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Some Properties of Stable Rank-2 Vector Bundles on $\mathbb{P}_n$

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0. Introduction

0.1. At the moment little is known about the classification of algebraic vector bundles on $\mathbb{P}_n$. This paper will not improve much upon this situation, although it presents some properties of *stable* rank-2 bundles over $\mathbb{P}_n$, which might be useful for classification purposes, at least in the case of this kind of bundles.

The ground field will always be $\mathbb{C}$. Rank-r bundle will always mean *algebraic* $\mathbb{C}^r$-bundle. The definition of stability to be used is due to Mumford and Takemoto [13]: A rank-2 vector bundle is stable (resp. semistable) if $c_1(V) < 2c_1(\mathcal{Q})$ for every torsion-free rank-1 quotient sheaf $\mathcal{Q}$ of V (resp. $\leqq$). Stability of V on $\mathbb{P}_n$ is equivalent with $\mathrm{End}(V) \simeq \mathbb{C}$.

0.2. The following is the basic technical tool (if one wants to describe vector bundles on $\mathbb{P}_n$ and not to use resolutions): Whenever V is a rank-2 bundle on $\mathbb{P}_n$ and $L \subset \mathbb{P}_n$ is a line, then by Grothendieck's theorem [3, p. 126], the restriction $V|L$ splits

$$V|L \simeq \mathcal{O}_L(k_1) \oplus \mathcal{O}_L(k_2), \qquad k_1 + k_2 = c_1(V).$$

The integers k_1 and k_2 are uniquely determined by $V|L$. Put

$$d(V|L) := |k_1 - k_2|$$

$$d(V) := \min d(V|L), \qquad L \subset \mathbb{P}_n.$$

Schwarzenberger [11, §8] constructed quite a lot of bundles on $\mathbb{P}_2$ with $d(V)$ arbitrarily large, but all of them are unstable. The reason for this fact was recently given by Grauert and Mülich [2, §6, Satz 2]. Their result can be formulated as follows.

Theorem 1 (Grauert-Mülich). *If V is a stable (or even semi-stable)* rank-2 *bundle on $\mathbb{P}_n$, then*

$$d(V) = 0 \quad \textit{for} \quad c_1(V) \textit{ even}$$

$$d(V) = 1 \quad \textit{for} \quad c_1(V) \textit{ odd}.$$

Since this result is quite basic for my purposes, and since Grauert and Mülich formulated their theorem in a less general way, for the convenience of the reader I shall include here a somewhat simplified version of their proof.

0.3. It is a consequence of the semi-continuity theorems for proper flat morphisms, that $d(V|L)=d(V)$ for all lines L parametrized by some Zariski-open set in the Grassmannian $\mathrm{Gr}(1,n)$. Let me call *jumping lines* those L for which $d(V|L)>d(V)$. If $d(V)=0$, these lines form a divisor S on $\mathrm{Gr}(1,n)$. A consequence of Theorem 1 is

Theorem 2. *Let V be a semi-stable* rank-2 *bundle over $\mathbb{P}_n$ with even first Chern-class $c_1(V)$. The divisor $S\subset\mathrm{Gr}(1,n)$ of jumping lines then has degree $-\Delta(V)/4$, where*

$$\Delta(V)=c_1(V)^2-4c_2(V)$$

is the discriminant of the bundle V.

It is a byproduct of the proof of this theorem, that this divisor S depends algebraically on V. This becomes particularly interesting in the case of $\mathbb{P}_2$, where Maruyama [9] established the existence of coarse moduli-schemes $M(\Delta)$ for stable vector bundles with fixed Chern classes. For even c_1, the map $V\mapsto S$ defines a morphism

$$M(\Delta)\to\mathbb{P}_N, \quad N=\binom{2+\Delta_0}{2}-1, \quad \Delta_0=-\Delta/4.$$

This morphism cannot be surjective if $-\Delta\geqq 16$, since $M(\Delta)$ is a manifold of dimension $-\Delta-3$. Although it is not injective either in general, one may hope that this morphism bears some significance.

A similar statement for odd Chern classes fails, because the jumping lines in $\mathrm{Gr}(1,n)$ may then lie in codimension 1 and/or 2.

0.4. Stability of rank-2 bundles V is preserved under

(1) small deformations of V;

(2) lifting V via ramified coverings $\pi:\mathbb{P}_n\to\mathbb{P}_n$;

(3) extending V from $\mathbb{P}_n$ to $\mathbb{P}_{n+1}$ (if this is possible);

(4) restricting V to general hyperplanes $\mathbb{P}_{n-1}\subset\mathbb{P}_n$, if $n\geqq 4$.

Invariance of stability under (1), (2), (3) is quite obvious. In fact, invariance (1) should be viewed as part of any reasonable definition of stability. Nevertheless, (2) combined with the theorem of Grauert-Mülich, describes the splitting of V on the general (not necessarily linear) rational curve in $\mathbb{P}_n$. (This is made precise in 5.2.)

Invariance (4) however seems quite interesting to me, mainly because it fails for $n=3$, and then for one stable bundle only. The precise formulation is

Theorem 3. *Let V be some stable* rank-2 *bundle on $\mathbb{P}_n$. If $n\geqq 4$, then there is an open subset $\emptyset\neq U\subset\mathbb{P}_n^*$, such that the restriction of V to all hyperplanes parametrized by U is stable again. The same holds for $n=3$, unless $V=V_0$ is a null-correlation bundle.*

A null-correlation bundle V_0 is homogeneous under the complex symplectic group $\mathrm{Sp}(2,\mathbb{C})\subset\mathrm{GL}(4,\mathbb{C})$, V_0 is uniquely determined up to tensoring by line bundles and up to automorphisms of $\mathbb{P}_3$. The restriction of V_0 to any plane in $\mathbb{P}_3$ is semi-stable. From Theorem 3 one obtains the following corollaries.

Corollary 1. *If V is a stable* rank-2 *bundle on* $\mathbb{P}_n$, *then necessarily* $\Delta(V)<0$. (For $n=2$ this was observed by Schwarzenberger [11, Theorem 10].)

Corollary 2. *If V, W are stable* rank-2 *bundles on* $\mathbb{P}_n$, $n \geqq 4$, *which become isomorphic when restricted to all hyperplanes in* $\mathbb{P}_n$ *parametrized by some open* $\emptyset \neq U \subset \mathbb{P}_n^*$, *then* $V \simeq W$. (For $n=2$ this is not true, and for $n=3$, I do not know.)

0.5. The reader will notice that most of the methods in this paper are generalizations of ideas from Van de Ven's article [14].

Additionally many conversations with Van de Ven had on this paper an influence for which I am very grateful. I also should like to thank H. Grauert for explaining to me the details from [2].

2. Preliminaries

2.1. *Notation.* I do not want to distinguish between a vector bundle and its associated locally free sheaf of sections. A subsheaf however is not necessarily a subbundle (=subsheaf of a locally free sheaf, which locally is a direct summand).

