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Ergodic Theory and Virtual Groups

To GOTTFRIED KÖTHE on his 60th Birthday

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1. Introduction

In a recent note [11] the author has given a highly condensed and largely unmotivated account of the basic notions in a theory whose aim is to bring to light and exploit certain apparently far reaching analogies between group theory and ergodic theory. Somewhat more motivation is contained on pages 652—654 of [10], but the brief account given there is embedded in a description of certain rather technical developments in the theory of infinite dimensional group representations. Moreover, some essential conceptual improvements in the theory were made between the times at which [10] and [11] were written. It is the purpose of this paper to give a leisurely account, with a minimum of mathematical technicalities, of the considerations which led to the formulation of the definitions and theorems in the first three sections of [11].

2. Subgroups and transitive actions

Let G be a group and let H be a subgroup. Let $S_H = G/H$ be the set of all right H cosets. For each $s = Hy$ in S_H and each $x \in G$ let $sx = Hyx$. Then the mapping taking s, y into sy is a mapping of $S_H \times G$ into S_H which defines an *action* of G on S_H in the sense that conditions (i) and (ii) below are satisfied.

(i) $(sx_1)x_2 = sx_1x_2$ for all x_1, x_2 in G and all s in S_H,

(ii) $se = s$ for all s in S_H where e is the identity of G.

This action is clearly *transitive* in the sense that for each pair s_1, s_2 of elements of S_H there exists an element x of G such that $s_1 x = s_2$. Equivalently S_H has no invariant subsets except the empty set and its complement. (An invariant subset is a subset S^1 such that $sx \in S^1$ whenev $s \in S^1$ and $x \in G$.)

Conversely let $s, x \to sx$ define a transitive action of G on the set S and let H_s be the subgroup of all x with $sx = s$. Then the mapping $x \to sx$ defines a many-one mapping of G on S which is constant on each right coset. Thus it defines a one-to-one mapping θ of S_{H_s} on S. Evidently θ has the property that $\theta(sx) = \theta(s)x$ so that the actions of G on S and S_{H_s} are equivalent in an obvious sense. Thus every transitive G action is equivalent to one defined by a subgroup. One verifies easily that two subgroups define equivalent actions if and only if they are conjugate and hence that there is a natural one-to-one correspondence between the transitive actions of a group G and the conjugacy classes of its subgroups. These facts are of course elementary and well known. We review them in detail because they play such a central role in motivating what is to follow.

3. Measure theoretic actions and ergodicity

In the group actions that occur in analysis, the group G and the space S come equipped with topologies and the mapping $s, x \to sx$ is continuous. We shall be mainly interested in measure theoretic questions and accordingly find it convenient to replace the topologies of S and G by their underlying Borel structures. By a *Borel space* we shall mean a set together with a distinguished σ field of subsets called its Borel sets. By a Borel mapping of the Borel space S_1 into the Borel space S_2 we shall mean a mapping θ such that $\theta^{-1}(E)$ is a Borel subset of S_1 whenever E is a Borel subset of S_2. Two Borel spaces S_1 and S_2 will be said to be isomorphic if there exists a one-to-one Borel mapping θ of S_1 on S_2 such that θ^{-1} is also a Borel mapping. Let G be a separable locally compact group. By a Borel G space we shall mean a G space S which is also a Borel space and in such a way that $s, x \to sx$ is a Borel function from $S \times G$ to S. When H is a closed subgroup of G then the coset space S_H is a topological space in a natural way and hence is a Borel space. Since $s, x \to sx$ is continuous it is certainly a Borel function and S_H is a transitive Borel G space. Moreover S_H as a Borel space is "standard" in the sense that it is isomorphic as a Borel space to a Borel subset of a separable complete metric space. Conversely let S be any transitive Borel G space where S is standard and G is separable and locally compact. It is known ([8], p. 284) that for each $s \in S$ the subgroup H_s of all x with $sx = s$ is *closed* and that the mapping θ of § 2 is a Borel isomorphism between S_{H_s} and S. Thus when G is a separable locally compact group we have the following analogue of the result reviewed in § 2: The possible transitive actions of G on standard Borel spaces correspond one-to-one to the conjugacy classes of closed subgroups of G.

Let S be a standard Borel G space where G is separable and locally compact and let μ be a *measure* defined in S; that is let μ be a function from the Borel subsets of S to the positive real numbers augmented by $+\infty$ which is additive on countable unions of disjoint sets and which is σ *finite* in the sense that S is a countable union of subsets S_j such that $\mu(S_j) < \infty$. If $\mu(Ex) = \mu(E)$ for all $x \in G$ and all Borel subsets E of S we shall say that μ is *invariant*. If $\mu(Ex) = 0$ whenever $x \in G$ and $\mu(E) = 0$ we shall say that μ is *quasi invariant*. Two measures with the same sets of measure zero will be said to be in the same class, and a class of measures (= measure class) will be said to be invariant if $E \to \mu(Ex)$ is in the class whenever x is in G and μ is in the class. It is clear that the class of a quasi invariant measure is invariant and that every member of an invariant measure class is quasi invariant. It is known ([6], p. 106) that $S_H = G/H$ admits a unique invariant measure class whenever H is a closed subgroup of G but that S_H admits an invariant measure only for rather special choices of H. One is also lead naturally to invariant measure classes which do not need to contain invariant measures whenever S is a C_∞ manifold and the mappings $s \to sx$ preserve the differential structure. Thus, although classical ergodic theory is chief concerned with those G spaces S which have an invariant measure, it seems natural and is convenient to deal here with the more general situation in which our G spaces have only an invariant measure class.

Let G be as above and let C be an invariant measure class in the standard Borel G space S. Suppose that the action is not transitive and that E is an invariant subset. Then $S-E$ is also invariant and E and $S-E$ will be independent G spaces of which S is a direct sum in a natural way. If E is a Borel set they will both the standard and if E (resp. $S-E$) is not of measure zero we obtain a non trivial invariant measure class in it by taking any member μ of C and then taking the class of its restriction to E (resp. $S-E$). Thus if E is a Borel set and neither E nor $S-E$ has measure zero we obtain S, C as the direct sum of two invariant subsystems. However, even when G is the additive group of the integers or of the real line it is possible to choose S and C in such a way that

(1) Every measurable invariant subset of S is either of measure zero or the complement of a set of measure zero

(2) Every invariant subset of S on which G acts transitively is of measure zero.

For example if G is the additive group of the integers we may choose S to be the unit circle $|z|=1$ in the complex plane, C to be the measure class of the length measure and the action of G on S to be such that $zn = ze^{in\alpha 2\pi}$ where α is an irrational number. Thus mere lack of transitivity does not ensure the possibility of a direct sum decomposition — even if we ignore invariant sets of measure zero. One is forced to replace transitivity as the basic notion by a more inclusive and sophisticated one which is sometimes called ergodicity and sometimes metric transitivity. We say that the action of G on S is *ergodic* (or metrically transitive) with respect to C if condition (1) above holds. This condition holds in particular when the action of G on S is transitive. It also holds when the action is *essentially transitive* in the sense that there exists an invariant subset of measure zero on whose complement G acts transitively. When conditions (1) and (2) above both hold, that is when the action is ergodic but not essentially transitive we shall say that the action is *strictly ergodic.*

4. The virtual subgroup point of view

Let G be a separable locally compact group. Since the ergodic actions of G constitute a natural generalization of the transitive actions (§ 3) and since the transitive actions correspond one-to-one in a natural way to the conjugacy classes of closed subgroups of G (§ 2) it is natural to ask the following question.

(a) Is there a generalization of the notion of closed subgroup such that the ergodic actions bear the same relationship to these generalized subgroups that the transitive ones do to actual subgroups? We shall show in the sequel that such a notion does exist and in further publications that a surprisingly large number of the notions and theorems of group theory have analogues which apply to it. We shall not only be able to define the notion of virtual subgroup of a genuine group G but also the notion of "virtual group" in the abstract. Before doing either of these things however we shall show that many of the developments which the notion of virtual group make possible may be carried out without actually defining virtual groups. The idea is this. Since a transitive action determines and is determined by a conjugacy class of closed subgroups we may hope to be able to translate properties of these closed subgroups into properties of the action. When this is possible one may hope in addition that these translated properties continue to have meaning when applied to ergodic actions. To the extent that they do, we may think of them as being the definitions of properties of the "virtual subgroups" defining the ergodic actions.

