Math. Ann. 223, 183—190 (1976) © by Springer-Verlag 1976

A Partial Solution of the Pompeiu Problem*

Stephen A. Williams
Mathematics Department, Wayne State University, Detroit, Mich. 48202, USA

A nonempty bounded open subset D of $\mathbb{R}^n$ is said to have the Pompeiu property if and only if for every continuous complex-valued function f on $\mathbb{R}^n$ which does not vanish identically there is a rigid motion σ of $\mathbb{R}^n$ onto itself – taking D onto $\sigma(D)$ – such that the integral of f over $\sigma(D)$ is not zero. This article gives a partial solution of the Pompeiu problem, the problem of finding all sets D with the Pompeiu property.

In the special case that D is the interior of a homeomorphic image of an $(n-1)$-dimensional sphere, the main result states that if D has a portion of an $(n-1)$-dimensional real analytic surface on its boundary, then either D has the Pompeiu property or any connected real analytic extension of the surface also lies on the boundary of D. Thus, for example, any such region D having a portion of a hyperplane as part of its boundary must have the Pompeiu property, since the entire hyperplane cannot lie in the boundary of the bounded set D.

The Pompeiu Property

It will be assumed throughout this paper that D is a nonempty bounded open subset of $\mathbb{R}^n$ with $n \geqq 2$. Let Σ denote the group of rigid motions of $\mathbb{R}^n$ onto itself. Thus Σ is generated by the translations and rotations of $\mathbb{R}^n$, and contains no reflection. We say that D has the *Pompeiu property* if and only if the only continuous complex-valued function f on $\mathbb{R}^n$ for which

$$\int_{\sigma(D)} f(x_1, x_2, \ldots, x_n)dx = 0 \quad \text{for every} \quad \sigma \in \Sigma$$

is the function $f \equiv 0$. Here $\sigma(D)$ denotes the image of D under the rigid motion σ. Here and for the remainder of the paper, $x = (x_1, x_2, \ldots, x_n)$ denotes the generic point of $\mathbb{R}^n$.

Sets not Having the Pompeiu Property

Sets constructed by the methods of this section are the only ones known by the author not to have the Pompeiu property.

* The research for this paper was done in part while on sabbatical at the Courant Institute of Mathematical Sciences, New York University.

A (solid open) ball of any radius $R>0$ fails to have the Pompeiu property. The corresponding function f may be taken to be $f(x_1, x_2, \ldots, x_n)=\sin(ax_1)$ for any $a>0$ satisfying $J_{n/2}(aR)=0$, where J_λ denotes the Bessel function of order λ. (In fact, the calculation of [1, pp. 141, 142] which proves this for $n=2$ can be generalized to the case $n\geqq 2$ using [2, p. 482, formula (19)], the identity $J_{\lambda-1}(z)z^\lambda=(d/dz)(z^\lambda J_\lambda(z))$, and the fact that $J_\lambda(z)/z^\lambda$ is an analytic function of z whose power series involves only even powers of z.) Thus for fixed $a>0$ there is an infinite sequence $0<R_1<R_2<\ldots$ such that the integral of $\sin(ax_1)$ over any ball of radius R_i $(i=1, 2, \ldots)$ is zero. (For information about zeroes of Bessel functions sufficient to give this, see [2, pp. 416, 496].)

Clearly if $B_1, B_2, \ldots, B_N$ are disjoint balls with radii in this sequence, all contained in a ball B with radius in this sequence, then the region $B\sim(\bar{B}_1\cup\bar{B}_2\cup\ldots\cup\bar{B}_N)$ fails to have the Pompeiu property with $\sin(ax_1)$ a corresponding f. Clearly disjoint unions of such regions fail to have the Pompeiu property again with $\sin(ax_1)$ a corresponding f.

