE (cf. [12]). Then we take the quotient variety of $\mathscr{M}(\Gamma_g(\lambda))$ by $\Gamma'/\Gamma_g(\lambda)$ and then its quotient variety by Γ/Γ'. If we denote this quotient variety by $\mathscr{M}(\Gamma)$, it is a normal projective variety, and, clearly, it is uniquely determined up to an isomorphism by Γ. Furthermore, we have a morphism $\mathscr{M}(\Gamma) \to \mathscr{S}(\Gamma)$, and this defines a natural transformation of functors $\mathscr{M} \to \mathscr{S}$. In fact, the birational transformation $\mathscr{M}(\Gamma) \to \mathscr{S}(\Gamma)$ which induces the identity on $\Gamma \backslash \mathfrak{S}_g$ is at most finitely many valued. Since $\mathscr{M}(\Gamma)$ is normal, therefore, the birational transformation is a morphism. The naturality of $\mathscr{M}(\Gamma) \to \mathscr{S}(\Gamma)$ is clear. Now the problem is to examine $\mathscr{M}(\Gamma)$. In the following theorem, the group $\Gamma_g(\lambda, 2\lambda)$ denotes the subgroup of $\Gamma_g(\lambda)$ introduced in (9):

Theorem 4. *The projective variety $\mathscr{M}(\Gamma)$ is almost non-singular and the morphism $\mathscr{M}(\Gamma) \to \mathscr{S}(\Gamma)$ is a blowing up of $\mathscr{S}(\Gamma)$ which is an isomorphism over $\Gamma \backslash \mathfrak{S}_g$. Moreover $\mathscr{M}(\Gamma_g(\lambda, 2\lambda))$ does have singular points and $\mathscr{M}(\Gamma_g(\lambda, 2\lambda)) \to$ $\to \mathscr{S}(\Gamma_g(\lambda, 2\lambda))$ is not a monoidal transformation along the boundary for $\lambda \geqq 3$.*

Proof. Since $\mathscr{M}(\Gamma)$ has a global non-singular covering, it is almost non-singular. On the other hand, the birational transformation $\mathscr{S}(\Gamma) \to \mathscr{M}(\Gamma)$ is finitely many valued in $\Gamma\backslash\mathfrak{S}_g$ but not on the boundary. Therefore, the morphism $\mathscr{M}(\Gamma)\to\mathscr{S}(\Gamma)$ is an isomorphism over $\Gamma\backslash\mathfrak{S}_g$, and this is the largest open subset over which the morphism is an isomorphism. Since $\mathscr{S}(\Gamma)$ and $\mathscr{M}(\Gamma)$ are projective varieties over $\mathbf{C}$, the morphism $\mathscr{M}(\Gamma)\to\mathscr{S}(\Gamma)$ is a blowing up of $\mathscr{S}(\Gamma)$ (6, III, p. 106). This proves the first part. We shall prove the second part. In order to use the previously introduced notations without alterations, we take an even integer λ at least equal to 6, and put $\Gamma=\Gamma_g((\frac{1}{2})\lambda,\lambda)$. We also take a point t_0 of $\mathfrak{S}_{g_0}$ for $g_0=g-2$. Then, we know that $t-t_0$, $z-z_0$, $\xi-\xi_0$ form a set of local parameters of $\mathscr{M}(\Gamma_g(\lambda))$ at the point ω determined by $(t,z)\to(t_0,z_0)$ and by $\mathrm{Im}(\omega)\to\infty$ with the normal coordinates of $\mathrm{Im}(w)$ bounded above. This sequence also determines a point ω', say, of $\mathscr{M}(\Gamma)$. We shall assume that $\xi_0=0$. Then, we see that the stabilizer of ω in $\Gamma/\Gamma_g(\lambda)$ contains one and only one element different from the identity, and the corresponding transformation keeps $t-t_0$, $z-z_0$ invariant while changing ξ to $-\xi$. Therefore $t-t_0$, $z-z_0$, $(\xi_{ij})^2$ and ζ/ξ_{ij} for $\zeta=\xi_{12}\xi_{13}\xi_{23}$ form a base of the maximal ideal of the analytic local ring of $\mathscr{M}(\Gamma)$ at ω'. Furthermore none of these elements is redundant. Therefore, the Zariski tangent space of $\mathscr{M}(\Gamma)$ at ω' is of dimension larger than the dimension of the variety, and hence ω' is singular on $\mathscr{M}(\Gamma)$. On the other hand, the analytic local ring $\mathcal{O}'$, say, of $\mathscr{S}(\Gamma)$ at the image point of t_0 can be considered as a subring of the analytic local ring $\mathcal{O}$. The condition for an element