Here is an example. Let H be a closed subgroup of G. Every transitive action of H is defined by a closed subgroup K of H. But every closed subgroup K of H is also a closed subgroup of G and hence defines a transitive action of G. Hence every transitive action of H is canonically associated with a transitive action of G. If every "virtual subgroup" of H is automatically a virtual subgroup of G as it will be if the notion of virtual subgroup has reasonable properties then the above canonical association should have a generalization which associates an ergodic action of G to every ergodic action of H. It does and here it is. Given an invariant measure class C in the standard Borel space G space S form the auxiliary space $S \times G$. Make $S \times G$ into an $H \times G$ space by setting $(s, x)(h, y) = sh, y^{-1}xh$. The action of H alone defines an equivalence relation in $S \times G$. Let $\widetilde{S \times G}^H$ denote the space of all equivalence classes and let φ be the mapping taking s, x in $S \times G$ into its equivalence class; that is into the set of all sh, xh for $h \in H$. Since the action of G on $S \times G$ commutes with the action of H the action of G maps each H equivalence class into another and we obtain a well defined action of G on $\widetilde{S \times G}^H$ by setting $\varphi(s, x)y = \varphi(s, y^{-1}x)$. We give $\widetilde{S \times G}^H$ a Borel structure by defining a subset E of $\widetilde{S \times G}^H$ to be a Borel set whenever $\varphi^{-1}(E)$ is a Borel subset of $S \times G$. It is not hard to show that $\widetilde{S \times G}^H$ is a standard Borel space. Let C_G denote the measure class of Haar measure in G. Then the measures $\mu \times \nu$ where $\mu \in C$ and $\nu \in C_G$ all lie in a common measure class which we denote by $C \times C_G$. It is easy to see that this measure class is invariant under the $H \times G$ action. Using it we obtain an invariant measures class $\tilde{C}$ in $\widetilde{S \times G}^H$. $\tilde{C}$ is the common class of all measures $\tilde{\omega}$ where ω varies over the finite measures in $C \times C_G$ and $\tilde{\omega}(E) = \omega(\varphi^{-1}(E))$. It is straightforward to prove that $\tilde{C}$ is invariant under G and is ergodic under G if and only it C is ergodic under H. Thus we have found a canonical way of associating an ergodic action of G to every ergodic action of a closed subgroup H. In the special case in which $S = H/K$ for some closed subgroup K of H so that the H action is transitive one verifies at once that the G action on $\widetilde{S \times G}^H$ is also transitive and that this transitive action is defined by the subgroup of K of G.

Consider the special case in which G is the additive group of the real line. The most general non-trivial H is the set of all integer multiplies of a positive real number λ and hence is isomorphic to the additive group of the integers. Thus our general construction yields in particular an assignment of an ergodic action of the additive group of the real line (a so called ergodic flow) to every pair consisting of a positive real number λ and an ergodic action of the integers. Such an assignment is well known in ergodic theory. Given an ergodic action of the integers on a space S with invariant measure class C one forms the product $S \times I_\lambda$ where I_λ is the interval $0 \leqq x < \lambda$. Then one defines $(s, x)t$ for each s, x in $S \times I_\lambda$ and each real number t as sn, t' where $0 \leqq t' < \lambda$, n is an integer and

$x+t=\lambda n+t'$. The resulting flow is sometimes known as the flow (of height λ) built over the transformation $s \to s \cdot 1$. It is easy to see that our general construction, specialized as indicated, yields precisely the flow built over $s \to s \cdot 1$ and having height λ. The virtual subgroup point of view has thus led us to a natural and rather far reaching generalization of this apparently rather special construction. Strictly speaking the virtual subgroup point of view has only led us to conjecture the existence of our construction. The actual construction was simply written down ad hoc. However we shall show below that we can be led in a natural way to this construction as well as much more general ones by seeking to assign a meaning to the notion of "homomorphism" for virtual subgroups.

5. Homomorphisms of "virtual subgroups" into groups

Let G_1 and G_2 be separable locally compact groups. Let H be a closed subgroup of G_1 and let φ be a continuous homomorphism of H into G_2. Then the group H_φ of all pairs x, $\varphi(x)$ with x in H is clearly a closed subgroup of $G_1 \times G_2$ which uniquely determines H and φ. As such H_φ determines (and up to conjugacy is determined by) a certain transitive action of the product group $G_1 \times G_2$; namely the natural action on the coset space $(G_1 \times G_2)/H_\varphi$. It is easy to see that the transitive actions of $G_1 \times G_2$ so obtained from subgroups H_φ have the following properties:

(i) The $e \times G_2$ action is "free"; that is e, y leaves a point of $(G_1 \times G_2)/H_\varphi$ fixed only when $y=e$.

(ii) The orbits in $(G_1 \times G_2)/H_\varphi$ under the $e \times G_2$ action form a standard Borel space.

(iii) The action of $G_1 \times e$ on the $e \times G_2$ orbits is equivalent to the action of G_1 on G_1/H.

Conversely, let S be any standard Borel space on which $G_1 \times G_2$ acts transitively in such a manner that (i), (ii) and (iii) hold. Because the action is transitive it is the natural action of $G_1 \times G_2$ on $G_1 \times G_2/H'$ for some closed subgroup H' of $G_1 \times G_2$. The fact that (i) holds implies that $H' \cap (e \times G_2) = e, e$. Thus if $x, y \in H'$ then y is uniquely determined by x so that H' is the set of all x, $\varphi(x)$ for x ranging in some subset H'' of G_1 and φ is some function from H'' to G_2. Since H_φ is a group, H'' is a subgroup of G_1 and φ is a homomorphism. There is a natural mapping of the $e \times G_2$ orbits on the H'' cosets in G_1 and it follows from (ii) and (iii) that H'' is closed and conjugate to H. Finally the closedness of H_φ can be used in conjunction with the continuity of Borel homomorphisms to show that φ must be continuous. In other words the properties (i), (ii) and (iii) characterize the transitive actions of $G_1 \times G_2$ defined by the pairs H, φ where H is a closed subgroup of G_1 and φ is a homomorphism of H into G_2. We can use this fact to "describe" the homomorphisms φ of a fixed H into G_2 in a manner which refers only to the G_1 space G_1/H and never to H. They "are" the standard transitive actions of $G_1 \times G_2$ which have properties (i), (ii) and (iii).

Replacing G_1/H by an arbitrary standard ergodic G_1 space S_1 we may "define" the homomorphisms of the virtual subgroups of G_1 associated with the action on S_1 to "be" the ergodic $G_1 \times G_2$ spaces having the obvious analogues of properties (i), (ii) and (iii). Of course two different $G_1 \times G_2$ spaces may define the same homomorphism and we must indicate the appropriate equivalence relation. However, we shall find it convenient to proceed slightly differently and replace the ergodic $G_1 \times G_2$ spaces having properties (i), (ii) and (iii) by certain mappings which define them.

Let S be a standard $G_1 \times G_2$ space which is ergodic with respect to a measure class C in S. Let the $e \times G_2$ actions be "free" in the sense described in (i) above and let the $e \times G_2$ orbits form a standard Borel space. Then, as follows from Theorem 2.9 of [3] there exists a Borel subset S_1 of S which meets each $e \times G_2$ orbit just once. Thus each element s of S may be written uniquely in the form $s = ty$ where $t \in S_1$ and $y \in G_2$. Thus $sz = tyz$ for each $z \in G_2$. In other words, we may replace S by $S_1 \times G_2$ and when we do $(t, y)z = t, yz$. Applying the map $t, y \to t, y^{-1}$ we may also replace S by $S_1 \times G_2$ in such a way that $(t, y)z = t, z^{-1}y$ and we chose to realize S in this way. The action of G_1 on S is somewhat more complicated. Since $(t, y) = (t, e)y^{-1}$ we have $(t, y)x = (t, e)y^{-1}x = ((t, e)x)y^{-1}$ $= \theta(t, x), y\pi(t, x)$ where θ is a Borel function from $S_1 \times G_2$ to S_1 and π is a Borel function from $S_1 \times G_1$ to G_2. Thus $(t, y)(x, z) = \theta(t, x), z^{-1}y\pi(t, x)$ for all $t, y \in S_1 \times G_2$ and all $x, z \in G_1 \times G_2$. Moreover, a straightforward computation proves the following: Let θ be an arbitrary Borel function from $S_1 \times G_1$ to S_1 and let π be an arbitrary Borel function from $S_1 \times G_1$ to G_2. Then the definition $(t, y)(x, z) = \theta(t, x), z^{-1}y\pi(t, x)$ converts $S_1 \times G_2$ into a $G_1 \times G_2$ space if and only if the following conditions hold:

(a) $t, x \to \theta(t, x)$ converts S_1 into a Borel G_1 space.

(b) $\pi(t, x_1 x_2) = \pi(t, x_1)\,\pi(tx_1, x_2)$ for all $t \in S_1$ and all $x_1, x_2 \in G_1 \times G_2$.