Federbush gave a more complicated example: let R_i and R_j be as above with $R_i<R_j$; consider two balls B_1 and B_2 of radius R_j whose boundaries intersect in an $(n-2)$-sphere of radius R_i; let B be the unique ball of radius R_i with that $(n-2)$-sphere on its boundary; then the set whose characteristic function is $\chi_{B_1}+\chi_{B_2}-\chi_B$ clearly fails to have the Pompeiu property with $\sin(ax_1)$ a corresponding f. (Here and for the remainder of the paper, for any set W, χ_W denotes the characteristic function of W.)

Finally, if a (relatively) closed set of measure zero is deleted from a set failing to have the Pompeiu property, obviously the resulting set also fails to have the Pompeiu property.

Sets Having the Pompeiu Property

The sets that will be discussed in this section are the only ones known to the author that previous literature has proved to have the Pompeiu property.

In [1, Theorem 5.11, p. 150] it is proved in the case $n=2$ that if there is an open half-plane H whose intersection with ∂D, the boundary of D, is a single Lipschitz curve, if there is a unique point $\boldsymbol{p}$ on this curve of maximal distance from ∂H, and if one-sided tangent rays to the curve exist at $\boldsymbol{p}$, do not coincide, and intersect ∂H, then D has the Pompeiu property. Thus, roughly, a planar region has the Pompeiu property if it has a "corner" that "sticks out" from the rest of the region. As a very special case, the interior of any simple closed polygon in the plane has the Pompeiu property. Theorem 5.11 of [1] and its proof generalize directly to the case $n\geqq 2$.

In [1, Theorem 5.1, p. 143] it is proved in the case $n=2$ that any ellipse has the Pompeiu property. This theorem and its proof generalize easily to ellipsoids for $n\geqq 2$. (Here we use the generalization of the calculation of [1, pp. 141, 142] discussed above.)

Finally, if D has the Pompeiu property, then any bounded open set differing from it by a set of measure zero clearly also has the Pompeiu property.

A Conjecture

Conjecture. *If ∂D is homeomorphic to the unit sphere in $\mathbb{R}^n$, then D has the Pompeiu property if and only if it is not a ball.*

This conjecture (which the author believes quite likely to be true) is consistent with the known facts and indicates the type of theorem one would hope to be able to prove eventually in this area.

If ∂D is homeomorphic to the unit sphere in $\mathbb{R}^n$ and fails to have the Pompeiu property, then Theorems 1 and 2 below show the existence of a function T and a complex number $\alpha \neq 0$ such that $\Delta T + \alpha T = -1$ on D, with $T=0$ and $\partial T/\partial n = 0$ on ∂D. (If ∂D is smooth, it can be shown that α is real, so that T may be taken to be real-valued.) In view of this, the conjecture is very closely related to the result of James Serrin [10] that if Ω is a bounded open connected set with smooth boundary $\partial\Omega$ on which there exists a function u satisfying $\Delta u = -1$ on Ω, with $u=0$ and $\partial u/\partial n = \text{constant}$ on $\partial\Omega$, then Ω must be a ball.

Notation, Definitions, and Preliminary Results

Let $\mathscr{E}'(\mathbb{R}^n)$ denote the set of distributions of compact support on $\mathbb{R}^n$. Let $\mathscr{E}(\mathbb{R}^n)$ denote the set of all infinitely-differentiable complex-valued functions on $\mathbb{R}^n$. For $f \in \mathscr{E}(\mathbb{R}^n)$ and $T \in \mathscr{E}'(\mathbb{R}^n)$, let $T(f(x_1, x_2, \ldots, x_n))$ denote the complex number which results from applying the distribution T to the function $f = f(x_1, x_2, \ldots, x_n)$. For any T in $\mathscr{E}'(\mathbb{R}^n)$, the Fourier-Laplace transform $T\hat{}$ of T is defined by

$$(T\hat{})(z_1, z_2, \ldots, z_n) = T(e^{i(z_1x_1 + z_2x_2 + \ldots + z_nx_n)})$$

for all $z_1, z_2, \ldots, z_n$ in $\mathbb{C}$. If g is a Lebesgue-integrable function of compact support in $\mathbb{R}^n$, then g may be considered an element of $\mathscr{E}'(\mathbb{R}^n)$ by defining