$$\sum_\sigma H_\sigma(t,z,w)$$

of $\mathcal{O}$ to belong to $\mathcal{O}'$ is that σ is not only a half-integer matrix but an integer matrix. After this remark, we consider all sequences in $\mathfrak{S}_g$ of the form $(t,z)\to$ $\to(t_0,z_0)$ and the normal coordinates of $\mathrm{Im}(w)$ tending to $-\infty$ satisfying the condition $y_{12}\geqq y_{13}, y_{23}$. Then we certainly have $\xi\to 0$. Moreover, if we take $\theta_{\sigma_{12}}(t,z)$ such that we have $\theta_{\sigma_{12}}(t_0,z_0)\neq 0$, the quotient $(H_\sigma/H_{\sigma_{12}})(t,z,w)$ is finite along the sequences for every integral, positive, non-degenerate matrix σ. Among these quotients, we find

$$(H_{\sigma_{i3}}/H_{\sigma_{12}})(t,z,w)=(\theta_{\sigma_{i3}}/\theta_{\sigma_{12}})(t,z)(\xi_{i3}/\xi_{12})(1+\cdots)$$

for $i=1,2$. Moreover $(\xi_{13}/\xi_{12},\xi_{23}/\xi_{12})$ can approach to any point of the product of two copies of a closed unit disc in $\mathbf{C}$ provided we take the limit $\mathrm{Im}(w)\to\infty$ suitably. This implies that the fiber at the image point of t_0 of the monoidal transform of $\mathscr{S}(\Gamma)$ along its boundary is of dimension one more than the fiber of $\mathscr{M}(\Gamma)\to\mathscr{S}(\Gamma)$ at the image point of t_0. Therefore $\mathscr{M}(\Gamma)$ is not the monoidal transform of $\mathscr{S}(\Gamma)$ along its boundary. This completes the proof.

We note that the blowing up $\mathscr{M}(\Gamma)\to\mathscr{S}(\Gamma)$ can be analyzed along the quasi projective varieties of the form $\Gamma_0\backslash\mathfrak{S}_{g_0}$ for $g_0=g-1$, contained in the boundary. In fact, it is the partial desingularization of $\mathscr{S}(\Gamma)$ along the image of $\Gamma_0\backslash\mathfrak{S}_{g_0}$, provided Γ operates without fixed points on $\mathfrak{S}_g$ (cf. 10). We also note that most of the groups in the commensurability class of $\mathrm{Sp}(g,\mathbf{Z})$ has "undesirable" properties stated for $\Gamma_g(\lambda,2\lambda)$. We shall not try to give a precise form to this statement.

We shall examine $\mathcal{M}(\Gamma_g(\lambda))$ for $\lambda=1,2$. Although a part of the following investigations can be carried out also for $g=3$, we shall limit ourselves to the case $g=2$. We are interested in $\mathcal{M}(\Gamma_2(\lambda))$ for $\lambda=1,2$, because we know $\mathcal{S}(\Gamma_2(\lambda))$ explicitly for the same values of λ(8). We keep in mind that they are defined as quotient varieties and not as monoidal transforms. We take a point t_0 of $\mathfrak{S}_1$. Since t_0 is not a fixed point of any element of $\Gamma_1(2)$ different from $\pm 1_2$, the analytic local ring $\mathcal{O}$ of $\mathcal{M}(\Gamma_2(2))$ at the image point of t_0 consists of convergent series of the form