Evidently, the action of G_1 on S_1 defined by θ is equivalent to the action of G_1 on the $e \times G_2$ equivalence classes.

It is now clear that in order to find the most general $G_1 \times G_2$ space which has properties (i), (ii) and (iii) with respect to a G_1 space S_1 we have only to find the most general Borel mapping π from $S_1 \times G_1$ to G_2 which satisfies (b) above. Then our space will be $S_1 \times S_2$ and $G_1 \times G_2$ will act upon it according to the rule. $(s, y)(x, z) = sx, z^{-1}y\pi(s, x)$.

It is not difficult to see that whenever C_1 is an ergodic measure class in S_1 and C_2 is the Haar measure class in G_2 then $C_1 \times C_2$ will be an ergodic measure class in $S_1 \times G_2$. Finally then, if we are to identify homomorphisms of the "virtual" subgroup of G_1 defined by S_1, C_1 with certain ergodic $G_1 \times G_2$ spaces we may equally well identify them with functions satisfying the identity (b). Of course, two different functions satisfying (b) may lead to the same $G_1 \times G_2$ space (changing the cross section will change π) and we have already alluded to an equivalence relation between $G_1 \times G_2$ spaces. Thus the correspondence between homomorphisms and functions π satisfying (b) is not one-to-one. It turns out that both causes of non one-to-one-ness produce the same identification of π's. We shall not go into detail but simply announce the result. Following terminology familiar in homological algebra let us call a Borel function π from $S_1 \times G_1$ to G_2 a *one cocycle* whenever is satisfies the identity (b) above. If π is any one cocycle and a is any Borel function from S_1 to G_2 let $\pi^a(s, x) = a(s)\,\pi(s, x)\,a(sx)^{-1}$. Then π^a is clearly a one cocycle. If $\pi_2 = \pi_1^a$ for some a we shall say that π_1 and π_2 are cohomologous. An analysis which we shall omit shows that two one cocycles define the "same" homomorphism if and only if they are cohomologous. In the sequel then we shall think of a cohomology class of one cocycles as "being" a homomorphism into G_2 of the "virtual subgroup" of G_1 defined by the ergodic action of G_1 on S_1, C_1.

As a sort of a check let us consider the special case in which $S_1 = G_1/H$ for some closed subgroup H of G_1. Let $s_0 = H$ so that H is the subgroup leaving s_0 fixed. Then for $x \in H$ the identity (b) reduces to $\pi(s, x_1 x_2) = \pi(s, x_1)\,\pi(s, x_2)$. Thus π defines an honest homomorphism of H_1 into G_2 (which is continuous because it is a Borel homomorphism). It is straightforward to verify that two π's are cohomologous if and only if they define homomorphisms of H into G_2 which are conjugates.

In the general ergodic case one should of course make certain further identifications when things are equal almost everywhere and one should generalize (b) by replacing identity by identity almost everywhere. However, technical difficulties arise when one tries to do this in a straightforward manner and the resolution of these difficulties seems to require that we cast the theory in the more sophisticated form toward which we are gradually working. Moreover, the central notions can be explained more simply if we have only genuine identities to deal with. For these reasons we shall, for the time being, avoid dealing with this ultimately necessary refinement.

6. A generalization of the concept of a flow built under a function

Let H, $G_1 \cdot G_2$ and φ be as in § 5. Then the kernel K of φ is a closed subgroup of G_1 which defines a transitive action of G_1 and the range $\varphi(H)$ of φ is a subgroup of G_2 whose closure $\overline{\varphi(H)}$ defines a transitive action of G_2. The virtual subgroup point of view suggests that we seek formulations of the definitions of these actions in which reference to H and φ is replaced by reference to G_1/H and a one cocycle π defining φ. If these definitions make sense when G_1/H is replaced by an arbitrary ergodic G_1 space S_1, C_1 and lead to ergodic actions of G_1 and G_2 respectively then we may think of these ergodic actions as being defined by "virtual subgroups" which are the kernel and the closure of the range respectively, of the "homomorphism" defined by π. In this section we shall show that such a formulation is possible for the action of G_2 defined by $\overline{\varphi(H)}$ and that the general construction to which it leads contains the well known construction of a "flow built under a function" as a very special case.

Consider the action of $G_1 \times G_2$ defined by the closed subgroup H_φ. The $G_1 \times e$ orbits correspond one-to-one to the $H_\varphi : G_1 \times e$ double cosets and hence to the $\varphi(H)$ right cosets in G_2. Now by Theorem 7.2 of [7] the space of all $\varphi(H)$

right cosets in G_2 is a standard Borel space if and only if $\varphi(H)$ is closed and it follows easily that the space of all $G_1 \times e$ orbits (with this natural Borel structure) is a standard Borel space if and only if $\varphi(H)$ is closed. Now, since the $G_1 \times e$ action commutes with the $e \times G_2$ action each $G_1 \times e$ orbit is carried into another by each element of $e \times G_2$. Thus G_2 acts on the space of all $G_1 \times e$ orbits and when $\varphi(H)$ is closed this is a standard Borel space. An easy calculation shows then that this G_2 action is transitive and equivalent to the G_2 action on the coset space $G_2/\varphi(H)$. When $\varphi(H)$ is not closed, things are a little more complicated. Let us say that two $G_1 \times e$ orbits are *conjugate* if the $\varphi(H)$ right cosets which define them belong to the same $\overline{\varphi(H)}$ right coset. Then the set of all conjugacy classes of $G_1 \times e$ orbits is a standard Borel G_2 space. Moreover the action of G_2 on this space is equivalent to its action on $G_2/\overline{\varphi(H)}$. The action of G_2 on the conjugacy classes of $G_1 \times e$ orbits may be defined in a more readily generalizable manner as follows. Let B denote the Boolean algebra of Borel sets mod null sets in $G_1 \times G_2/H_\varphi$. Let B' denote the subalgebra of all elements left fixed by each $x, e \in G_1 \times e$. Then the action of G_2 takes each element of B' into another element of B' and defines an action of G_2 on B'. As shown in [9] such a Boolean algebra action is derivable in an essentially unique way from a point action and in this case the point action is equivalent to that on the conjugacy classes of $G_1 \times e$ orbits. From another point of view (and somewhat vaguely stated) one writes the $G_1 \times e$ action as a direct integral of ergodic parts and considers the action of G_2 on the space of ergodic parts.

Guided by the preceding, we now make the following construction. Let G_1 and G_2 be as above and let C_1 be an ergodic invariant measure class in the standard Borel G_1 space S_1. Let π be a Borel one cocycle from $S_1 \times G_1$ to G_2 and convert $S_1 \times G_2$ into a $G_1 \times G_2$ space by setting $(s, y)(x, z) = sx, z^{-1} y \pi(s, x)$. Then (as indicated above) if C_2 is the Haar measure class in G_2, $C_1 \times C_2$ will be an ergodic invariant measure class in $S_1 \times G_2$. Let S' be the space of all $G_1 \times e$ orbits. If this is standard under the quotient Borel structure it will be a standard G_2 space in a natural way and the measure class C' defined in S' by $C_1 \times C_2$ will be ergodic and invariant. We shall speak of this ergodic action of G_2 as defined by the "virtual subgroup" of G_2 which is the image under the "homomorphism" defined by π of the "virtual subgroup" of G_1 defined by the given ergodic action of G_1. If S' is not standard we pass to the Boolean algebra of Borel sets mod null sets in $S_1 \times G_2$ and consider the action of G_2 on B' the subalgebra of all elements which are left fixed by all x, e in $G_1 \times e$. This action is associated with an essentially unique point action which will be ergodic. We shall speak of this ergodic action of G_2 as defined by a "virtual subgroup" of G_2 which is the *closure* of the image under the "homomorphism" defined by π of the "virtual subgroup" of G_1 defined by the given ergodic action of G_1.

If ϱ is any Borel homomorphism of G_1 into G_2 we obtain a one cocycle π^ϱ by setting $\pi^\varrho(s, x) = \varrho(x)$. Clearly the Borel cocycles of the form π^ϱ are just those which are independent of s. In the special case in which G_1 is a closed subgroup of G_2 and ϱ is the identity, the construction of an ergodic action using π^ϱ reduces to that of § 4.