$$g(f(x_1, x_2, \ldots, x_n)) = \int_{\mathbb{R}^n} f(x)g(x)dx$$

for every f in $\mathscr{E}(\mathbb{R}^n)$. The proof in [1, Theorem 4.1, p. 136] in the case $n=2$ has a straightforward extension to the case $n \geqq 2$ which states that D *fails* to have the Pompeiu property if and only if there is an $\alpha \in \mathbb{C}$ with $\alpha \neq 0$ such that $(\chi_D\hat{})(z_1, z_2, \ldots, z_n) \equiv 0$ on the set $M_\alpha = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n; z_1^2 + z_2^2 + \ldots + z_n^2 = \alpha\}$. (An easy calculation shows that if r is the reflection of $\mathbb{R}^n$ about any hyperplane in $\mathbb{R}^n$ and if $r(D)$ denotes the image of D under this reflection, then $\chi_D\hat{} \equiv 0$ on M_α if and only if $\chi_{r(D)}\hat{} \equiv 0$ on M_α. Thus if the definition of the Pompeiu property is modified by allowing reflections in Σ, we obtain a new definition equivalent to the original.)

The author is indebted to B. Schreiber for informing him of the following theorem and to L. Brown, B. Schreiber, and B. A. Taylor for providing a detailed proof:

Theorem 1. *The set D fails to have the Pompeiu property if and only if there is an $\alpha \in \mathbb{C}$ with $\alpha \neq 0$ and a $T \in \mathscr{E}'(\mathbb{R}^n)$ such that*

$$\Delta T + \alpha T = -\chi_D .$$

Proof. By the Paley-Wiener-Schwartz theorem [3, Theorem 1.7.7, p. 21], $\{W\hat{}; W \in \mathscr{E}'(\mathbb{R}^n)\}$ is the set of all entire functions F on $\mathbb{C}^n$ such that for some real constants A, C, and N we have $|F(z)| \leqq C(1+|z|)^N e^{A|\operatorname{Im} z|}$ for all $z=(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, where $|z|=[|z_1|^2+|z_2|^2+\ldots+|z_n|^2]^{1/2}$ and $|\operatorname{Im} z|=[|\operatorname{Im} z_1|^2+|\operatorname{Im} z_2|^2+\ldots+|\operatorname{Im} z_n|^2]^{1/2}$. The map $W \mapsto W\hat{}$ of $\mathscr{E}'(\mathbb{R}^n)$ onto $\{W\hat{};\ W \in \mathscr{E}'(\mathbb{R}^n)\}$ is one-to-one.

Suppose that D fails to have the Pompeiu property. Then there is an $\alpha \in \mathbb{C}$ with $\alpha \neq 0$ such that $\chi_D\hat{} \equiv 0$ on M_α. We claim that there exists a unique entire function F on $\mathbb{C}^n$ such that $\chi_D\hat{}(z)=(z_1{}^2+z_2{}^2+\ldots+z_n{}^2-\alpha)F(z)$ for every $z=(z_1, z_2, \ldots, z_n)$ in $\mathbb{C}^n$. Clearly this is only a local question. Since $\chi_D\hat{}(z)$ is entire, the claim is obviously true for points not on M_α. Let z^0 be any point of M_α and consider germs of holomorphic functions about that point. Since $z_1{}^2+z_2{}^2+\ldots+z_n{}^2-\alpha$ is irreducible as a polynomial in z_n, it is also irreducible as a holomorphic function about z^0 by [4, Lemma 5, p. 71]. The fact that F is holomorphic on some neighborhood of z^0 then follows from the *Nullstellensatz* for principal ideals [4, Theorem 18, p. 90]. Since z^0 was an arbitrary point of M_α, F is entire.