$$\sum_{k=0}^{\infty} \theta_k(t,z)\, e(kw/2).$$

The coefficient $\theta_k(t,z)$ is holomorphic in $V\times\mathbf{C}$ for some V and $z\to\theta_k(t,2z)$ is a theta-function of order $4k$, which is even in the sense it is invariant under $z\to -z$. This follows from Theorem 1. Moreover, the ideal $\mathcal{I}$ consists of convergent series which start from $k=1$. Furthermore, there exists a base $(\theta_1(t_0,z))$ over $\mathbf{C}$ of the vector space of even theta-functions of order four such that each $\theta_1(t,z)$ is holomorphic in some $V\times\mathbf{C}$ and such that the correspondence $z\to(\theta_1(t,z))$ gives rise to a complex-analytic embedding of the quotient variety of $T_1(t)$ by its involution $z\to -z$, which is $P_1(\mathbf{C})$, as a plane curve of order two. Therefore, if we take monoidal transform of $\mathcal{S}(\Gamma_2(2))$ along its boundary in the neighborhood of the image point of t_0, we get the quotient variety of the monoidal transform of $\mathcal{S}(\Gamma_2(\lambda))$, for any even λ at least equal to 3, along its boundary in the neighborhood of the image point of t_0. In the same way, we can conclude that the morphism $\mathcal{M}(\Gamma_2(1))\to\mathcal{S}(\Gamma_2(1))$ is a monoidal transformation in the neighborhood of the image point of t_0 provided t_0 is not a fixed point of any element of $\Gamma_1(1)$ different from $\pm 1_2$. We shall next consider the case when t_0 is a fixed point of an element of $\Gamma_1(1)$ different from $\pm 1_2$. We can assume that t_0 is the fixed point of either $t\to -1/t$ or $t\to -1-1/t$.

We shall first consider the case when t_0 is the fixed point of $t\to -1/t$. We start from an analytic local ring of $\mathcal{S}(\Gamma_2(\lambda))$ at the image point of t_0 for some $\lambda\geqq 3$. Then we take its invariant subring $\mathcal{O}'$, say, with respect to the subgroup of the group $P_{\mathbf{Z}}$, introduced in Section 2, consisting of elements M with $M_0=\pm 1_2$. The ring $\mathcal{O}'$ consists of convergent series of the form

$$\sum_{k=0}^{\infty} \theta_k(t,z)\, e(kw),$$

in which $\theta_k(t,z)$ is holomorphic in $V\times\mathbf{C}$ for some V and $z\to\theta_k(t,z)$ is an even theta-function of order $2k$. In this ring, the set of convergent series which start from $k=1$ forms an ideal $\mathcal{I}'$, say. Moreover, if we blow up $\mathcal{O}'$ with respect to $\mathcal{I}'$, we get the quotient variety of $\mathcal{M}(\Gamma_2(\lambda))$ by the subgroup of $P_{\mathbf{Z}}$ defined by $M_0=\pm 1_2$ over some neighborhood of the image point of t_0 in a similar quotient variety of $\mathcal{S}(\Gamma_2(\lambda))$. In order to obtain $\mathcal{M}(\Gamma_2(1))$ over some neighborhood of the image point of t_0, we have to take further quotient under the automorphism coming from the transformation of the form $(t,z,w)\to(-1/t,\ z/t,\ w-z^2/t)$. Now the analytic local ring $\mathcal{O}'$ is regular. In fact, if we denote by $\theta_{ij}(t,z)$ for

$i, j=0, 1$ the wellknown elliptic theta-functions, a set of local parameters in $\mathcal{O}'$ is given by $t-t_0$ and any two linear combinations of $\theta_{ij}(t, z)^2 e(w)$ with coefficients which are holomorphic in some V such that they are linearly independent over $\mathbf{C}$ at $t=t_0$. We shall try to obtain local parameters from globally defined functions.

We know that, if ψ_4, ψ_6 denote modular forms of weights 4, 6 and χ_{10}, χ_{12} the cusp forms of weights 10, 12 respectively belonging to $\Gamma_2(1)$, they are unique up to constant factors. We also know that $A(\Gamma_2(1))^{(2)}$ coincides with $\mathbf{C}[\psi_4, \psi_6, \chi_{10}, \chi_{12}]$. Therefore $\mathcal{S}(\Gamma_2(1))$ is the projective variety associated with this graded ring. We have seen also that the constant factors can be normalized in some way, and the normalized forms can be expanded into Fourier-Jacobi series as follows