Now consider the special case in which G_1 is the additive group Z of all the integers. Let x_0 be a generator. Then for any cocycle π, $\pi(s, x)$ is determined uniquely by the Borel function $s \to \pi(s, x_0)$. Indeed

$$\pi(s, x_0^n) = \pi(s, x_0^{n-1})\, \pi(s x_0^{n-1}, x_0) \quad 1 = \pi(s, e) = \pi(s, x_0^{-n}) \pi(s x_1^{-n}, x_0^n).$$

Furthermore given any Borel function g from S to G_2 there exists a unique Borel cocycle π_g such that $\pi_g(s, x_0) \equiv g(s)$. In other words, in this case, the Borel one cocycles correspond one-to-one to the Borel functions from S to G_2 and our general procedure allows us to construct an ergodic action of G_2 whenever we are given an ergodic action of Z and a Borel function from the Z space S to G_2. The special case in which this function is a constant, $g(s) \equiv y_0$, coincides with that in which $\pi = \pi_\varrho$ where $\varrho(n) = y_0^n$. If the cyclic subgroup generated by y is closed we may identify Z with this subgroup and our construction reduces to that of § 4.

Let α be a Borel automorphism of the standard Borel space S and let C be a measure class in S which is invariant and ergodic under the action of α. Let f be a positive real valued Borel function on S. In the special case in which C contains a finite invariant measure μ there is a standard construction in ergodic theory allowing one to pass from the triple S, α, f to an ergodic action of the real line — a so-called ergodic flow. This flow is called the flow *built under* the function f. We wish to show that this construction is included as a very special case in the general construction described above. Generalized slightly so as not to demand a finite invariant measure this standard construction may be described as follows. Let S' be the set of all pairs of points s, y where $s \in S$ and y is a real number. Let C' be the product of the measure class C in S and the Lebesgue measure class in the line. Let S'' be the set of all $s, y \in S'$ with $0 \leqq y < f(s)$ and let C'' be the restriction of C' to S''. We obtain the required ergodic flow by letting the real line act on S'' as follows. For each $s, y \in S''$ and each $x > 0$ choose the unique positive integer n such that $f(s) + f(s\alpha) + \cdots + f(s\alpha^{n-1}) \leqq y + x < < f(s) + f(s\alpha) + \cdots + f(s\alpha^n)$ and let $(s, y)\, x = s\alpha^n$, $y + x - f(s) - f(s\alpha) \cdot \cdots - - f(s\alpha^{n-1})$. Then define $(s, y) x^{-1}$ so that $(s, y) x^{-1} x = s, y$. It follows from the ergodic theorem that $\sum_{k=1}^{\infty} f(s\alpha^{k-1})$ diverges for almost all s so that we may assume that n exists. To compare this construction with our general one make S into a Z space by setting $sn = (s)\alpha^n$ and form the product of S with R the additive group of all real numbers. For each $s, y \in S \times R$ let $\beta(s, y) = s\alpha, y - f(s)$. Then β is a Borel automorphism of $S \times R$ and setting $(0, y)\, n = \beta^n(s, y)$ converts $S \times R$ into a Z space in such a manner that $S'' \subseteqq S' = S \times R$ meets each Z orbit just once. In other words we may identify S'' with the space of all Z orbits in $S \times R$ under the indicated action. Now $(s, y)\, x = s, y + x$ makes $S \times R$ into an R space and this action of R commutes with the Z action. Thus the space of Z orbits is an R space and we verify at once that the action of R on S'' which we thus obtain by identifying it with the space of Z orbits is the same as that defined above. In other words the construction of a flow built under a function can always be obtained as the action of R on the $Z \times e$ orbits in the $Z \times R$ action on $S \times R$

defined by setting $(s, y)(1, x) = s\alpha, y - f(s) + x$ and via the automorphism $s, y \to s, -y$ this action is equivalent to that defined by setting $(s, y)(1, x) = s\alpha$, $y + f(s) - x$. But this latter action is precisely what our general construction yields when $G_1 = Z$, $G_2 = R$ and $\pi = \pi_f$. Thus the flow built under the function f is just the ergodic action of the real line whose associated "virtual subgroup" is the range of the "homomorphism" defined by π_f. Since there is a cross section for the Z orbits the orbit space is standard and we may speak of the range itself rather than of its closure.

A remarkable theorem of AMBROSE [1] states that every ergodic flow (at least when there is a finite invariant measure in the measure class) is equivalent to a flow built under a function. Let us call a "virtual subgroup" *unimodular* when the associated ergodic action has an invariant measure and let us call a unimodular "virtual subgroup" *big* when this invariant measure is finite. Then AMBROSE's result has a corollary which may be stated as follows. Every big unimodular "virtual subgroup" of the additive group of the real line is a "homomorphic" image of a big unimodular "virtual subgroup" of the additive group of the integers. This corollary is only weaker than the theorem itself to the extent that that "homomorphisms" defined by cocycles π_g with $g > 0$ are more special than those in which g is not restricted. Could it be that π_h is cohomologous to π_g for some positive g whenever the "homomorphism" defined by π_g has a closed range?

This formulation of AMBROSE's theorem suggests a number of questions and possible generalizations.

(1) Can one remove the restriction to "virtual subgroups" which are big and unimodular?

(2) For how large a class of separable locally compact groups G is it true that every (big, unmodular) "virtual subgroup" of G is a homomorphic image of a (big, unmodular) "virtual subgroup" of Z?

(3) Same question as (2) with Z replaced by "some countable closed subgroup of G."

(4) Same question as (2) with Z replaced by "some proper closed subgroup of G."

The author has not yet given any serious thought to these questions. However, he believes them to be interesting and worthy of investigation.

7. Kernels of "homomorphisms" and the skew products of ANZAI

We now consider the other of the two questions raised at the beginning of § 6; that of defining an analogue of the kernel of a homomorphism when the domain of the "homomorphism" is a "virtual subgroup." Let H, G_1, and G_2 and φ be as in § 5 and § 6 and let K be the kernel of φ. If $\varphi(H)$ is closed and we replace G_2 by $\varphi(H)$ (which of course does not change the kernel of φ) we see that the action of $G_1 \times e$ on $G_1 \times G_2/H_\varphi$ is transitive and that the defining closed subgroup of G_1 is K. If we do not replace G_2 by $\varphi(H)$ then the action of $G_1 \times e$ on $G_1 \times G_2/H_\varphi$ is a direct integral [over the $\varphi(H)$ right cosets in G_2] of transitive actions all having K as defining subgroup. Now suppose that $K = \{e\}$

but that $\varphi(H)$ is not closed but is dense in G_2. Then the action of $G_1 \times e$ on $G_1 \times G_2/H_\varphi$ will be ergodic and not transitive and in view of the above it will be useful to think of the "virtual subgroup" of G_1 defined by this ergodic action as the kernel of φ even though this kernel as usually defined is trivial. In other words, if one introduces "virtual subgroups" then an honest homomorphism can have a non-trivial kernel even when it is one-to-one provided that it actually weakens the topology; that is, fails to have a continuous inverse. More generally, whether or not $K = \{e\}$ the action of $G_1 \times e$ on $G_1 \times G_2/H_\varphi$ will be ergodic if and only if $\varphi(H)$ is dense in G_2 and we shall agree to call the "virtual subgroup" of G_1 defining this action the kernel of φ. This "virtual subgroup" will be a real subgroup if and only if $\varphi(H) = G_2$ and then will coincide with the actual kernel of φ.

Now let S_1, C_1 be any ergodic G_1 space and let π be a Borel one cocycle from $S_1 \times G_1$ to G_2. As in § 6 we convert $S_1 \times G_2$ into a $G_1 \times G_2$ space (generalizing $G_1 \times G_2/H_\varphi$) by setting $(s, y)(x, z) = sx,\ z^{-1} y\pi(s, x)$. Also, we introduce the ergodic invariant measure class $C_1 \times C_2$. Guided by the above considerations, we shall say that the "homomorphism" defined by π has range which is dense in G_2 whenever the $G_1 \times e$ action on $S_1 \times G_2$ is ergodic. Moreover, if this range is dense we shall call the "virtual subgroup" of G_1 defined by the action the *kernel* of the "homomorphism." Of course, even when the range is not dense it may be dense in some subgroup. That is, π may be cohomologous to π' where π' takes its values in a subgroup H_2 of G_2 and where the range of the "homomorphism" defined by π' is dense in H_2. In that event we may define the kernel of our "homomorphism" using π' and H_2 instead of π and G_2. When G_2 is *compact* such a pair π', H_2 always exists and the action of $G_1 \times e$ on $S_1 \times G_2$ will itself be ergodic if and only if no π' exists with values in a proper closed subgroup of H_2.