Next we claim that $F=T\hat{}$ for some $T \in \mathscr{E}'(\mathbb{R}^n)$. By the Paley-Wiener-Schwartz theorem there are real constants A, C, and N such that $|\chi_D\hat{}(z)| \leqq C(1+|z|)^N e^{A|\operatorname{Im} z|}$ for all z in $\mathbb{C}^n$. Clearly for z in the set $G \equiv \{z \in \mathbb{C}^n;\ |z_1{}^2+z_2{}^2+\ldots+z_n{}^2-\alpha| \geqq 1\}$ we have $|F(z)| \leqq C(1+|z|)^N e^{A|\operatorname{Im} z|}$. Now consider any $z^0=(z^0{}_1, z^0{}_2, \ldots, z^0{}_n)$ in $\mathbb{C}^n \sim G$. Let $H=\{z \in \mathbb{C}^n;\ z_1=z^0{}_1, z_2=z^0{}_2, \ldots, z_{n-1}=z^0{}_{n-1}\}$. Let $a+bi$ be either square root of $\alpha-(z^0{}_1)^2-(z^0{}_2)^2-\ldots-(z^0{}_{n-1})^2$. Since $|z_n{}^2-(a+bi)^2|=|z_n-(a+bi)|\,|z_n-(-a-bi)|$, if $z \in H \sim G$, then either z_n is less than one unit from $a+bi$ or z_n is less than one unit from $-a-bi$. Thus there is an r with $0<r<4$ such that for all z in H with $|z_n-z^0{}_n|=r$ we have $z \in G$. Since F is an analytic function of z_n on H we have by the maximum modulus theorem that

$$|F(z^0)| \leqq \max\{|F(z)|;\ z \in H \quad \text{and} \quad |z_n-z^0{}_n|=r\}.$$

But for $z \in H$ with $|z_n-z^0{}_n|=r$ we have $|z-z^0| \leqq 4$, so

$$|F(z)| \leqq C(5+|z^0|)^N e^{A|\operatorname{Im} z^0|+4A}$$

so

$$|F(z^0)| \leqq C 5^N e^{4A}(1+|z^0|)^N e^{A|\operatorname{Im} z^0|}.$$

Thus by the Paley-Wiener-Schwartz theorem there is a $T \in \mathscr{E}'(\mathbb{R}^n)$ such that $T\hat{}=F$.

Finally we note that

$$\begin{aligned}(\Delta T+\alpha T)\hat{}(z) &= (-z_1{}^2-z_2{}^2-\ldots-z_n{}^2+\alpha)(T\hat{}(z)) \\ &= (z_1{}^2+z_2{}^2+\ldots+z_n{}^2-\alpha)(-F(z)) \\ &= -\chi_D\hat{}(z).\end{aligned}$$

Since $W \mapsto W\hat{}$ is one-to-one, we have $\Delta T+\alpha T=-\chi_D$.

Conversely, if there is an $\alpha \in \mathbb{C}$ with $\alpha \neq 0$ and a $T \in \mathscr{E}'(\mathbb{R}^n)$ such that $\Delta T+\alpha T=-\chi_D$, then $\chi_D\hat{}(z)=(-\Delta T-\alpha T)\hat{}(z)=(z_1{}^2+z_2{}^2+\ldots+z_n{}^2-\alpha)(T\hat{}(z))$ which obviously vanishes on M_α, so that D fails to have the Pompeiu property. This completes the proof.

Theorem 2. *Any solution $T \in \mathscr{E}'(\mathbb{R}^n)$ of $\Delta T+\alpha T=-\chi_D$ for $\alpha \in \mathbb{C}$ and $\alpha \neq 0$ is a* function *of compact support. This function (after redefinition, if necessary, on a set of measure zero) is given by*

$$T(x)=-\int_D \gamma(|x-y|)\,dy\,, \tag{1}$$

where $y=(y_1, y_2, \ldots, y_n)$, $|x-y|=[(x_1-y_1)^2+(x_2-y_2)^2+\ldots+(x_n-y_n)^2]^{1/2}$, $\sqrt{\alpha}$ *is either square root of* α, *and for* $r>0$,

$$\gamma(r)=\frac{[\sqrt{\alpha}]^{(n-2)/2}}{2^{n/2+1}\pi^{n/2-1}}\, r^{-(n-2)/2} N_{(n-2)/2}(\sqrt{\alpha}\, r), \tag{2}$$

where N_β *is the Neumann function of order* β.