A *decomposable* bundle is a direct sum of line bundles. The hyperplane bundle on $\mathbb{P}_n$ is denoted by $\mathcal{O}_{\mathbb{P}}(1)$. For any $\mathcal{O}_{\mathbb{P}}$-sheaf $\mathcal{F}$ one puts $\mathcal{F}(k)=\mathcal{F}\otimes_{\mathcal{O}_{\mathbb{P}}}\mathcal{O}_{\mathbb{P}}(1)^{\otimes k}$, $k\in\mathbb{Z}$. In particular $\mathcal{O}_{\mathbb{P}}(k)=\mathcal{O}_{\mathbb{P}}(1)^{\otimes k}$. The same notation can be applied to the Grassmann variety $\mathrm{Gr}(m,n)$ of projective m-planes $E\subset\mathbb{P}_n$. Indeed, $\mathrm{Pic}(\mathrm{Gr})\simeq\mathbb{Z}$, and we may denote the positive generator by $\mathcal{O}_{\mathrm{Gr}}(1)$. If one uses the Plücker-embedding $\mathrm{Gr}(m,n)\subset\mathbb{P}_N$, $N=\binom{n+1}{m+1}-1$, then $\mathcal{O}_{\mathrm{Gr}}(1)$ is the restriction of $\mathcal{O}_{\mathbb{P}_N}(1)$. The flag-variety $F=F(0,m,n)$ can be embedded in $\mathbb{P}_n\times\mathrm{Gr}(m,n)$ with canonical projections $p:F\to\mathbb{P}_n$, $q:F\to\mathrm{Gr}$. Sometimes I shall employ the notation $\mathcal{O}_F(k_1,k_2)$ for the bundle $p^*\mathcal{O}_{\mathbb{P}}(k_1)\otimes q^*\mathcal{O}_{\mathrm{Gr}}(k_2)$.

Since $H^2(\mathbb{P}_n,\mathbb{Z})=H^4(\mathbb{P}_n,\mathbb{Z})=\mathbb{Z}$, the Chern-classes $c_1(V)$ and $c_2(V)$ can be viewed as integers, which will be done throughout this article. The first Chern class without ambiguity already is defined on the complement of any codimension-2 subvariety in $\mathbb{P}_n$. Since any torsion-free rank-1 sheaf $\mathcal{Q}$ is locally free on such a complement we may associate with $\mathcal{Q}$ a Chern-class $c_1(\mathcal{Q})$ too. This classs coincides with $c_1(\mathcal{Q}^{**})$.

If $\mathcal{F}$ is any sheaf, then put

$$\mathcal{F}^*=\mathcal{H}om_{\mathcal{O}}(\mathcal{F},\mathcal{O})$$

$$r\mathcal{F}=\mathcal{F}\oplus\ldots\oplus\mathcal{F}\quad (r\text{-}times).$$

If V is a vector bundle of rank r, put

$P(V)$ = the $\mathbb{P}_{r-1}$-bundle associated with V;
S^lV = l-fold symmetric product of V;
$\det V=\wedge^r V$.

If V, W are vector bundles of rank r both, and $\alpha\in\mathrm{Hom}(V,W)$, then $\det(\alpha)\in\mathrm{Hom}(\det V,\det W)$ is the *determinant* morphism.

If not specified otherwise, then "variety" will mean both, a reduced quasi-projective scheme over $\mathbb{C}$ or a reduced complex space. If $Y\subset X$ is a subspace and $\mathcal{F}$ some $\mathcal{O}_X$-sheaf, then $\mathcal{F}|Y:=\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{O}_Y$. A morphism between two varieties can mean both, morphism of schemes or complex spaces. If $f: X \to Y$ is such a morphism with X irreducible, then rank f $= \dim f(X)$. If Y is nonsingular, this rank coincides with the rank of the *differential* $df: T_x(X) \to T_{fx}(Y)$ in almost all points x of the nonsingular part of X.

2.2. *Direct Image Sheaves.* If $f: X \to Y$ is a morphism and $\mathscr{F}$ an $\mathcal{O}_X$-sheaf, then we denote by $f_*\mathscr{F} = f_{*0}\mathscr{F}$ the direct image sheaf and by $f_{*i}\mathscr{F}$ the higher direct images. The same notation will be applied in the case $\mathscr{F} = V$, a vector bundle.

Sometimes without further notice the following fundamental theorems are used.

Coherence. If f is proper, the direct images are coherent.

Semi-continuity. If additionally $\mathscr{F}$ is $\mathcal{O}_Y$-flat via f and if Y is reduced, then the functions $h^i(\mathscr{F}|f^{-1}y)$, $y \in Y$, are upper-semi-continuous in the Zariski-topology of Y. I.e., the sets $\{y \in Y: h^i(\mathscr{F}|q^{-1}y) \geqq \text{const}\}$ are Zariski-closed.

Base-change. If in addition the function $h^i(\mathscr{F}|q^{-1}y)$ is constant on Y, then $q_{*i}\mathscr{F}$ is locally free and the canonical morphism $(q_{*i}\mathscr{F})|Y_0 \to (q|q^{-1}Y_0)_{*i}\mathscr{F}$ is an isomorphism for every subvariety $Y_0 \subset Y$. In particular $(q_{*i}\mathscr{F})/\mathscr{I}_y \cdot q_{*i}\mathscr{F} \to H^i(\mathscr{F}|q^{-1}y)$ is bijective for every $y \in Y$ ($\mathscr{I}_y \subset \mathcal{O}_Y$ the ideal sheaf of y).

As a reference one can use [10].

There will also be needed the following very primitive criterion for sheaves $f_*\mathscr{F}$ to be locally free:

Definition. Let X be nonsingular. A coherent $\mathcal{O}_X$-sheaf $\mathscr{F}$ is called *normal, if restriction* $\Gamma(\mathscr{F}|U) \to \Gamma(\mathscr{F}|U \setminus A)$ is bijective for every (Zariski-)open $U \subset X$ and every closed subvariety $A \subset U$ of codimension $\geqq 2$.

Lemma 1. *A coherent* rank-1 $\mathcal{O}_X$*-sheaf* $\mathscr{F}$ *is locally free if and only if it is free of torsion and normal.*

Proof. Let $\mathscr{F}$ be of rank 1, free of torsion, and normal. The assertion is obvious, if $\mathscr{F}$ is locally an ideal sheaf, i.e., a subsheaf of some locally free rank-1 sheaf. Since $\mathscr{F}$ is free of torsion, it canonically embeds in its double dual $\mathscr{F}^{**}$. We only have to show that $\mathscr{F}^{**}$ is locally free, or more general that $\mathscr{F}^*$ is. Take a local resolution $k\mathcal{O} \to l\mathcal{O} \to \mathscr{F} \to 0$. Then $\mathscr{F}^*$ embeds in $l\mathcal{O}$, and choosing a suitable projection $l\mathcal{O} \to \mathcal{O}$, we may embed it in $\mathcal{O}$. So $\mathscr{F}^*$ locally is an ideal sheaf. Using that $l\mathcal{O}$ is normal and $k\mathcal{O}$ torsion-free, one easily checks that $\mathscr{F}^*$ is normal too.

2.3. *The Subspace Defined by a Section s.* Let V be some rank-2 bundle on $\mathbb{P}_n$ and $0 \neq s \in \Gamma(V)$.

Lemma 2. *If s vanishes on a hypersurface, then* $h^0(V(-1)) \neq 0$ *and* $h^0(V) \geqq n+1$.

Proof. Assume that s vanishes on the hypersurface with equation $f=0$, $f \in \Gamma(\mathcal{O}_{\mathbb{P}}(k))$, $k>0$. Then s/f is a nontrivial section in $V(-k)$, in particular $h^0(V(-k)) \neq 0$ and therefore $h^0(V(-1)) \neq 0$ too. If g varies in $\Gamma(\mathcal{O}_{\mathbb{P}}(k))$, then gs/f varies in a subspace of $\Gamma(V)$ of dimension $h^0(\mathcal{O}_{\mathbb{P}}(k)) \geqq n+1$.