Consider now the special case in which G_1 is the additive group Z of all the integers and G_2 is the multiplicative group of all complex numbers of modulus one. Then G_2 is compact and its only proper closed subgroups are the finite subgroups H_p where p is an integer and H_p is the group of all p-th roots of unity. Since $G_1 = Z$ each π may be written in the form π_g for some Borel function g from the Z space S to G_2. It is easy to see that π_{g_1} is cohomologous to π_{g_2} if and only if $g_1(s) \equiv g_2(s)\, h(s)/h(s\alpha)$ where $s\alpha = s \cdot 1$. Thus, by the result quoted above, π_g will define a homomorphism whose range is dense in G_2 if and only if for no positive integer p do we have $g(s)^p \equiv h(s)/h(s\alpha)$ for some h. Moreover when this condition holds we obtain an ergodic action of Z on $S \times G_2$ by setting

$$(s, y)\, n = (s, y)\, \beta^n \qquad \text{where} \qquad (s, y)\, \beta = s\alpha,\ yg(s).$$

In the subspecial case in which $S = G_2$ and α is rotation through an irrational angle β is what ANZAI [2] calls the skew product of γ and g and our condition for ergodicity reduces to ANZAI's. We see then that ANZAI's ergodic skew product transformations may be considerably generalized and as "virtual subgroups" are the kernels of certain "homomorphisms." It could be of some interest to investigate the extent to which the other results of ANZAI's paper may be generalized.

8. Further examples of "homomorphisms" in ergodic theory

(A) If one looks at the range of a "homomorphism" in the special case in which both G_1 and G_2 are Z one is led to KAKUTANI's concept of "induced measure preserving transformation" [5]. We leave details to the reader.

(B) Let S be a standard Borel G space and let C be an ergodic invariant measure class. Let μ be any member of C. For each $x \in G$ let $\mu_x(E) = \mu(Ex)$. Then μ_x and μ are measures with the same null sets and we may form the Radon-Nikodym derivative $s \to \varrho_x(s)$ of μ_x with respect to μ. Of course, this derivative is only determined almost everywhere and it can be shown that the choices can be made so that $s, x \to \varrho_x(s)$ is a measurable function from $S \times G$ to the positive real numbers. As such it is equal almost everywhere to a Borel function. Moreover, it follows at once from the definitions that for each x_1 and x_2 we have $\varrho_{x_1 x_2}(s) = \varrho_{x_1}(s)\, \varrho_{x_2}(s x_1)$ for almost all s. Let us set $\pi(s, x) = \varrho'(s)$ where $\varrho'_x(s) = \varrho_x(s)$ for almost all pairs s, x and $s, x \to \varrho'_x(s)$ is a Borel function. Then, except for the fact that the relevant identities hold only almost everywhere, π is a Borel one cocycle from $S \times G$ to the multiplicative group of all positive real numbers. Of course π depends upon the choice of μ but an easy calculation shows that changing μ to another measure in the same class changes π to a cohomologous cocycle. If we ignore almost everywhere considerations, as we have agreed to do for the time being, we see that our ergodic action is canonically associated with a "homomorphism" of the associated "virtual subgroup" into the multiplicative group R^x of all positive real numbers. Moreover, this "homomorphism" is trivial (in the sense of containing the cocycle π which is identically one) if and only if the measure class C contains an invariant measure. In the special case in which $S = G/H$ for some closed subgroup H of G our "virtual subgroup" becomes the genuine subgroup H and our canonical homomorphism becomes $h \to \delta(h)/\Delta(h)$ where δ and Δ are the modular homomorphisms of H and G into R^x; that is $\Delta(x)$ is the (constant) Radon-Nikodym derivatives of right invariant Haar measure with respect to left translation by x and similarly for $\delta(h)$. Thus the known fact ([12], pages 43—44) that G/H admits an invariant measure if and only if Δ and δ agree in H becomes unified with the functional equation condition for the existence of an invariant measure in an invariant measure class ([4], page 751).

(C) Let G_1, G_2, S_1, and C_1 be as in § 5 and consider the one cocycles from $S_1 \times G_1$ to G_2 of the form π^ϱ as defined in § 6; that is $\pi^\varrho(s, x) = \varrho(x)$ where ϱ is a Borel (and hence continuous) homomorphism of G_1 into G_2. We may think of the "homomorphism" defined by π^ϱ as the restriction to a "virtual subgroup" of G_1 of the genuine homomorphism ϱ. The question then arises as to when π^ϱ defines the trivial homomorphism, that is as to when the kernel of ϱ "contains" the "virtual group" defined by the ergodic action of G_1 on S_1, C. When G_2 is commutative the set of all such homomorphisms ϱ is a group A_{G_2} and this group of homomorphisms of G_1 into G_2 is an invariant of the given ergodic action of G_1. In the special case in which G_1 is commutative and in which G_2 is the group T of all complex numbers of unit modulus A_{G_2} is a subgroup of the character group $\hat{G}_1$ of G_1. Let us see what this subgroup looks like. By definition $X \in \hat{G}_1$ is in A_T if and only if $X(x) = f(sx)/f(s)$ for some Borel function f in S and all s and x. In the special case in which S admits a finite invariant measure μ such an f has a constant absolute value. Hence it is in $\mathscr{L}^2(S, \mu)$. But the identity $X(x) = f(sx)/f(s)$ may be also written as $f(sx) \equiv X(x)f(s)$ and this says that f defines a one dimensional invariant subspace of $\mathscr{L}^2(S, \mu)$. Conversely, every one dimensional invariant subspace of $\mathscr{L}^2(S, \mu)$ is associated with a unique $X \in A_{G_2}$ and we see that A_{G_2} in this case is precisely the so-called *point spectrum* of the given action of G_1. Now those actions of G_1 in which $\mathscr{L}^2(S, \mu)$ is a direct sum of one dimensional invariants subspaces are said to have pure point spectrum and it is known that an action with pure point spectrum is determined to within equivalence by its spectrum. This suggests that it might be possible to classify a large class of actions by showing that they are determined to within equivalence by the family of subgroups A_{G_2}, $A_{G'_2}$, $A_{G''_2}$ etc. for suitably chosen groups G_2, G'_2, G''_2 .... Note however, that if the point spectrum is trivial then every $A^{(k)}_{G_2}$ will also be trivial. Indeed if ϱ is a non identity member of A_{G_2} for some G_2 and X is any member of G_2 which is not identically one in the range of ϱ then $X \circ \varrho$ will be a non identity member of A_T. Thus one can only hope to apply the method to actions with non trivial point spectrum.

9. Toward a formal definition of virtual subgroup

Until now the phrase "virtual subgroup" has been used as a suggestive locution rather than as the name of a well defined mathematical concept. We have shown that many concepts in ergodic theory are suggested by the analogy between transitive actions and ergodic actions and have used terminology involving the undefined term "virtual subgroup" in order to emphasize this analogy. In this section and the next two sections we shall show that we need not continue to use suggestive undefined terms in order to pursue the analogy in question. One can introduce an honest mathematical object which plays the role we have implicitly defined by our development of the virtual subgroup point of view. In order to motivate the definition let us note that the notion of isomorphism between topological groups leads to a natural equivalence relation between transitive actions of (possibly different) separable locally compact groups. Indeed each transitive action of G is defined by a conjugacy class of closed subgroups and one obtains an equivalence relation (which we shall call similarity) if we state that the G_1 action on G_1/H_1 is similar to the G_2 action on G_2/H_2 whenever H_1 and H_2 are isomorphic as topological groups. Now every Borel homomorphism of one separable locally compact group into another is necessarily continuous. Thus we may define similarity equivalently as follows. The G_1 action on G_1/H_1 is similar to the G_2 action of G_2/H_2 if there exist Borel homomorphisms θ_1 and θ_2 of H_1 into H_2 and of H_2 into H_1 such that $\theta_1 \circ \theta_2$ and $\theta_2 \circ \theta_1$ are both the identity. If we had a notion of homomorphisms from one "virtual subgroup" to another we could use it as just indicated to define a notion of similarity for ergodic actions. We could then define a virtual group to be a similarity class of ergodic actions and a virtual subgroup of a group G

to be a virtual group together with a certain homomorphism of this virtual group into G. This is the path we shall follow. To do so we must develop the virtual subgroup point of view a little further and define the notion of homomorphism when both the range and the domain are "virtual." Now we have already defined a "homomorphism" into G_2 of the "virtual subgroup" of G_1 defined by an ergodic action on S_1, C_1 to be a cohomology class of one cocycles from $S_1 \times G_1$ to G_2. Moreover, we have seen that the "virtual subgroup" of G_2 defined by the action of G_2 on a certain "quotient space" of $S_1 \times G_2$ may be regarded as the "closure of the range" of this "homomorphism." Let us denote this quotient space of $S_1 \times G_2$ by S_2 and the mapping taking each element of $S_1 \times G_2$ into its equivalence class by θ. Then θ is a Borel map of $S_1 \times G_2$ onto S_2 which satisfies the identity.

$$(*) \qquad \theta((s, y)(x, z)) = \theta(s, y)z \qquad \text{for all} \qquad s, x, y, z \in S_1 \times G_2 \times G_1 \times G_2 .$$

Conversely given any Borel map θ_1 satisfying * from $S_1 \times G_2$ to an arbitrary ergodic G_2 space S'_2, C'_2 it will define a Borel map θ'_1 from S_2 to S'_2 such that $\theta'_1(sx) = \theta'_1(s)x$ for all $s \in S_2$ and all $x \in G_2$. In the transitive case such a map θ' will exist if and only if S_2 and S'_2 may be defined by subgroups H_2 and H'_2 such that $H'_2 \subseteq H_2$. Thus we may think of π, θ_1 as defining a "homomorphism" of the "virtual subgroup" of G_1 defined by its action on S_1, C_1 *into* the "virtual subgroup" of G_2 defined by its action on S'_2, C'_2.