$$\begin{aligned}
\psi_4(t, z, w) &= \phi_4(t) + \cdots \\
\psi_6(t, z, w) &= \phi_6(t) + \cdots \\
\chi_{10}(t, z, w) &= (1/2^8)(\theta_{00}\theta_{01}\theta_{10})(t)^6 \theta_{11}(t, z)^2 e(w) + \cdots \\
\chi_{12}(t, z, w) &= (1/2^8 3)(\theta_{00}\theta_{01}\theta_{10})(t)^4 (\theta_{00}(t)^{10}\theta_{00}(t, z)^2 - \\
&\quad - \theta_{01}(t)^{10}\theta_{01}(t,z)^2 - \theta_{10}(t)^{10}\theta_{10}(t,z)^2) e(w) + \cdots,
\end{aligned}$$

in which both ϕ_4 and ϕ_6 have 1 as their constant terms as power-series in $e(t)$. All these, except for the Fourier-Jacobi series expansions, are in (8). We mention also that, in calculating the Fourier-Jacobi series for $\chi_{12}(t, z, w)$, we used the following expression

$$\chi_{12} = (1/2^{17} 3^2) \sum (\theta_{m_1}\theta_{m_2} \cdots \theta_{m_6})^4,$$

in which each θ_m is a theta-constant of degree two. Also, the product is taken over six even characteristics whose sum is zero mod 2 and four of which form an azygous quadruple, and the summation is extended over forty-five sets of such six even characteristics. We note that the coefficients of $e(w)$ in $\chi_{10}(t, z, w)$ and $\chi_{12}(t, z, w)$ are linearly independent as functions of z for every t. This is based on the fact that $\theta_{00}\theta_{01}\theta_{10}$ does not vanish at any point of $\mathfrak{S}_1$.

Now, we recall that $\phi_6(t)^2/\phi_4(t)^3$ has a zero of order two at $t=t_0$. Therefore $\phi_6(t)/\phi_4(t)^{3/2}$ and $t-t_0$ differ by a unit in the ring of convergent power-series in $t-t_0$ with coefficients in $\mathbf{C}$. Consequently, if we put

$$\eta_1 = \psi_6/(\psi_4)^{3/2}, \quad \eta_2 = \chi_{10}/(\psi_4)^{5/2}, \quad \eta_3 = \chi_{12}/(\psi_4)^3,$$

the analytic local ring $\mathcal{O}'$ becomes the ring of convergent power-series in η_1, η_2, η_3 with coefficients in $\mathbf{C}$. Moreover, the blowing up of $\mathcal{O}'$ with respect to $\mathcal{I}'$ is defined by $\mathcal{O}'[\eta_3/\eta_2]$ and $\mathcal{O}'[\eta_2/\eta_3]$. We observe that the transformation $(t, z, w) \to (-1/t, z/t, w - z^2/t)$ simply changes the signs of η_1, η_2, keeping η_3 invariant. Therefore, the analytic local ring $\mathcal{O}$ of $\mathcal{S}(\Gamma_2(1))$ at the image point of t_0 is the ring of convergent power-series in

$$(\eta_1)^2, \ \eta_1\eta_2, \ (\eta_2)^2, \ \eta_3.$$

Moreover, the invariant subrings of $\mathcal{O}'[\eta_3/\eta_2]$ and $\mathcal{O}'[\eta_2/\eta_3]$ are respectively $\mathcal{O}[\eta_1\eta_3/\eta_2, (\eta_3/\eta_2)^2]$ and $\mathcal{O}[\eta_1\eta_2/\eta_3, (\eta_2/\eta_3)^2]$. These rings define the blowing up of $\mathcal{O}$ with respect to the ideal generated by

$$(\eta_2)^2,\ \eta_1\eta_2\eta_3,\ (\eta_3)^2,$$

and this blowing up describes the morphism $\mathcal{M}(\Gamma_2(1))\to\mathcal{S}(\Gamma_2(1))$ over some neighborhood of the image point of t_0.