The function T *is real analytic in* D, *where it satisfies* $\Delta T+\alpha T=-1$ *in the classical sense. It is also real analytic in the complement of* $\bar{D}$, *where it satisfies* $\Delta T+\alpha T=0$ *in the classical sense. It vanishes identically on the unique unbounded component of the complement of* $\bar{D}$.

The function T *is in* $C^1(\mathbb{R}^n)$, *and first order derivatives are given by differentiation of* (1) *under the integral sign.*

Proof. By [5, p. 193], any $T\in\mathscr{E}'(\mathbb{R}^n)$ which solves $\Delta T+\alpha T=-\chi_D$ is given by $T=\gamma\star(-\chi_D)$, where γ is a fundamental solution of the operator $\Delta+\alpha$ and where $\star$ denotes convolution. By [5, p. 259] with $C=0$ and $\sqrt{\lambda}=-i\sqrt{\alpha}$, γ can be taken to be the function given by (2) above, interpreting r as $|x|$.

By [6, Corollary 4, p. 1708], since $-\chi_D$ is infinitely differentiable on D and on $\sim\bar{D}$ (the complement of $\bar{D}$), since $\Delta+\alpha$ is an elliptic formal partial differential operator, and since T is a distribution with $(\Delta+\alpha)T=-\chi_D$ on D and on $\sim\bar{D}$, T is (after redefinition if necessary on a set of measure zero) an infinitely-differentiable function on D and $\sim\bar{D}$. That T satisfies $\Delta T+\alpha T=0$ on $\sim\bar{D}$ and $\Delta T+\alpha T=-1$ on D in the classical sense then follows from $\Delta T+\alpha T=-\chi_D$ together with [6, Lemma 6, p. 1647]. By [7], T is (real) analytic on D and on $\sim\bar{D}$. Since T is of compact support, it vanishes identically in some neighborhood of infinity and hence, by the uniqueness of analytic continuation, vanishes on the unique unbounded component of $\sim\bar{D}$.

It remains to prove that T is in $C^1(\mathbb{R}^n)$, with first-order partial derivatives computable by differentiation of (1) under the integral sign. Define

$$T_{x_i}(x)=-\int_D \gamma'(|x-y|)\frac{x_i-y_i}{|x-y|}\,dy, \tag{3}$$

for $1\leq i\leq n$. From [2, pp. 484, 496, 500, 501] we have that $\gamma(r)=A_n(r)r^{2-n}$ for odd n, $\gamma(r)=A_2(r)\ln r+B_2(r)$ for $n=2$, and $\gamma(r)=A_n(r)\ln r+B_n(r)+C_n(r)r^{2-n}$ for n even with $n>2$, where for any n the functions $A_n(r)$, $B_n(r)$ and $C_n(r)$ are entire analytic functions of r. By differentiating these expressions, we get corresponding formulas for $\gamma'(r)$. For any $R>0$ we can get upper bounds for $|\gamma(r)|$ and $|\gamma'(r)|$, valid for $0<r\leq R$, by replacing $A_n(r)$, $A_n'(r)$, $B_n(r)$, $B_n'(r)$, $C_n(r)$, and $C_n'(r)$ by the maxima of their absolute values on $[0, R]$ and replacing $\ln r$ by $|\ln r|$. Fix any point x^0 in $\mathbb{R}^n$, choose R so large that the ball of radius $R-1>0$ centered at x^0 contains the set D. Using the upper bounds just derived for $|\gamma(r)|$ and $|\gamma'(r)|$, and doing the integration with these upper bounds using spherical coordinates with origin at x, for any x in $\mathbb{R}^n$ with $|x-x^0|<1$, proves that the integrals of (1) and (3) for $T(x)$ and $T_{x_i}(x)$ converge and that the contributions to these integrals from any subset of $B_\zeta(x)\cap D$ (here $B_\zeta(x)$ denotes the open ball of radius $\zeta>0$ centered at x) can be made small (uniformly for $|x-x^0|<1$) by taking $\zeta>0$ small. Let $\varepsilon>0$ be given. Choose ζ with $0<\zeta<1$ so that the contributions to the integrals of (1) and (3) from any subset of $B_\zeta(x)\cap D$ are in absolute value less than $\varepsilon/3$. For $|x-x^0|<\zeta/4$ we then have