Lemma 2'. *If* $c_1(V) \leqq 0$ *and* $h^0(V(-1))=0$, *then either* $h^0(V)=1$, *or* $c_1(V)=0$ *and* $V \simeq 2\mathcal{O}_{\mathbb{P}}$.

Proof. Assume first, that s does not vanish at all. Then $\mathbb{C}\cdot s\subset V$ is a trivial line subbundle and V fits into an extension $0\to\mathcal{O}\xrightarrow{s}V\to\mathcal{O}(c_1)\to 0$. Since $c_1\leqq 0$, this extension splits, showing $V\simeq\mathcal{O}\oplus\mathcal{O}(c_1)$. Then either $c_1=0$ and $V\simeq 2\mathcal{O}$, or $h^0(V)=1$. Assume that s does have zeros. Let $s'\in\Gamma(V)$ be an arbitrary other section. $s\wedge s'\in\Gamma(\det V)=\Gamma(\mathcal{O}_{\mathbb{P}}(c_1))$ vanishes identically. Hence s'/s is some rational function f. Since s has zeros in codimension 2 only (Lemma 2), f is constant and $s'=\text{const}\cdot s$. This shows $h^0(V)=1$.

Assume now, that s vanishes in codimension 2 only. Use local trivializations $V\simeq 2\mathcal{O}_{\mathbb{P}}$ to write locally $s=(s_1,s_2)$, with s_1 and s_2 coprime in each point of $s=0$. Define the ideal sheaf $\mathcal{J}$ locally as $\mathcal{O}s_1+\mathcal{O}s_2\subset\mathcal{O}$. It is easy to see that $\mathcal{J}\subset\mathcal{O}_{\mathbb{P}}$ gives a global ideal sheaf defining a subspace $(Y,\mathcal{O}_{\mathbb{P}}/\mathcal{J})$ supported on $s=0$.

Lemma 3. *There is an exact sequence*

$$0\to\mathcal{O}_{\mathbb{P}}\xrightarrow{s}V\to\mathcal{J}(k)\to 0,\qquad k=c_1(V).$$

Proof. Put $\mathcal{Q}:=V/\mathcal{O}\cdot s$. Then we have locally commutative diagrams of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O} & \xrightarrow{s} & 2\mathcal{O} & \longrightarrow & \mathcal{Q} & \longrightarrow & 0\\
 & & \Vert & & \downarrow{\scriptstyle\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}} & & \downarrow\wr & & \\
0 & \longrightarrow & \mathcal{O} & \xrightarrow{\begin{pmatrix}-s_2\\ s_1\end{pmatrix}} & 2\mathcal{O} & \xrightarrow{(s_1,s_2)} & \mathcal{J} & \longrightarrow & 0
\end{array}$$

and find $\mathcal{Q}\simeq\mathcal{J}$ locally. Then globally $\mathcal{Q}\simeq\mathcal{J}\cdot\mathcal{Q}^{**}$ with $\mathcal{Q}^{**}$ a line bundle of degree $c_1(V)$.

2.4. *Jumping Lines.* Let V be some rank-2 bundle on $\mathbb{P}_n$ and define the integers $d(V|L)$ and $d(V)$ as in the introduction. Call those lines L with $d(V|L)>d(V)$ jumping lines.

Lemma 4. *The jumping lines form a closed subvariety $S\subset\mathrm{Gr}(1,n)$. If $d(V)=0$, then S is a hypersurface (or empty).*

Proof. We may replace V by a bundle $V(k)$, $k\in\mathbb{Z}$, to obtain $V|L\simeq\mathcal{O}_L(-1)\oplus\mathcal{O}_L(-d-1)$ for those lines L with $d(V|L)=d(V)$. In particular L is a jumping line if and only if $h^0(V|L)>0$. Next use the standard morphisms $\mathrm{Gr}(1,n)\xleftarrow{q}F(0,1,n)\xrightarrow{p}\mathbb{P}_n$. Since q is locally a product, p^*V is flat over $\mathcal{O}_{\mathrm{Gr}}$. Semi-continuity applied to q_*p^*V then shows, that the jumping lines form a closed subvariety. Assume next that $d(V)=0$. Then $V|L\simeq 2\mathcal{O}_L(-1)$ for the general line L. From deformation theory follows, that proper deformations of $2\mathcal{O}_L(-1)$ can occur in codimension 1 only (compare [1, Satz 6.2]).

The set of jumping lines for a given bundle V can in general be determined only with some efforts, except in the following special case:

Lemma 5. *Assume that $c_1(V)\leqq 0$ and that V admits a non-trivial section s vanishing in codimension 2 only. Put $Y=\{s=0\}\subset\mathbb{P}_n$. Then L is a jumping line, if and only if it intersects Y. In particular one has $d(V)=-c_1$.*

Proof. If $L\cap Y=\emptyset$, then one has an exact sequence

$$0\to\mathcal{O}_L\xrightarrow{s}V|L\to\mathcal{O}_L(c_1)\to 0.$$

Since $c_1 \leqq 0$, this extension splits and $d(V|L) = -c_1$. If L intersects Y in finitely many points, put $V|L \simeq \mathcal{O}_L(k) \oplus \mathcal{O}_L(k-d)$, $d = d(V|L)$. If $k \leqq 0$, then $V|L$ would not admit nontrivial sections with zeros. Hence $k > 0$ and from $c_1 = 2k - d$ follows $d(V|L) = -c_1 + 2k > d(V) = -c_1$. Assume finally that $L \subset X$. Fix one point $x \in L$ and some plane $E \subset \mathbb{P}_n$ not contained in Y. Consider the 1-dimensional family of lines through x in E. Its general member L' is not contained in Y, though it intersects Y in x. We know already that L' is a jumping-line and since the set of jumping lines is closed, L belongs to this set too.

2.5. In the proof of Theorem 3 there will be needed the following simple consequence of Lemma 4:

Lemma 6. *Let V be some* rank-2 *bundle on $\mathbb{P}_n$. If there is at least one plane $E \subset \mathbb{P}_n$ such that $V|E$ is trivial, then so is V itself.*

Proof. In view of Van de Ven's theorem [14], it suffices to show that the set $S \subset \mathrm{Gr}(1, n)$ of jumping lines for V is empty. But if S would not be empty, it would be a hypersurface by Lemma 4. Since every hypersurface on $\mathrm{Gr}(1, n)$ intersects every subvariety in $\mathrm{Gr}(1, n)$ of positive dimension, there would be jumping lines for V contained in the plane E. This would contradict the triviality of $V|E$.

2.6. Lemma 7. *Let $X \subset \mathbb{C}$ be the unit disc and*

$$0 \to V \xrightarrow{h} W \to \mathcal{Q} \to 0$$

be an exact $\mathcal{O}_X$-sequence with V, W free and $\mathcal{Q}$ supported in $0 \in X$. Then $|\mathcal{Q}| := h^0(\mathcal{Q})$ equals the vanishing order of $\det(h)$ *at* 0.