In a similar manner, we can describe the morphism $\mathcal{M}(\Gamma_2(1))\to\mathcal{S}(\Gamma_2(1))$ over some neighborhood of the image point of the fixed point t_0 of $t\to -1-1/t$. If we put

$$\zeta_1=\psi_4/(\psi_6)^{2/3},\quad \zeta_2=\chi_{10}/(\psi_6)^{5/3},\quad \zeta_3=\chi_{12}/(\psi_6)^2,$$

the analytic local ring $\mathcal{O}$ of $\mathcal{S}(\Gamma_2(1))$ at the image point of t_0 is the ring of convergent power-series in

$$(\zeta_1)^3,\ (\zeta_1)^2\zeta_2,\ \zeta_1(\zeta_2)^2,\ (\zeta_2)^3,\ \zeta_3$$

with coefficients in $\mathbf{C}$. Moreover, over some neighborhood of the image point of t_0, the morphism $\mathcal{M}(\Gamma_2(1))\to\mathcal{S}(\Gamma_2(1))$ is described as the blowing up of $\mathcal{O}$ with respect to the ideal generated by

$$(\zeta_2)^3,\ \zeta_1(\zeta_2)^2\zeta_3,\ (\zeta_1)^2\zeta_2(\zeta_3)^2,\ (\zeta_3)^3.$$

This determines the morphism $\mathcal{M}(\Gamma_2(\lambda))\to\mathcal{S}(\Gamma_2(\lambda))$ for $\lambda=1,2$ except at the image point of $\mathfrak{S}_0$ and at its conjugates.

We shall denote $\mathfrak{S}_0$ symbolically by ∞. Then the morphism $\mathcal{M}(\Gamma_2(2))\to$ $\to\mathcal{S}(\Gamma_2(2))$ over some neighborhood of the image point of ∞ looks almost the same as in the case of higher levels. It is a monoidal transformation locally around the image point of ∞ and the fiber over the image point of ∞ consists of three projective lines $P_1(\mathbf{C})$ meeting at one point just like three coordinate axes in $\mathbf{C}^3$. In fact, if we put

$$\xi_{ij}=e(-w_{ij}/2)\qquad (ij)=(12),(13),(23)$$
$$\zeta=\xi_{12}\xi_{13}\xi_{23},$$

we have

$$H_{\sigma_0}(w)=\zeta(1+\cdots),\quad H_{\sigma_{ij}}(w)=\zeta\xi_{ij}(1+\cdots),$$

provided $\mathrm{Im}(w)$ is very large with the normal coordinates of $\mathrm{Im}(w)$ bounded above. The rest is the same as in the case of higher levels.

We shall finally consider the morphism $\mathcal{M}(\Gamma_2(1))\to\mathcal{S}(\Gamma_2(1))$ over some neighborhood of the image point of ∞. The result is that it is locally monoidal along the boundary. In the following, we shall give our first proof for this result. We put

$$t_1=\psi_6/(\psi_4)^{3/2},\quad t_2=\chi_{10}/(\psi_4)^{5/2},\quad t_3=\chi_{12}/(\psi_4)^3.$$

Then the analytic local ring of $\mathcal{S}(\Gamma_2(1))$ at the image point of ∞ is $\mathbf{C}\langle\!\langle t_1-1, t_2, t_3\rangle\!\rangle$, which means the ring of convergent power-series in t_1-1, t_2, t_3 with coefficients in $\mathbf{C}$. Moreover, the ideal of the boundary is generated by t_2, t_3. Therefore, the monoidal transformation is defined by the two rings $\mathbf{C}\langle\!\langle t_1-1,\ t_2, t_3\rangle\!\rangle\,[t_3/t_2]$ and $\mathbf{C}\langle\!\langle t_1-1,\ t_2, t_3\rangle\!\rangle\,[t_2/t_3]$. We shall determine the integral closures of these rings in the field of fractions of the analytic local ring $\mathcal{O}$ of $\mathcal{S}(\Gamma_2(2))$ at the image point of ∞. In order to state the result, we introduce the following functions

$$x_1 = (\theta_{0100}\theta_{1000}/\theta_{1100}\theta_{0000})^2 \qquad x_2 = (\theta_{1000}\theta_{1100}/\theta_{0100}\theta_{0000})^2$$
$$x_3 = (\theta_{1100}\theta_{0100}/\theta_{1000}\theta_{0000})^2 .$$

Then the result can be stated as follows:

Lemma 13. *The integral closure of* $\mathbf{C}\langle\!\langle t_1 - 1, t_2, t_3\rangle\!\rangle\,[t_3/t_2]$ *in the field of fractions of* $\mathcal{O}$ *is*

$$\mathcal{O}\,[x_1, x_2, x_3, 1/(1-x_1), 1/(1-x_2), 1/(1-x_3)] .$$

In the spectrum of the integral closure of $\mathbf{C}\langle\!\langle t_1 - 1, t_2, t_3\rangle\!\rangle\,[t_2/t_3]$ *in the same field, there exist six points lying over the point given by the residue homomorphism* $(t_1, t_2, t_3, t_2/t_3) \to (1, 0, 0, 0)$ *over* $\mathbf{C}$, *and the corresponding analytic local rings are given by* $\mathbf{C}\langle\!\langle x_i - 1, x_j, x_k\rangle\!\rangle$, $\mathbf{C}\langle\!\langle 1/x_i, x_j, x_k\rangle\!\rangle$, *in which* (i, j, k) *is a permutation of* $(1, 2, 3)$.