The identity (*) is somewhat complicated when written out and it us useful to observe that $\theta(s, y) \equiv \varphi(s)y^{-1}$ where $\varphi(s) \equiv \theta(s, e)$ and that the condition on φ necessary and sufficient for the satisfaction of * is quite simple. Indeed if we write * out and set $x = e$, $y = z$ it becomes $\theta(s, e) = \theta(s, y)y$ and if we replace $\theta(s, y)$ in * by $\varphi(s)y^{-1}$ it reduces to

$$(**) \qquad \varphi(sx) \equiv \varphi(s)\,\pi(s, x) .$$

We are now ready to make our definitions. A *homomorphism* of the "virtual subgroup" defined by an ergodic action of G_1 on S_1, C_1 into the "virtual subgroup" defined by an ergodic action of G_2 on S_2, C_2 is a class of pairs π, φ where π is a Borel one cocycle from $S_1 \times G_1$ to G_2. φ is a Borel function from S_1 to S_2 and π and φ satisfy the identity (**). π_1, φ_1 and π_2, φ_2 belong to the same class if and only if there exists a Borel function a from S_1 to G_2 such that $\varphi_1(s) \equiv \varphi_2(s)\,a(s)$ and $a(s)\,\pi_1(s, x) \equiv \pi_2(s, x)\,a(s, x)$. Now let G_1, G_2 and G_3 be three separable locally compact groups and let C_i be an ergodic invariant measure class in the standard Borel G_i space S_i. Let π_1, φ_1 define a "homomorphism" of the first associated "virtual subgroup" into the second and let π_2, φ_2 define one of the second into the third. Then $s, x \to \varphi_1(s)$, $\pi_1(s, x)$ is a mapping of $S_1 \times G_1$ into $S_2 \times G_2$ which determines the pair φ_1, π_1 and $s, x \to \varphi_2(s)$, $\pi_2(s, x)$ is a mapping of $S_2 \times G_2$ into $S_3 \times G_3$ which determines the pair φ_2, π_2. The composition of these two mappings is a mapping of $S_1 \times G_1$ into $S_3 \times G_3$ which maps s, x into $\varphi_2(\varphi_1(s)), \pi_2(\varphi_1(s), x)$. Moreover, one can prove that $s, x \to \pi_2(\varphi_1(s), x)$ is a Borel one cocycle, that this cocycle and $\varphi_2 \circ \varphi_1$ satisfy ** and that the cohomology class of the pair $s, x \to \pi_2(\varphi_1(s), x)$, $\varphi_2 \circ \varphi_1$ depends only on the cohomology classes of π_1 and π_2. This suggests that we define the composition of the homomorphisms defined by π_1, φ_1 and π_2, φ_2 to be the homomorphism defined by φ_3, π_3 where $\varphi_3 = \varphi_2 \circ \varphi_1$ and $\pi_3(s, x) = \pi_2(\varphi_1(s), x)$. That this is the "right" definition is confirmed by the fact that it agrees with the usual definition when our G_2 spaces are all transitive. Our goal seems to be at hand. Having defined homomorphism and composition we may now define similarity as indicated at the beginning of this section and define a virtual group as a similarity class of ergodic actions. Finally, we may define a virtual subgroup of a group G as a virtual group together with a particular homomorphism of it into G. However, we have not made our peace with almost everywhere questions and the technical difficulties that arise become quite troublesome when one tries to deal with composition of almost everywhere defined homomorphisms. In the next two sections we shall present a new point of view toward our definition which not only makes it easier to deal with almost everywhere questions, but suggests a broader framework for the whole theory. We shall see that a virtual group need not come to us as a subgroup of a group — indeed that there may be virtual groups not embeddable in any group. In particular, one can define ergodicity for equivalence relations, whether or not the relation is defined by a group, and every ergodic equivalence relation defines a virtual group.

10. Ergodic equivalence relations

Let C_1 be an ergodic invariant measure class in the standard Borel G_1 space S_1 and let us consider the important special case in which the action of G_1 on S_1 is free; that is in which for all $s \in S_1$ $sx = s$ implies $x = e$. We begin by observing that the "virtual subgroup" of G_1 defined by the G_1 action on S_1, C_1 depends only on the equivalence relation in S_1 set up by the action and not at all upon the other features of the action. Indeed it follows from the identity (**) that in this case π is uniquely determined by φ and that a π will exist for a given φ if and only if $\varphi(s_1)$ and $\varphi(s_2)$ lie in the same orbit whenever s_1 and s_2 lie in the same orbit. Given φ_1 and φ_2 the question arises as to when the uniquely determined one cocycles π_1 and π_2 are such that π_1, φ_1 and π_2, φ_2 belong to the same class and an easy calculation yields the following answer. φ_1 and φ_2 define pairs belonging to the same class if and only if $\varphi_1(s_1)$ and $\varphi_2(s_2)$ are in the same G_2 orbit whenever s_1 and s_2 are in the same G_1 orbit; that is if and only if φ_1 and φ_2 define *identical* maps of the orbit space of S_1 under G_1 into the orbit space of S_2 under G_2. In other words in this case a homomorphism of the S_1, C_1, G_1 virtual group into the S_2, C_2, G_2 virtual group *is* a mapping of the space of all G_1 orbits into the space of all G_2 orbits. However, not every mapping will do and the mappings which will do can *not* be described as those which are Borel mappings with respect to the natural Borel structure in the orbit spaces. These orbit Borel spaces are much too irregular to be useful. The mappings which define homomorphisms are those which can be "lifted" to be Borel maps of S_1 into S_2.

To say that the G_1 action on S_1 is free is the same as to say that the mapping $s, x \to s, sx$ is one to one from $S_1 \times G_1$ to a certain subset $\mathscr{E}_1$ of $S_1 \times S_1$. This subset $\mathscr{E}_1$ is the set of all pairs s_1, s_2 such that s_1 and s_2 lie in the same G_1 orbit; that is the set of ordered pairs defining the equivalence relation set up by the G_1 action. If we wish to emphasize the equivalence relation rather than the action it is useful to transfer our attention from $S_1 \times G_1$ to $\mathscr{E}_1$ using the mapping $s, x \to s, sx$ to transform functions with domain $S_1 \times G_1$ into functions with domain $\mathscr{E}_1$. From this point of view a one cocycle π from $S_1 \times G_1$ to G_2 becomes a Borel function from $\mathscr{E}_1$ to G_2 satisfying the identity

$$\pi(s_1, s_2)\,\pi(s_2, s_3) = \pi(s_1, s_3)$$

whenever s_1, s_2 and s_2, s_3 are both in $\mathscr{E}_1$. Moreover the one-cocycles π_1 and π_2 are cohomologous if and only if there exists a Borel function a from S_1 to G_2 such that $\pi_1(s_1, s_2) = a(s_1)\,\pi_2(s_1, s_2)\,a(s_2)^{-1}$.

The fact that the above reformulations are possible suggests that we attempt to generalize our theory to include "ergodic equivalence relations" which do not come from an ergodic action of a group. Let us see how such a notion might be defined. Keeping in mind our ultimate confrontation with almost everywhere questions we shall want not only a measure class in S which is in some sense "invariant" but also a measure class in $\mathscr{E}$ generalizing the product of the Haar measure class in G with the invariant measure class in S. Actually the measure class in S is uniquely recoverable from its product with the Haar measure class so it will suffice to be given a certain measure class in $\mathscr{E}$. An ergodic equivalence relation may be defined as the system consisting of a standard Borel space S, a Borel subset $\mathscr{E}$ of $S \times S$ defining an equivalence relation in S and a measure class C in $S \times S$ satisfying certain conditions which we shall now discuss.