$$|T(x)-T(x^0)|\leqq \left|\int_{D\sim B_{\zeta/2}(x^0)} \gamma(|x-y|)-\gamma(|x^0-y|)dy\right|+2\varepsilon/3,$$

with a similar formula for $T_{x_i}(1\leqq i\leqq n)$. By the uniform continuity in (x, y) of $\gamma(|x-y|)$ for $y\in D\sim B_{\zeta/2}(x^0)$ and $|x-x^0|<\zeta/4$, the integral on the right-hand side of the above inequality may be made smaller than $\varepsilon/3$ by taking x sufficiently close to x^0. The same may be done for the corresponding integral for T_{x_i} using the uniform continuity in (x, y) of $\gamma'(|x-y|)(x_i-y_i)|x-y|^{-1}$ for $y\in D\sim B_{\zeta/2}(x^0)$ and $|x-x^0|<\zeta/4$. Thus T and $T_{x_i}(1\leqq i\leqq n)$ are continuous at x^0. Since x^0 was arbitrary, T and $T_{x_i}(1\leqq i\leqq n)$ are continuous on $\mathbb{R}^n$.

It remains to show that $\partial T/\partial x_i=T_{x_i}$ for $1\leqq i\leqq n$. Denoting the i-th coordinate unit vector by e_i we have by the Fubini theorem that for given $\tau_0<\tau_1$ and $1\leqq i\leqq n$ we have

$$\int_{\tau_0}^{\tau_1} -\int_D \gamma'(|x+\tau e_i-y|)\frac{x_i+\tau-y_i}{|x+\tau e_i-y|}dyd\tau=T(x+\tau_1 e_i)-T(x+\tau_0 e_i).$$

Since for fixed x in $\mathbb{R}^n$ and $1\leqq i\leqq n$ the function $T_{x_i}(x+\tau e_i)$ is continuous in τ, this last equality shows by the fundamental theorem of calculus that $(d/d\tau)(T(x+\tau e_i))=T_{x_i}(x+\tau e_i)$. Thus $\partial T/\partial x_i=T_{x_i}$ as required.

This completes the proof.

The Main Result

Definition. *By the* outer boundary *of D we will mean that subset of ∂D which is in the closure of the unbounded component of $\sim\bar{D}$. We will denote it by $\partial^* D$.*

Since the function T of Theorem 2 vanishes identically on the unbounded component of $\sim\bar{D}$ and is in $C^1(\mathbb{R}^n)$, T and ∇T are zero on $\partial^* D$.

Note that by the Jordan-Brouwer separation theorem [8, Theorem 15, p. 198], if a set B in $\mathbb{R}^n$ is homeomorphic to the unit sphere in $\mathbb{R}^n$ and if we take D to be the unique bounded component of $\mathbb{R}^n\sim B$, then $\partial^* D=B=\partial D$.