Since the proof of this lemma involves long and technical calculations depending on our results in (8), we shall not reproduce it here. We observe that, if $\mathrm{Im}(w)$ is very large with the normal coordinates of $\mathrm{Im}(w)$ bounded above, we have

$$x_1 = 4\xi_{12}(1 + \cdots), \quad x_2 = 4\xi_{13}(1 + \cdots), \quad x_3 = 4\xi_{23}(1 + \cdots),$$

in which the unwritten parts are convergent power-series in ξ_{ij} without constant terms. Therefore, we see that, if we take the monoidal transform of $\mathcal{S}(\Gamma_2(1))$ along the boundary in some neighborhood of the image point of ∞, the part of $\mathcal{M}(\Gamma_2(2))$ over some neighborhood of the image point of ∞ in $\mathcal{S}(\Gamma_2(2))$ is a covering of the monoidal transform. Since the processes of taking Galois coverings and taking quotient varieties by finite groups are the inverses of each other, we can conclude that the morphism $\mathcal{M}(\Gamma_2(1)) \to \mathcal{S}(\Gamma_2(1))$ is the monoidal transformation of $\mathcal{S}(\Gamma_2(1))$ along its boundary in some neighborhood of the image point of ∞. We shall summarize our results in the following way:

Theorem 5. *The blowing up* $\mathcal{M}(\Gamma_2(2)) \to \mathcal{S}(\Gamma_2(2))$ *is the monoidal transformation of* $\mathcal{S}(\Gamma_2(2))$ *along its boundary. Moreover* $\mathcal{M}(\Gamma_2(2))$ *is non-singular and the fiber over the image point of* ∞ *consists of three projective lines* $P_1(\mathbf{C})$ *meeting just like three axes in* $\mathbf{C}^3$. *The blowing up* $\mathcal{M}(\Gamma_2(1)) \to \mathcal{S}(\Gamma_2(1))$ *is a monoidal transformation of* $\mathcal{S}(\Gamma_2(1))$ *along its boundary except at the two singular points of* $\mathcal{S}(\Gamma_2(1))$ *on the boundary. The sheaf of ideals with respect to which this blowing up is defined is given "homogeneously" by*

$$(\chi_{10}, \chi_{12})^6$$

in general

$$((\chi_{10})^2, \psi_6\chi_{10}\chi_{12}, (\chi_{12})^2)^3 \quad \textit{at} \quad \psi_6 = \chi_{10} = \chi_{12} = 0$$
$$((\chi_{10})^3, \psi_4(\chi_{10})^2\chi_{12}, (\psi_4)^2\chi_{10}(\chi_{12})^2, (\chi_{12})^3)^2 \quad \textit{at} \quad \psi_4 = \chi_{10} = \chi_{12} = 0 .$$

We note that $\mathcal{M}(\Gamma_2(1))$ appeared in Satake's first paper on compactifications as a V-manifold (15). It was shown later by Baily that $\mathcal{M}(\Gamma_2(1))$ is a normal projective variety (cf. 16, p. 259; Notice also an incorrect statement on p. 260 that $\mathscr{S}(\Gamma_2(1))$ is not a V-manifold), and a problem was left open as to whether $\mathscr{M}(\Gamma_2(1))$ is a monoidal transform of $\mathscr{S}(\Gamma_2(1))$ along its boundary or not (cf. 1, p. 363). Theorem 5 gives this blowing up explicitly. It also gives the structure of $\mathscr{M}(\Gamma_2(1))$ because the structure of $\mathscr{S}(\Gamma_2(1))$ is known by our earlier work.

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Professor Jun-ichi Igusa
Department of Mathematics, The Johns Hopkins University
Baltimore, Maryland 21218

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