Let S be a standard Borel space and let $\mathscr{E}$ be a Borel subset of $S \times S$ which defines an equivalence relation in S in the usual sense of the word. Let $\sigma(s_1, s_2) = s_1$ so that σ is a Borel mapping of $\mathscr{E}$ on S and let $\theta(s_1, s_2) = s_2, s_1$ so that θ is an involuntary automorphism of $\mathscr{E}$ as a Borel space. For each finite measure μ in $\mathscr{E}$ let $\tilde{\mu}$ denote the measure in S such that $\tilde{\mu}(F) = \mu(\sigma^{-1}(F))$. It follows from the "quotient measure theorem" (cf. [6], page 124 for a formal statement with references to proofs) that there exists an essentially unique assignment to each $s \in S$ of a measure μ_s in $\sigma^{-1}(s)$ so that (in an obvious sense) we have $\mu = \int \mu_s \, d\tilde{\mu}(s)$. It is easy to see that the class of $\tilde{\mu}$ depends only upon the class of μ and that for $\tilde{\mu}$ almost all s in S the class of μ_s depends only upon the class of μ. Thus for each measure class C in $\mathscr{E}$ we have a canonically defined measure class $\tilde{C}$ in S and an essentially unique assignment to each $s \in S$ of a measure class C_s in $\sigma^{-1}(s)$. We shall speak of $\tilde{C}$ and the C_s as the *decomposition* of C defined by σ. Note that $s, t \to t$ is a one-to-one map of $\sigma^{-1}(s)$ into the $\mathscr{E}$ equivalence class of s in S. Thus each C_s defines a unique measures class C_s^0 which is concentrated in the $\mathscr{E}$ equivalence class of s.

Now consider the special case in which $\mathscr{E}$ is the set of all pairs s, sx for some free action of a separable locally compact group G on S. Then each C_s^0 may be identified with a measure class in G via the mapping $x \to sx$. Moreover a necessary and sufficient condition that $C^0_{s_1} = C^0_{s_2}$ whenever $s_1, s_2 \in \mathscr{E}$ is that each C^0_s coincides with the Haar measure class when this identification is made. When these equivalent conditions do hold it follows that C, transferred to $S \times G$ by the mapping $s, x \to s, sx$, is of the form $C_1 \times C_G$ where C_G is the Haar measure class and $C_1 = \tilde{C}$ is some measure class in S. Finally, it can be proved that when C (transferred to $S \times G$) is of the form $C_1 \times C_G$ then C_1 is invariant under the G action if and only if C is invariant under θ.

These facts suggest that we complete our definition of an ergodic equivalence relation as follows. It is a system consisting of a standard Borel space S, a Borel subset $\mathscr{E}$ of $S \times S$ defining an equivalence relation in S and a measure class C in $\mathscr{E}$ provided that C has the properties listed as (i), (ii) and (iii) below.

(i) C is invariant under θ.

(ii) In the decomposition of C defined by the mapping σ we have $C^0_{s_1} = C^0_{s_2}$ for C almost all pairs $s_1, s_2 \in \mathscr{E}$.

(iii) If f is a Borel function on S such that $f(s_1) = f(s_2)$ for C almost all pairs $s_1, s_2 \in \mathscr{E}$ then f is a constant $\tilde{C}$ almost everywhere in S.

If $S, \mathscr{E}, C$ is an ergodic equivalence relation and S_1 is any Borel subset of S such that neither S_1 nor $S - S_1$ is of $\tilde{C}$ measure zero then we may define a new ergodic equivalence relation $S_1, \mathscr{E}_1, C_1$ by taking $\mathscr{E}_1$ to be the set of all $s_1, s_2 \in \mathscr{E}$ with $s_1 \in S_1$ and $s_2 \in S_1$ and taking C_1 to be the restriction of C to $\mathscr{E}_1$. Clearly then there exist ergodic equivalence relations whose equivalence classes are not manifestly the orbits of some group action.

Having reformulated our virtual group concepts in terms of the ergodic equivalence relation canonically associated with a free ergodic action, it is clear that they may be formulated for any ergodic equivalence relation whether or not it comes from a free ergodic action of a group. If only all ergodic actions were free we would have a natural generalization of the theory outlined in the first nine sections which in particular would free us from the necessity of demanding that virtual groups be subgroups of honest groups. In the next section we shall introduce the notion of an ergodic groupoid and show that it bears the same relationship to the notion of ergodic action that the notion of ergodic equivalence relation bears to that of free ergodic action. Thus we shall obtain a notion of virtual group that includes both the virtual subgroups associated with arbitrary ergodic actions and the virtual groups defined by ergodic equivalence relations. At the same time we shall see that the notions of homomorphism and similarity take on a simple and more natural form when defined in terms of the groupoid associated with the underlying ergodic action.

11. Virtual groups as similarity classes of ergodic groupoids

Speaking loosely, a groupoid is a group in which multiplication is not necessarily everywhere defined. More precisely a *groupoid* is a set $\mathscr{G}$ together with a mapping $x, y \to xy$ from some subset $\mathscr{D}$ of $\mathscr{G} \times \mathscr{G}$ to $\mathscr{G}$ such that the following axioms are satisfied.

(i) For each x in $\mathcal{G}$ there is a unique e in $\mathcal{G}$ such that ex is defined and $ex=x$ and there is a unique e' in $\mathcal{G}$ such that xe' is defined and $xe'=x$. We call e and e' the *left* and *right units* of x respectively.

(ii) Each left or right unit is its own left and right unit.

(iii) xy is defined if and only if the right unit of x coincides with the left unit of y.

(iv) If xy and yz are defined then $(xy)z$ and $x(yz)$ are defined and $(xy)z=x(yz)$.

(v) For each x in $\mathcal{G}$ there exists a unique x^{-1} in G (called the inverse of x) such that xx^{-1} and $x^{-1}x$ are the left and rights units of x respectively.

Clearly a groupoid is a group if and only if $\mathcal{D}=\mathcal{G}$.

It is clear that an element of $\mathcal{G}$ is a left unit if and only if it is a right unit and hence we may speak of the set $S_{\mathcal{G}}$ of all units of $\mathcal{G}$. (The reader familiar with the theory of categories may find it useful to observe that a groupoid may also be defined as a category in which every morphism has an inverse. More precisely the elements of a groupoid are the morphisms of a category whose objects are the units). Given two units e_1 and e_2 there may or may not be an element x in $\mathcal{G}$ such that $e_1xe_2=x$. If there is, we shall say that e_1 and e_2 are *conjugate*. It is easy to show that conjugacy is an equivalence relation. Thus the groupoid structure defines an equivalence relation in the space $S_{\mathcal{G}}$ of all units. Let us say that our groupoid is *principal* if for each pair e_1, e_2 of conjugate units there is a unique x such that $e_1xe_2=x$ and let $\mathcal{E}$ denote the subset of $S\times S$ consisting of all conjugate pairs e_1, e_2. Then $x\to x^{-1}x$, xx^{-1} is a one-to-one map of $\mathcal{G}$ onto $\mathcal{E}$. Moreover if we use this map to transfer the groupoid structure in $\mathcal{G}$ over to $\mathcal{E}$ we convert $\mathcal{E}$ into the groupoid defined as follows: $(e_1, e_2)(e_3, e_4)$ is defined if and only if $e_2=e_3$ and then $(e_1, e_2)(e_3, e_4)=e_1, e_4$. Conversely given any equivalence relation $\mathcal{E}'$ in any set S' we may make $\mathcal{E}'$ into a groupoid by copying the above definition. In this groupoid the units are the elements (s, s) and the units (s_1, s_1) and (s_2, s_2) are conjugate if and only if $(s_1, s_2)\in\mathcal{E}'$. In other words one has a natural one-to-one correspondence between principal groupoids on the one hand and sets with an equivalence relation on the other.

Consider the special case in which our equivalence relation is defined by a free action of a group G on our space S. Then the mapping $s, x\to s, sx$ permits us to transfer the groupoid structure from $\mathcal{E}$ back to $S\times G$. We thus convert $S\times G$ into a groupoid in which multiplication is defined as follows: $(s_1, x_1)(s_2, x_2)$ is defined if and only if $s_1x_1=s_2$ and then $(s_1, x_1)(s_2, x_2)=s_1, x_1x_2$. Now we make the important observation that this definition makes sense and converts $S\times G$ into a groupoid *whether the action is free or not*. Whenever S is a G space the above prescription converts $S\times G$ into a groupoid and it is easy to see that this groupoid is principal if and only if the action of G on S is free.