Proof. The statement is obvious if either $|\mathcal{Q}| = 0$ or $r := \operatorname{rank} V = \operatorname{rank} W = 1$. To prove the lemma for $\operatorname{rank} r > 1$ and $q := |\mathcal{Q}| > 0$ by induction, assume that it has been proven for

$$\begin{array}{ll} \operatorname{rank} r' < r & \text{and all } q' \\ \operatorname{rank} r & \text{and all } q' < q. \end{array}$$

Then distinguish two cases:

i) h vanishes at 0, i.e., $h(V)$ is contained in $z \cdot W$, z the coordinate function. Then we have a diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V & \xrightarrow{h'} & W & \longrightarrow & \mathcal{Q}' & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle z} & & \downarrow & & \\
0 & \longrightarrow & V & \xrightarrow{h} & W & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{C}^r & = & \mathbb{C}^r & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Hence, $|\mathcal{Q}|=|\mathcal{Q}'|+r$ and $h=z\cdot h'$, so $\det(h)=z^r\cdot\det(h')$. We are reduced to the case $q'<q$.

ii) h does not vanish at 0, i.e., there is a section $s:\mathcal{O}\to V$ such that $h(s)$ does not vanish at all. Then we have a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O} & = & \mathcal{O} & & & & \\
 & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle h(s)} & & & & \\
0 & \longrightarrow & V & \xrightarrow{h} & W & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & V' & \xrightarrow{h'} & W' & \longrightarrow & \mathcal{Q}' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $h=\begin{pmatrix}1 & 0\\ * & h'\end{pmatrix}$, we have $\det h=\det h'$. So we are reduced to the case $r'<r$.

3. On the Definition of Stability over $\mathbb{P}_n$

3.1. Here are given some equivalent definitions of stability and some easy consequences. For $\mathbb{P}_2$ the equivalences are well-known (Takemoto [13, Corollary 18 and Proposition 4.1], Schwarzenberger [11, 12]), but for $n>2$, there seems to be no reference. Although it is no problem, to generalize from $\mathbb{P}_2$ to $\mathbb{P}_n$, let me do it here for the sake of completeness.

Definition (Mumford, Takemoto [13]): A rank-2 bundle V on $\mathbb{P}_n$ is called *stable* (resp. *semi-stable*) if

$$c_1(V)<2c_1(\mathcal{Q}) \quad (\text{resp. } \leqq) \tag{1}$$

for every torsion-free rank-1 quotient $\mathcal{Q}$ of V. The bundle V is called *unstable*, if it is not stable, and *properly unstable*, if it is not semi-stable.

Every semi-stable V with odd Chern class $c_1(V)$ must be stable. So the distinction between semi-stable and unstable becomes illusory *unless* $c_1(V)$ *is even*. Inequality (1) is invariant under tensoring with line-bundles $\mathcal{O}_{\mathbb{P}}(k)$, so V is stable (resp. semi-stable) if and only if $V(k)$ is. Every 2-bundle on $\mathbb{P}_1$ is properly unstable, except the semi-stable bundles $2\mathcal{O}_{\mathbb{P}_1}(k)$, $k\in\mathbb{Z}$.

3.2. Lemma 8. *Let* $0\to\mathcal{K}\to V\to\mathcal{Q}\to 0$ *be an exact sequence with* $\operatorname{rank}\mathcal{K}=\operatorname{rank}\mathcal{Q}=1$. *Then* $\mathcal{Q}$ *is torsion-free if and only if* $\mathcal{K}$ *is a line-bundle with the inclusion-morphism* $s:\mathcal{K}\to V$ *vanishing in codimension* 2 *only.*

Proof. Assume $\mathcal{Q}$ to be torsion-free. Since V is normal, this implies that $\mathcal{K}$ is normal too, hence $\mathcal{K}\simeq\mathcal{O}_{\mathbb{P}}(k)$ is a line bundle by Lemma 1. If s would vanish on a hypersurface, say with equation $f=0$, $f\in\Gamma(\mathcal{O}_{\mathbb{P}}(l))$, then s/f *would embed* $\mathcal{O}_{\mathbb{P}}(k+l)$ as a subsheaf in V, and the image in $\mathcal{Q}$ of this subsheaf would be isomorphic with $\mathcal{O}_{\mathbb{P}}(k+l)/f\cdot\mathcal{O}_{\mathbb{P}}(k)$, a torsion-sheaf.

Assume now, that $s:\mathcal{O}_{\mathbb{P}}(k)\to V$ vanishes in codimension 2 only. From Lemma 3 follows that $\mathcal{Q}\simeq\mathcal{J}\cdot\mathcal{Q}^{**}$ is torsion-free.

Since $c_1(V)=c_1(\mathcal{K})+c_1(\mathcal{Q})$, from this lemma follows the equivalence: *V is stable (resp. semi-stable) if and only if for every morphism s: $\mathcal{O}_{\mathbb{P}}(k)\to V$* vanishing in codimension 2 only *we have* $2k<c_1(V)$ *(resp.* $\leqq$*).*

If s vanishes in codimension 1, we may replace s by some morphism s/f: $\mathcal{O}_{\mathbb{P}}(k+l)\to V$, $f\in\Gamma(\mathcal{O}_{\mathbb{P}}(l))$, $l>0$, vanishing at most in codimension 2. This shows, that we may replace the above restriction on s by: s is not vanishing identically.

3.3. The following description of stable bundles is less elegant, but in many cases easier to work with: Call V *normalized*, if either $c_1(V)=0$ or $c_1(V)=-1$. For every V, there is a unique normalized bundle $V_{\text{norm}}=V(l)$, some $l\in\mathbb{Z}$. Putting the integer k above equal to zero, one obtains:

V is stable (resp. semi-stable) if and only if $h^0(V_{\text{norm}})=0$ (resp. $h^0(V_{\text{norm}}(-1))=0$ in case $c_1(V)$ is even). In view of Lemma 2′ the condition $h^0(V_{\text{norm}}(-1))=0$ is equivalent with $h^0(V_{\text{norm}})\leqq 1$ or $V_{\text{norm}}\simeq 2\mathcal{O}_{\mathbb{P}}$. Occasionally I need the following characterization too: *V is stable if and only if* $\operatorname{End}(V)\simeq\mathbb{C}$, *i.e., if V does not admit endomorphisms other than homothethies.* (Bundles with this property were called *simple* by Maruyama [8]. Unstable bundles on $\mathbb{P}_2$ were called *almost-decomposable* by Schwarzenberger [11, 12] and „*der triviale Fall*“ by Grauert-Mülich [2].)

To prove this last equivalence, assume first that $\dim\operatorname{End}(V)\geqq 2$. If $x\in\mathbb{P}_n$ is an arbitrary point, restriction to the fibres over x (vertical arrows) and the determinant map (horizontal arrows) define a commutative diagram

$$\begin{array}{ccc}\operatorname{End}(V) & \longrightarrow & \Gamma(\mathcal{O}_{\mathbb{P}}) \\ \downarrow & & \| \\ \operatorname{End}(V(x)) & \longrightarrow & \mathbb{C}(x).\end{array}$$

This proves the existence of some non-trivial endomorphism α with $\det\alpha\equiv 0$. Put $\mathcal{Q}:=\operatorname{im}\alpha$ and $\mathcal{K}=\ker\alpha$. Since $\mathcal{Q}\subset V$, it is torsion-free and $\mathcal{K}$ is locally free. Together with $\mathcal{Q}$ also the locally free sheaf $\mathcal{Q}^{**}$ embeds in V. Furthermore $c_1(V)=c_1(\mathcal{K})+c_1(\mathcal{Q})$, so the conditions $2c_1(\mathcal{Q})<c_1(V)$ and $2c_1(\mathcal{K})<c_1(V)$ cannot hold simultaneously, i.e., V must be unstable.