Definition. *A subset S of $\mathbb{R}^n$ is an $(n-1)$*-dimensional real analytic surface *if and only if S is nonempty and if for every point x in S, there is a real analytic one-to-one map of the open unit ball $B_1(0)$ in $\mathbb{R}^n$ onto an open neighborhood J of x such that $B_1(0)\cap\{x\in\mathbb{R}^n; x_n=0\}$ maps onto $J\cap S$.*

The author would like to thank J. Ralston for two important suggestions that evolved rapidly into the following theorem:

Theorem 3. *If D fails to have the Pompeiu property, if an $(n-1)$-dimensional real analytic surface S is contained in $\partial^* D$, if there is a point y^0 in S and a $\delta>0$ such that $y\in B_\delta(y^0)\cap\partial D$ implies that $y\in S$, if W is a connected $(n-1)$-dimensional real analytic surface with $S\subseteq W$, and if the distance between W and $\partial D\sim(\partial^* D)$ is positive, then $W\subseteq\partial^* D$.*

Note. It may be helpful to the reader, in order to understand the statement of the above theorem, to see what it asserts about the set of the Federbush example.

Proof. At each point x in W, use the Cauchy-Kowalewsky theorem [9, pp. 39, 40] to solve the equation $\Delta U+\alpha U=-1$ on some open neighborhood N_x of x subject to the initial data $U=0$ and $\partial U/\partial n=0$ on $W\cap N_x$ (here $\partial U/\partial n$ denotes the normal derivative of U). Using the Holmgren uniqueness theorem [9, p. 238] and the uniqueness of real analytic continuation, we may piece together these local solutions to obtain a real analytic function U defined on an open set N containing W, with $\Delta U+\alpha U=-1$ on N and with $U=0$ and $\partial U/\partial n=0$ on W.

Fix any point x of W. Since $\Delta U(x)+\alpha U(x)=-1$ and $U(x)=0$, there is an i with $1\leqq i\leqq n$ such that $(\partial^2 U/\partial x_i^2)(x)\neq 0$. Thus by the implicit function theorem, $\partial U/\partial x_i=0$ defines the graph in a sufficiently small neighborhood of x of a continuously differentiable function of $(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$. But $\nabla U=\mathbf{0}$ and hence $\partial U/\partial x_i=0$ on W. Thus the points of W near x are in the graph of this function (therefore the hyperplane tangent to W at x is the hyperplane tangent to the graph of the function at x, which is known by the implicit function theorem not to be parallel to the x_i-axis). Thus there is an open neighborhood of x such that if $(\nabla U)(y)=\mathbf{0}$ for a point y of that neighborhood, then $y\in W$. Thus there is an open set $\tilde{N}$ with $W\subseteqq\tilde{N}\subseteqq N$ such that $(\nabla U)(y)=\mathbf{0}$ and $y\in\tilde{N}$ imply that $y\in W$.

Let y^0 be the point of S mentioned in the statement of Theorem 3. Let ϕ be a real analytic one-to-one map of $B_1(\mathbf{0})$ onto an open neighborhood J of y^0 such that $J\subseteqq B_\delta(y^0)$ and $B_1(\mathbf{0})\cap\{x\in\mathbb{R}^n; x_n=0\}$ maps onto $J\cap S$. Since $y\in B_\delta(y^0)\cap\partial D$ implies that $y\in S$, and since $y^0\in\partial^* D$ with D open, either $\phi(\{x\in\mathbb{R}^n; x_n>0\})\subseteqq D$ and $\phi(\{x\in\mathbb{R}^n; x_n<0\})\subseteqq\mathbb{R}^n\sim\bar{D}$ or vice versa. Since D fails to have the Pompeiu property, Theorems 1 and 2 give the existence of a function T as in Theorem 2. Since $S\subseteqq\partial^* D$, $T=0$ and $\nabla T=\mathbf{0}$ on S while $\Delta T+\alpha T=-1$ on D. Assume for definiteness that it is $\phi(\{x\in\mathbb{R}^n; x_n>0\})$ which is contained in D. By the Holmgren uniqueness theorem [9, p. 238] there is a $\lambda>0$ such that $T(x)=U(x)$ for all $x\in G_\lambda\equiv B_\lambda(y^0)\cap\phi(\{x\in\mathbb{R}^n; x_n>0\})$. Clearly we may take $\lambda>0$ so small that G_λ is connected. By the uniqueness of analytic continuation we have $T\equiv U$ on that component C of $D\cap\tilde{N}$ which contains G_λ.