We have now associated a groupoid structure both with an ergodic equivalence relation and with an ergodic action of a separable locally compact group. In each instance our groupoid is also a standard Borel space equipped with a measure class. Converting our definition of "ergodic equivalence relation" into a definition of "principal ergodic groupoid" by using the equivalence between the notions of groupoid and equivalence relation we find that it applies equally well to non principal groupoids and in particular to those of the form $S \times G$. Slightly modified to permit almost everywhere considerations and a wider class of spaces it takes the following form. Let $\mathscr{G}$ be a groupoid and at the same time an analytic Borel space. Let C be a measure class in $\mathscr{G}$. Let $\sigma(x)$ be the left unit of x for each $x \in \mathscr{G}$ and let $S_{\mathscr{G}}$ be the space of all units in $\mathscr{G}$. We shall say that $\mathscr{G}$, C is an *ergodic groupoid* if the following conditions are satisfied.

(i) The domain $\mathscr{D}$ of $x, y \to xy$ is a Borel subset of $\mathscr{G}$.

(ii) $x, y \to xy$ is a Borel function from $\mathscr{D}$ to $\mathscr{G}$.

(iii) $x \to x^{-1}$ is a Borel function from $\mathscr{G}$ to $\mathscr{G}$.

(iv) C is invariant under $x \to x^{-1}$.

(v) If T_x is the one-to-one map $y \to xy$ from $\sigma^{-1}(x^{-1}x)$ to $\sigma^{-1}(xx^{-1})$ and $\tilde{C}$, $\{C_s\}$ is the decomposition of C defined by σ then there exists a $\tilde{C}$ null set N in $S_{\mathscr{G}}$ such that for all x with $xx^{-1} \notin N$ and $x^{-1}x \notin N$ the map T_x carries $C_{x^{-1}x}$ onto $C_{xx^{-1}}$.

(vi) For every Borel function f on $S_{\mathscr{G}}$, $f(x^{-1}x) = f(xx^{-1})$ for almost all x in $\mathscr{G}$ implies that f is almost everywhere constant.

To complete our program and define a virtual group as a similarity class of ergodic groupoids we must show that the notion of homomorphism introduced in § 9 may be expressed in terms of the groupoid structure of $S \times G$ in such a way as to make sense in the general case. Now it is natural to define a homomorphism of a groupoid $\mathscr{G}_1$ into a groupoid $\mathscr{G}_2$ to be a function ψ from $\mathscr{G}_1$ to $\mathscr{G}_2$ such that $\psi(x_1)\,\psi(x_2)$ is defined and equal to $\psi(x_1 x_2)$ whenever $x_1 x_2$ is defined. When $\mathscr{G}_1$ and $\mathscr{G}_2$ are of the form $S_1 \times G_1$ and $S_2 \times G_2$ respectively where S_j is a G_j space and G_j is a group we may write $\psi(s, x) = a(s, x)$, $\pi(s, x)$ where $a(s, x) \in S_2$, $\pi(s, x) \in G_2$. Moreover a straightforward calculation shows that the pair a, π defines a homomorphism in the indicated sense if and only if π is a one cocycle, a depends only on s and $a(sx, e) \equiv a(s, e)\,\pi(s, x)$. In other words the pairs π, φ, equivalence classes of which were used in § 9 to define virtual group homomorphisms, are precisely those Borel functions from $S_1 \times G_1$ to $S_2 \times G_2$ which are groupoid homomorphisms. The condition that π_1, φ_1 and π_2, φ_2 define the same virtual group homomorphism may also be expressed in groupoid terms. When so expressed it says that there exists a Borel map θ from $S_{\mathscr{G}_1}$ to G_2 such that for all $s \in S_{\mathscr{G}_1}$ the left and right units of $\theta(s)$ are $\psi_1(s)$ and $\psi_2(s)$ respectively and $\theta(xx^{-1})\,\psi_1(x) = \psi_2(x)\,\theta(x^{-1}x)$ for all $x \in \mathscr{G}_1$. Here ψ_1 and ψ_2 are the groupoid homomorphisms defined by π_1, φ_1 and π_2, φ_2 respectively. We are indebted to Professor S. Eilenberg for pointing out that this relationship between ψ_1 and ψ_2 has a simple interpretation when $\mathscr{G}_1$ and $\mathscr{G}_2$ are regarded as categories. Indeed from the category point of view a groupoid homomorphism is a functor and the relationship in question is just what is known in category theory as "natural equivalence" of functors.

One is tempted to define a homomorphism from the ergodic groupoid $\mathscr{G}_1$, C_1 to the ergodic groupoid $\mathscr{G}_2$, C_2 to be a Borel function from $\mathscr{G}_1$ to $\mathscr{G}_2$

which is a groupoid homomorphism. However this is not enough. One needs a further restriction involving the measure classes C_1 and C_2 and of course a suitable "almost everywhere" relaxation has to be formulated. The actual definition depends upon two auxiliary notions which we now introduce. Let $\mathscr{G}$, C be an ergodic groupoid and let S_0 be a Borel subset of $S_{\mathscr{G}}$ such that $\sigma^{-1}(S_{\mathscr{G}} - S_0)$ is a C null set. Let $\mathscr{G} \restriction S_0$ be the set of all $x \in \mathscr{G}$ with $xx^{-1} \in S_0$ and $x^{-1}x \in S_0$. Then C restricted to $\mathscr{G} \restriction S_0$ makes $\mathscr{G} \restriction S_0$ into an ergodic groupoid which we shall call an *inessential contraction* of $\mathscr{G}$, C. Now let $\mathscr{G}_1$, C_1 and $\mathscr{G}_2$, C_2 be ergodic groupoids and let ψ be a Borel function from $\mathscr{G}_1$ to $\mathscr{G}_2$. We shall say that ψ is a *strict homomorphism* if (i) $\psi(z_1)\,\psi(z_2)$ is defined and equal to $\psi(z_1 z_2)$ whenever $z_1 z_2$ is defined and (ii) if $\tilde{\psi}$ is the restriction of ψ to $S_{\mathscr{G}_1}$ then $\tilde{\psi}^{-1}(E)$ is a $\tilde{C}_1$ null set whenever E is a $\tilde{C}_2$ null set which is not contained in some $S_{\mathscr{G}_2}$ equivalence class of positive measure. We remark that the definition of strict homomorphism given in [11] is not quite correct and should be replaced by the above. Finally we define a *homomorphism* of $\mathscr{G}_1$, C_1 into $\mathscr{G}_2$, C_2 to be a Borel function from $\mathscr{G}_1$ to $\mathscr{G}_2$ whose restriction to some inessential contraction is a strict homomorphism. We define two strict homomorphisms ψ_1 and ψ_2 to be *strictly similar* if there exists a Borel function θ from $S_{\mathscr{G}_1}$ to $\mathscr{G}_2$ such that for all $s \in S_{\mathscr{G}_1}$ the left and right units of $\theta(s)$ are $\psi_1(s)$ and $\psi_2(s)$ respectively and $\theta(xx^{-1})\,\psi_1(x) = \psi_2(x)\,\theta(x^{-1}x)$ for all $x \in \mathscr{G}_1$. We shall say that two homomorphisms are *similar* if they have strictly similar restrictions to a common inessential contraction of $\mathscr{G}_1$, C_1.

It is now easy to complete our program and define the notion of virtual group. Let ψ_1 and ψ_2 be homomorphisms of the ergodic groupoids $\mathscr{G}_1$, C_1 and $\mathscr{G}_2$, C_2 into one another. If $\psi_1 \circ \psi_2$ and $\psi_2 \circ \psi_1$ are each similar to the identity we shall say that $\mathscr{G}_1$, C_1 and $\mathscr{G}_2$, C_2 are similar ergodic groupoids. We define a *virtual group* to be a similarity class of ergodic groupoids.

12. Concluding remarks

The notion of homomorphism "commutes" with the notion of similarity in such a way that a similarity class of homomorphisms between ergodic groupoids may be regarded as a homomorphism between the corresponding virtual groups. Similarly one can define a notion of "homomorphism from a virtual group into a group." Let the separable locally compact group G act ergodically in S, C where C is an invariant measure class in the standard Borel G space S. Let $S \times G$, $C \times C_G$ be the corresponding ergodic groupoid. Then the mapping $s, x \to x$ defines a homomorphism θ of the virtual group containing $S \times G$, $C \times C_G$ into G. Moreover, as stated precisely in theorem 4 of [11], the given ergodic action is uniquely determined by the virtual group and its "imbedding" θ into G. In this way the problem of finding all ergodic actions of a given G breaks down into two problems:

(a) Find all virtual groups.

(b) Find all possible imbeddings of a given virtual group into G. It would be interesting to know whether this is a real division or whether one of problems (a) and (b) has a trivial solution.

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