Assume now that V is unstable and let $0\to\mathcal{O}_{\mathbb{P}}(k)\to V\to\mathcal{Q}\to 0$ be an exact sequence with Q torsion-free and $c_1(V)\geqq 2c_1(\mathcal{Q})$. Then $\mathcal{Q}$ embeds in $\mathcal{Q}^{**}$ and

$$c_1(\mathcal{Q}^{**})=c_1(\mathcal{Q})\leqq c_1(V)-c_1(\mathcal{Q})=k,$$

so there is a nontrivial morphism $\mathcal{Q}^{**}\to\mathcal{O}_{\mathbb{P}}(k)$. Composing

$$V\to\mathcal{Q}\to\mathcal{Q}^{**}\to\mathcal{O}_{\mathbb{P}}(k)\to V$$

one obtains an endomorphism of V which is not a homothethy.

3.4. Next there are given some simple consequences of stability.

Proposition 1. *Let S be a reduced variety and $\mathcal{V}$ some* rank-2 *bundle over $S\times\mathbb{P}_n$. The points $s\in S$ such that $\mathcal{V}|\{s\}\times\mathbb{P}_n$ is unstable, form a Zariski-closed subset of S.*

Proof. Since $\mathscr{V}\otimes\mathscr{V}^*$ is an $\mathcal{O}_S$-flat sheaf via the projection $S\times\mathbb{P}_n\to S$, the assertion follows from the semi-continuity of the function $\dim_{\mathbb{C}}\mathrm{End}(\mathscr{V}|\{s\}\times\mathbb{P}_n)$, $s\in S$.

Proposition 2. *Let V be a* rank-2 *bundle over $\mathbb{P}_n$ and $\pi:\tilde{\mathbb{P}}_n\to\mathbb{P}_n$ a finite covering. If V is stable (resp. semi-stable, unstable, properly unstable), then so is π^*V.*

Proof. Put $g=\deg\pi$. Since $\pi^*(V(k))=(\pi^*V)(gk)$, the bundle π^*V will be stable (resp. semi-stable etc.) if $\pi^*(V(k))$ is. So we may tensorize V by some line bundle to obtain

$$h^0(V)\neq 0, \quad \text{but} \quad h^0(V(-1))=0.$$

Assume first that V is unstable (resp. properly unstable). This implies $c_1(V)\leqq 0$ (resp. <0). Then $c_1(\pi^*V)=\pi^*c_1(V)=g\cdot c_1(V)\leqq 0$ (resp. <0) too. Since V admits a non-trivial section, so does π^*V, and π^*V will be unstable (resp. properly unstable).

Assume next that V is stable (resp. semi-stable). This means $c_1(V)>0$ (resp. $\geqq 0$). Then $c_1(\pi^*V)>0$ (resp. $\geqq 0$) too, and we have to show $h^0((\pi^*V)(-1))=0$. Since π is finite, the higher direct images vanish, and we have

$$\begin{aligned} H^0((\pi^*V)(-1)) &= H^0(\pi_*((\pi^*V)(-1))) \\ &= H^0(V\otimes\pi_*(\mathcal{O}_{\tilde{\mathbb{P}}}(-1))). \end{aligned}$$

Since the covering-space is nonsingular, the sheaf $\pi_*\mathcal{O}_{\tilde{\mathbb{P}}}(-1)$ is locally free of rank g. All cohomology groups

$$H^i((\pi_*\mathcal{O}_{\mathbb{P}}(-1))(k))=H^i(\mathcal{O}_{\tilde{\mathbb{P}}}(-1+gk))$$

vanish for $i=1,\dots,n-1$ and all $k\in\mathbb{Z}$. From Horrocks' theorem [5, Theorem 7.4] it follows that $\pi_*\mathcal{O}_{\tilde{\mathbb{P}}}(-1)$ is decomposable.

$$\pi_*\mathcal{O}_{\tilde{\mathbb{P}}}(-1)\simeq\mathcal{O}_{\mathbb{P}}(k_1)\oplus\dots\oplus\mathcal{O}_{\mathbb{P}}(k_g).$$

Since $h^0(\pi_*\mathcal{O}_{\tilde{\mathbb{P}}}(-1))=h^0(\mathcal{O}_{\mathbb{P}}(k_1))+\dots+h^0(\mathcal{O}_{\mathbb{P}}(k_g))$ vanishes, all integers $k_1,\dots,k_g$ are negative. This shows

$$H^0(V\otimes\pi_*\mathcal{O}_{\tilde{\mathbb{P}}}(-1))=H^0(V(k_1))\oplus\dots\oplus H^0(V(k_g))=0$$

and π^*V will be stable (resp. semi-stable) too.

Proposition 3. *Let V be some* rank-2 *bundle on $\mathbb{P}_n$ and $\mathbb{P}_{n-1}\subset\mathbb{P}_n$ some hyperplane. If $V|\mathbb{P}_{n-1}$ is stable (resp. semi-stable), then so is V.*

Proof. We show that $c_1(V(k))>0$ (resp. $\geqq 0$), whenever $h^0(V(k))\neq 0$. If s is a nontrivial section in $V(k)$, then $s|\mathbb{P}_{n-1}$ is a nontrivial section in the restricted bundle, or s vanishes along $\mathbb{P}_{n-1}$ with a certain multiplicity, say l. Then s defines a non-trivial section in $V(k-l)$ not vanishing identically on $\mathbb{P}_{n-1}$. This shows

$$c_1(V(k))=2l+c_1(V(k-l))>0 \qquad (\text{resp.}\geqq 0).$$

In 9.2 we also shall need "that the space of stable rank-2 bundles is hausdorff". This is proved in very general form in [7]. For the benefit of the reader, let me include here a short proof of this fact in our special case.

Proposition 4. *Let $X \subset \mathbb{C}$ be the unit disc with origin* 0. *Let $\mathscr{V}$ and $\mathscr{W}$ be two* rank-2 *bundles over $X \times \mathbb{P}_n$ and put $\mathscr{V}_x := \mathscr{V} | \{x\} \times \mathbb{P}_n$, $\mathscr{W}_x := \mathscr{W} | \{x\} \times \mathbb{P}_n$. Assume that*

for all $x \in X$ the bundles $\mathscr{V}_x$ and $\mathscr{W}_x$ are stable

for all $0 \neq x \in X$ the bundles $\mathscr{V}_x$ and $\mathscr{W}_x$ are isomorphic.

Then $\mathscr{V}_0$ and $\mathscr{W}_0$ are isomorphic too.

Proof. Consider on $X \times \mathbb{P}_n$ the rank-4 bundle $\mathscr{H} := \mathscr{H}om(\mathscr{V}, \mathscr{W})$. For all $0 \neq x \in X$, $h^0(\mathscr{H} | \{x\} \times \mathbb{P}_n) = 1$ by assumption. Let $\pi : X \times \mathbb{P}_n \to X$ be the projection. The coherent $\mathscr{O}_X$-sheaf $\pi_* \mathscr{H}$ then is of rank 1 and torsion-free, hence locally free. We even may assume $\pi_* \mathscr{H} \simeq \mathscr{O}_X$. Use the canonical morphism

$$h : \mathscr{O}_{X \times \mathbb{P}_n} = \pi^* \mathscr{O}_X = \pi^* \pi_* \mathscr{H} \to \mathscr{H} = \mathscr{H}om(\mathscr{V}, \mathscr{W})$$

to obtain a morphism $h : \mathscr{V} \to \mathscr{W}$, which is nontrivial, hence an isomorphism, outside the fibre $\{0\} \times \mathbb{P}_n$. Let $\mathscr{Q} \subset \mathscr{W}_0$ be the image of $\mathscr{V}_0$ under h. Then there are the following three possibilities:

1) rank $\mathscr{Q} = 2$. This means that $\mathscr{Q}$ coincides with $\mathscr{W}_0$ almost everywhere. The determinant $\det(h) \in \Gamma(\mathscr{O}_{X \times \mathbb{P}})$ then vanishes at most on a subvariety of codimension $\geqq 2$, i.e., it cannot vanish at all. But this implies, $h : \mathscr{V} \to \mathscr{W}$ is an isomorphism everywhere, in particular $\mathscr{V}_0 \simeq \mathscr{W}_0$.