Let $W^*=W\cap\partial^* D\cap\bar{C}$. Since $y^0\in W^*$, W^* is not empty. Since $\partial^* D\cap\bar{C}$ is closed in $\mathbb{R}^n$, W^* is closed in the relative topology of W. Since W is connected, once it is proved that W^* is open relative to W we have $W^*=W$, so that $W\subseteqq\partial^* D$ and the theorem is proved. Let x^0 be any point of W^*. Let $\varepsilon>0$ be less than the distance from W to $\partial D\sim(\partial^* D)$; we may assume, taking $\varepsilon>0$ smaller if necessary, that $B_\varepsilon(x^0)\subseteqq\tilde{N}$ and that $B_\varepsilon(x^0)\sim W$ consists of precisely two components, C_1 and C_2. Since C is open and $x^0\in\bar{C}$, C has nonempty intersection with one of these components, say C_1. We claim now that $C_1\subseteqq C$. If not, then there would be an $x\in C_1$ with $x\in\partial C$. Since $C_1\subseteqq\tilde{N}$, x would have to be in ∂D. By the choice of ε and since $C_1\subseteqq B_\varepsilon(x^0)$, x would have to be in $\partial^* D$. Thus $\nabla T(x)=\mathbf{0}$. But $T\equiv U$ on C, and both T and U are in $C^1(C_1)$, so also $\nabla U(x)=\mathbf{0}$. Since $x\in\tilde{N}$ we have $x\in W$, a contradiction of the fact that $x\in C_1$. Therefore the claim that $C_1\subseteqq C$ is established. We claim now that $C_2\cap C$ is empty. If not, then by the argument just given we would also have $C_2\subseteqq C$, so that $C_1\cup C_2\subseteqq D$ and thus $x^0\notin\partial(\mathbb{R}^n\sim\bar{D})$, contradicting the fact that $x^0\in\partial^* D$. Thus $C_2\cap C$ is empty. If any point x of $B_\varepsilon(x^0)\cap W$ were in D, then an open neighborhood of that point would be in $D\cap\tilde{N}$, and $C_2\cap C$ would not be empty. Thus $B_\varepsilon(x^0)\cap W\subseteqq\partial D$ and hence $B_\varepsilon(x^0)\cap W\subseteqq\partial^* D$. Thus $B_\varepsilon(x^0)\cap W\subseteqq W\cap\partial^* D\cap\bar{C}=W^*$. Since x^0 was an arbitrary point of W^*, this proves that W^* is open relative to W and completes the proof of the theorem.

References

1. Brown, L., Schreiber, B., Taylor, B. A.: Spectral synthesis and the Pompeiu problem. Ann. Inst. Fourier, Grenoble, **23**, 125—154 (1973)
2. Courant, R., Hilbert, D.: Methods of mathematical physics. Vol. I. New York: Interscience 1953
3. Hörmander, L.: Linear partial differential operators. Berlin, Heidelberg, New York: Springer 1969
4. Gunning, R., Rossi, H.: Analytic functions of several complex variables. Englewood Cliffs, N. J.: Prentice-Hall 1965
5. Trèves, F.: Linear partial differential equations with constant coefficients. New York: Gordon and Breach 1966
6. Dunford, N., Schwartz, J.: Linear operators. Vol. II. New York: Interscience 1963
7. Morrey, C. B., Jr.: On the analyticity of the solutions of analytic non-linear elliptic systems of partial differential equations. Part I. Amer. J. of Math. **80**, 198—218 (1958)
8. Spanier, E.: Algebraic topology. New York: McGraw-Hill 1966
9. Courant, R., Hilbert, D.: Methods of mathematical physics. Vol. II. New York: Interscience 1962
10. Serrin, J.: A symmetry problem in potential theory. Arch. Rat. Mech. Anal. **43**, 304—318 (1971)

Received February 18, 1976