2) rank $\mathscr{Q} = 1$. As a subsheaf of $\mathscr{W}_0$, $\mathscr{Q}$ is torsion-free over $\{0\} \times \mathbb{P}_n$. Since $\mathscr{V}_0$ was stable by assumption, $c_1(\mathscr{V}_0) < 2c_1(\mathscr{Q})$. If $\mathscr{Q}$ embeds in $\mathscr{W}_0$, then so does the line bundle $\mathscr{Q}^{**}$. Since $\mathscr{W}_0$ was stable too, $2c_1(\mathscr{Q}) = 2c_1(\mathscr{Q}^{**}) < c_1(\mathscr{W}_0)$. One obtains $c_1(\mathscr{V}_0) < c_1(\mathscr{W}_0)$, a contradiction, because the Chern classes of $\mathscr{V}_x$ and $\mathscr{W}_x$ are independent of x.

3) rank $\mathscr{Q} = 0$. This means that h vanishes identically on $\{0\} \times \mathbb{P}_n$, say of order μ. If z is the coordinate on X, one may replace h by h/z^μ, to obtain a regular morphism $\mathscr{V} \to \mathscr{W}$ which does not vanish identically on $\{0\} \times \mathbb{P}_n$. This however is covered by one of the two other cases.

4. The Standard Construction

4.1. Let $\mathrm{Gr} = \mathrm{Gr}(m, n)$ be the grassmann variety parametrizing m-planes $E \subset \mathbb{P}_n$ and $F = F(0, m, n)$ the flag variety of pairs (x, E) with $x \in E \subset \mathbb{P}_n$. There is the standard diagramm

$$(D_m) \qquad \begin{array}{ccc} \mathrm{Gr} & \xleftarrow{q} & F \\ & & \downarrow p \\ & & \mathbb{P}_n \end{array} \qquad \begin{array}{l} q : (x, E) \mapsto \{E\} \in \mathrm{Gr} \\ p : (x, E) \mapsto x \in \mathbb{P}_n . \end{array}$$

The projections p and q are locally trivial with fibres $\mathrm{Gr}(m-1, n-1)$ and $\mathbb{P}_m$ respectively.

Fix some r-bundle V on $\mathbb{P}_n$. Whenever $E \subset \mathbb{P}_n$ is some m-plane with $e \in \mathrm{Gr}$ its corresponding point, then p^* induces an isomorphism $V|E \to p^* V | q^{-1} e$. Since $p^* V$ is an $\mathscr{O}_{\mathrm{Gr}}$-flat sheaf via q, semi-continuity shows that the function

$$h^0(p^* V | q^{-1} e) = h^0(V|E), \qquad e \in \mathrm{Gr}$$

is upper-semi-continuous in the Zariski-topology of Gr. In particular there is some Zariski-open $U \subset \mathrm{Gr}$ where this function assumes its minimum

$$\min\{h^0(V|E), E \subset \mathbb{P}_n\}.$$

4.2. Consider now the situation where this minimal dimension is 1. The coherent sheaf $q_* p^* V$ on Gr then will be of rank 1. This sheaf shares with $p^* V$ the property of being free of torsion and normal, hence it will be locally free (Lemma 1). We already observed that $\mathrm{Pic}(\mathrm{Gr}) = \mathbb{Z}$ and that it is generated by $\mathcal{O}_{\mathrm{Gr}}(1)$. So we may put

$$q_* p^* V \simeq \mathcal{O}_{\mathrm{Gr}}(-l) \quad \text{for some} \quad l \in \mathbb{Z}.$$

Denote the canonical morphism

$$q^* \mathcal{O}_{\mathrm{Gr}}(-l) = q^* q_* p^* V \to p^* V$$

by s. For $e \in U$, this morphism induces an isomorphism

$$\mathcal{O}_{\mathrm{Gr}}(l)_e / m_e \mathcal{O}_{\mathrm{Gr}}(l)_e \to \Gamma(p^* V | q^{-1} e) \quad \text{(base change)}. \tag{2}$$

So s will not be trivial. If we fix some $x \in \mathbb{P}_n$ and write $V \simeq r\mathcal{O}_{\mathbb{P}}$ in a neighborhood of this point, then on the fibre $p^{-1}x \simeq \mathrm{Gr}(m-1, n-1)$ the morphism s restricts to a morphism.

$$s|p^{-1}x = (s_1, \ldots, s_r) : q^* \mathcal{O}_{\mathrm{Gr}}(-l) | p^{-1}x \simeq \mathcal{O}_{\mathrm{Gr}(m-1,n-1)}(-l) \to$$
$$\to V|p^{-1}x \simeq r\mathcal{O}_{\mathrm{Gr}(m-1,-1,n-1)}.$$

This morphism cannot be trivial for all $x \in \mathbb{P}_n$, showing that $l \geqq 0$ (and thus justifying the choice of the sign for l).

Observe that $l = 0$ if and only if

$$h^0(V) = h^0(p^* V) = h^0(q_* p^* V) = h^0(\mathcal{O}_{\mathrm{Gr}}(-l)) \neq 0. \tag{3}$$

Lemma 9. *The zero-set $Z \subset F$ of the morphism s does not contain a hypersurface.*

Proof. Assume that H is a hypersurface in F on which s vanishes. By assumption $h^0(p^* V | q^{-1} e) = 1$ for all $e \in U$. So $p^* V | q^{-1} e$ contains a non-trivial section (unique up to multiplication with constants) which can vanish only in codimension $\geqq 2$ (Lemma 2). Since $s|q^{-1}e$ is a nontrivial multiple of this section by (2), H cannot intersect the general fibre $q^{-1}e$, $e \in U$. Therefore $H = q^{-1}H'$ with $H' = q(H) \subset \mathrm{Gr}$ some hypersurface. Let $h \in \Gamma(\mathcal{O}_{\mathrm{Gr}}(k))$ be an equation for H'. Then $s/q^* h$ is a nontrivial, regular morphism of $q^* \mathcal{O}_{\mathrm{Gr}}(-l+k)$ into $p^* V$, inducing an injection

$$\mathcal{O}_{\mathrm{Gr}}(-l+k) \to q_* p^* V = \mathcal{O}_{\mathrm{Gr}}(-l).$$

This is impossible, unless $k = 0$ and $H = H' = \emptyset$.

4.3. Now replace V by its associated $\mathbb{P}_{r-1}$-bundle $P(V)$ and $p^* V$ by $P(p^* V) = p^* P(V)$. The image of s is a subsheaf of $p^* V$, locally free of rank 1 outside Z. It determines a rational cross-section $\Gamma \subset P(p^* V)$ which degenerates only over Z. We may assume Γ to be irreducible. If $E \subset \mathbb{P}_n$ corresponds to some point $e \in U$ and $x \in p^{-1} e \setminus Z$, then the meaning of Γ at the point $(x, E) \in F$ is this: Let $\tilde{p} : P(p^* V) \to P(V)$ be the map induced by p. This $\tilde{p}$ maps the value of Γ at (x, E) onto the point in $P(V(x))$ defined by the direction in $V(x)$ of the unique section in $V|E$.

Lemma 10. *$l>0$ if and only if $\tilde{p}(\Gamma)\subset P(V)$ has dimension $\geqq n+1$.*

Proof. Assume first that $l>0$. For general $x\in\mathbb{P}_n$ write as above

$$s|p^{-1}x=(s_1,\ldots,s_r),\ s_i\in\Gamma(\mathcal{O}_{\mathrm{Gr}(m-1,n-1)}(l)).$$

Since the common zero-set of $s_1,\ldots,s_r$ is $Z\cap p^{-1}x$, it is of codimension $\geqq 2$ in this fibre. The sections $s_1,\ldots,s_r$ thus span a vector space of dimension $\geqq 2$. Since the part of Γ over $p^{-1}x$ is mapped under $\tilde{p}$ by $(s_1:\ldots:s_r)$, its image has dimension $\geqq 1$, and $\dim\tilde{p}(\Gamma)\geqq n+1$.

Assume next $l=0$. Then $s=p^*s$, with a unique section $s_0\in\Gamma(V)$. Hence $\tilde{p}\Gamma$ is the rational cross-section in $P(V)$ determined by s_0, which has dimension n.

5. The Theorem of Grauert-Mülich

5.1. Here will be given the proof of Theorem 1 from the introduction, first in the case of stable bundles.

Let V be a stable rank-2 bundle on $\mathbb{P}_n$ with $d(V)\geqq 2$. After tensoring V by some line bundle, we may assume that

$$V|L\simeq\mathcal{O}_L\oplus\mathcal{O}_L(-d)$$

on the general line $L\subset\mathbb{P}_n$. In particular $c_1(V)=-d$ and $h^0(V)=0$. We perform the standard construction (§4) for the case $m=1$. Since $h^0(V|L)=1$ on the general line L, (3) tells us that $p_*p^*V\simeq\mathcal{O}_{\mathrm{Gr}(1,n)}(-l)$ with $l>0$. The map $\tilde{p}:\Gamma\to P(V)$ will therefore be surjective by Lemma 10. We shall obtain a contradiction by proving that the differential of $\tilde{p}|\Gamma$ at a general point of Γ cannot be surjective:

To this behalf fix some line $L\subset\mathbb{P}_n$ corresponding to a point $e\in U\subset\mathrm{Gr}(1,n)$. The fibre $L:=q^{-1}e\subset F$ then does not intersect Z and Γ will be a regular section over some neighborhood of L. Let $A\subset p^*P(V)|L$ be the part of Γ over L. Under $\tilde{p}$ this cross-section A is mapped onto the cross-section $B\subset P(V)|L$ determined by the trivial subbundle

$$\mathcal{O}_L\subset V|L=\mathcal{O}_L\oplus\mathcal{O}_L(-d).$$

We have the commutative diagram

$$\begin{array}{ccccc} A & \subset & \Gamma & \longrightarrow & F \\ \downarrow{\scriptstyle\tilde{p}} & & \downarrow{\scriptstyle\tilde{p}} & & \downarrow{\scriptstyle p} \\ B & \subset & P(V) & \longrightarrow & \mathbb{P}_n \end{array}$$

with horizontal arrows the bundle projection π in $p^*P(V)$, resp. $P(V)$. The differentials dp and $d\tilde{p}$ define linear maps of normal bundles

$$\begin{array}{ccc} N_{A/\Gamma} & \longrightarrow & N_{L/F} \\ \downarrow{\scriptstyle d\tilde{p}} & & \downarrow{\scriptstyle dp} \\ N_{B/P(V)} & \xrightarrow{d\pi} & N_{L/\mathbb{P}_n} \end{array}$$

Use the well-known identifications

$$\begin{array}{ccccccccc} & & & & N_{L/F} & \xrightarrow{dp} & N_{L/\mathbb{P}_n} & \longrightarrow & 0 \\ & & & & \| & & \| & & \\ 0 & \longrightarrow & (n-1)\mathcal{O}_{\mathbb{P}_1}(-1) & \longrightarrow & 2(n-1)\mathcal{O}_{\mathbb{P}_1} & \longrightarrow & (n-1)\mathcal{O}_{\mathbb{P}_1}(1) & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_{B/P(V|L)} & \longrightarrow & N_{B/P(V)} & \xrightarrow{d\pi} & N_{L/\mathbb{P}_n} & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}_1}(-d) & \longrightarrow & N_{B/P(V)} & \longrightarrow & (n-1)\mathcal{O}_{\mathbb{P}_1}(1) & \longrightarrow & 0 \end{array}$$

to obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (n-1)\mathcal{O}_{\mathbb{P}_1}(-1) & \longrightarrow & N_{A/\Gamma} & \longrightarrow & (n-1)\mathcal{O}_{\mathbb{P}_1}(1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle d\tilde{p}} & & \| & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}_1}(-d) & \longrightarrow & N_{B/P(V)} & \longrightarrow & (n-1)\mathcal{O}_{\mathbb{P}_1}(1) & \longrightarrow & 0 \end{array}$$

We assumed $d \geqq 2$. The left-hand vertical arrow vanishes therefore. The image of $N_{A/\Gamma}$ under $d\tilde{p}$ then will be some rank-$(n-1)$ subbundle of the rank-n bundle $N_{B/P(V)}$ and $d\tilde{p}$ cannot be surjective near A. This proves Theorem 1.

5.2. It was observed by Van de Ven, that one may use Proposition 2 to generalize Theorem 1 in order to determine the restriction of V to the general (not necessarily linear) rational curve in $\mathbb{P}_n$. To be precise: A rational curve in $\mathbb{P}_n$ is a non-constant morphism $\varphi: \mathbb{P}_1 \to \mathbb{P}_n$. A family of rational curves is a morphism $\Phi: X \times \mathbb{P}_1 \to \mathbb{P}_n, X$ a connected curve, which is not constant on any fibre $\{x\} \times \mathbb{P}_1$.

Corollary of Theorem 1. *Let V be a stable* rank-2 *bundle over $\mathbb{P}_n$ and $\varphi: \mathbb{P}_1 \to \mathbb{P}_n$ a rational curve in $\mathbb{P}_n$. Then there is a family of rational curves $\Phi: X \times \mathbb{P}_1 \to \mathbb{P}_n$ such that*

$\varphi = \Phi|\{0\} \times \mathbb{P}_1$ *for some point* $0 \in X$,
$d(\Phi^* V|\{x\} \times \mathbb{P}_1) = 0$ *(if $c_1(\Phi^* V)$ is even) or* $=1$ *(if $c_1(\Phi^* V)$ is odd)*

for all points x contained in some Zariski-open subset of X.

Proof. Take homogeneous coordinates $z_0, \ldots, z_n$ on $\mathbb{P}_n$ and polynomials $f_0, \ldots, f_n \in \Gamma(\mathcal{O}_{\mathbb{P}_1}(k))$, $k > 0$, with $f_\nu = \varphi^* z_\nu$, $\nu = 0, \ldots, n$. Take some linear embedding $\mathbb{P}_1 \hookrightarrow \mathbb{P}_n$. For $\nu = 0, \ldots, n$, successively one can find polynomials $F_0, \ldots, F_n \in \Gamma(\mathcal{O}_{\mathbb{P}_n}(k))$ without common zeroes such that $F_\nu | \mathbb{P}_1 = f_\nu$. Let $\pi: \tilde{\mathbb{P}}_n \to \mathbb{P}_n$ be the covering of degree k^n given by $F_0, \ldots, F_n$. Then $\pi | \mathbb{P}_1 = \varphi$.

By Proposition 2, the bundle $\pi^* V$ on $\tilde{\mathbb{P}}_n$ is stable again. Theorem 1 shows that $d(V|L) = 0$, resp. 1, for the general line $L \subset \tilde{\mathbb{P}}_n$. After fixing such a line L, one can choose an appropriate curve X in the grassmannian of lines in $\tilde{\mathbb{P}}_n$, to connect $\mathbb{P}_1$ with L. The restriction of π to the family of rational curves parametrized by X will define a morphism Φ with the properties wanted.

Special case. *If V is a stable* rank-2 *bundle on $\mathbb{P}_n$, then $d(V|C) = 0$ for the general conic section $C \subset \mathbb{P}_n$.*

5.3. *Semi-stable Bundles*

Theorem 1 holds for trivial reasons in the case of semi-stable bundles too: If V is semi-stable, then $h^0(V_{\text{norm}})=1$ or $V_{\text{norm}}\simeq 2\mathcal{O}_{\mathbb{P}}$ by 3.3 and Lemma 2'. In the first case Lemma 5 shows $d(V)=d(V_{\text{norm}})=0$, and in the second case $d(V|L)=0$ for all lines L.

6. The Jumping Lines of a (Semi-)Stable Rank-2 Bundle with even First Chern Class

6.1. Let V be a semi-stable rank-2 bundle over $\mathbb{P}_n$ with $c_1(V)$ even. We just saw that $d(V|L)=0$ for the general line $L\subset\mathbb{P}_n$ (Theorem 1). By Lemma 4 the set $S\subset\mathrm{Gr}(1,n)$ of jumping lines of V must be of codimension 1 everywhere (or perhaps empty). We want to determine the degree of S. This can be done in a very simple way, if one views S as a divisor, i.e., if one admits components with higher multiplicity.

Definition of the Multiplicities. Tensor V such that $c_1(V)=-2$. Then L is a jumping line if and only if $h^1(V|L)\neq 0$. So $S\subset\mathrm{Gr}(1,n)$ is the support of the sheaf

$$\mathcal{L}:=q_{*1}p^*V,$$

where p and q are the projections in diagram (D_1).

Take a resolution of V

$$0\to U\to\bigoplus_{i=1}^{r}\mathcal{O}_{\mathbb{P}}(k_i)\to V\to 0 \tag{4}$$

with all $k_i<0$. Since $V|L=2\mathcal{O}_L(-1)$ on the general line L (theorem of Grauert-Mülich) the sheaf q_*p^*V vanishes. One obtains an exact sequence

$$0\to q_{*1}p^*U\xrightarrow{\lambda}\bigoplus_{i=1}^{r}q_{*1}p^*\mathcal{O}_{\mathbb{P}}(k_i)\to\mathcal{L}\to 0.$$

Now $h^1(\mathcal{O}_L(k_i))$ is independent of L, and base-change implies that the sheaf in the middle of this exact sequence is locally free on $\mathrm{Gr}(1,n)$. But from the exact sequence

$$0\to H^0(V|L)\to H^1(U|L)\to\oplus H^1(\mathcal{O}_L(k_i))$$
$$\to H^1(V|L)\to 0$$

and from the constancy of $h^0(V|L)-h^1(V|L)$ it follows that $q_{*1}p^*U$ is locally free too.

So λ is a morphism between two locally free sheaves of the same rank, and one can define the divisor

$$S:=\text{divisor of zeroes of }\det(\lambda).$$

That the multiplicities assigned to the components of S in this way are independent of the resolution (4), this follows from the

Interpretation of these Multiplicities. Let $e\in S$ be an arbitrary point. Choose local coordinates $x_1,\dots,x_{2(n-1)}$ concentrated at e such that the disc $X=\{x_1=0\}$ intersects S transversally at e. The multiplicity $\mu_e(S)$ of the divisor S at e then equals the vanishing order of $\det(\lambda)|X$ at e.

Now restriction onto X is right-exact, hence

$$(q_{*1}p^*U)|X\to\oplus(q_{*1}p^*\mathcal{O}_{\mathbb{P}}(k_i))|X\to\mathcal{L}|X\to 0$$

is exact. But $\lambda|X$ is injective, since it is so outside S. Lemma 7 now implies that the vanishing order at e of $\det(\lambda|X) = \det(\lambda)|X$ equals $h^0(\mathscr{L}|X)$, an integer independent of the resolution (4). Furthermore, if L is the line corresponding to e, then one has the inequality

$$\begin{aligned}\mu_e(S) &= h^0(\mathscr{L}|X)\\ &= h^0(\mathscr{L}/(x_2, \ldots, x_{2(n-1)})\mathscr{L})\\ &\geqq h^0(\mathscr{L}/(x_1, \ldots, x_{2(n-1)})\mathscr{L})\\ &\geqq h^1(p^*V|q^{-1}e)\\ &= h^1(V|L) = d(V|L)/2\,. \end{aligned} \tag{5}$$

6.2. Theorem 2. *The divisor S on* $\mathrm{Gr}(1,n)$ *defined this way has the properties:*

i) $\deg S = \Delta_0(V) := -\Delta(V)/4$.

(Recall that $\Delta(V) = c_1^2 - 4c_2$ and that $\Delta_0(V)$ is an integer if c_1 is even.)

ii) *$d(V|L) \leqq 2\mu_e(S)$ if L is the jumping line corresponding to $e \in S$.*

iii) *$S = S(V)$ depends analytically, resp. algebraically on V.*

Proof of i). It is no loss of generality to assume here that $n=2$. Then one can choose a resolution of V by line bundles

$$0 \to \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}}(k_i) \to \bigoplus_{i=1}^{r+2} \mathcal{O}_{\mathbb{P}}(l_i) \to V \to 0\,, \tag{6}$$

with k_i, $l_i < 0$. If one puts

$$K := \bigoplus_{i=1}^{r} q_{*1} p^* \mathcal{O}_{\mathbb{P}}(k_i)$$

$$L := \bigoplus_{i=1}^{r+2} q_{*1} p^* \mathcal{O}_{\mathbb{P}}(l_i)\,,$$

then the direct image of this sequence under $q_{*1}p^*$ will be

$$0 \to \mathrm{K} \xrightarrow{\lambda} L \to q_{*1} p^* V \to 0\,.$$

Since the divisor of $\det(\lambda)$ has the degree

$$c_1(L) - c_1(K)$$

one only has to compute first Chern classes. Comparing Chern classes in (6) one finds

$$\Sigma k_i - \Sigma l_i = 2$$

$$\sum_{i<j} k_i k_j - \sum_{i<j} l_i l_j = 2\Sigma k_i - c_2(V)$$

and consequently

$$\begin{aligned}\Sigma k_i^2 - \Sigma l_i^2 &\\ &= (\Sigma k_i)^2 - (\Sigma l_i)^2 - 2\sum_{i<j} k_i k_j + 2\sum_{i<j} l_i l_j\\ &= 4\Sigma l_i + 4 - 4\Sigma k_i + 2c_2(V)\\ &= 2c_2(V) - 4\,.\end{aligned}$$

Next embed the flag variety $F(0,1,2)$ in $\Pi:=\mathbb{P}_2\times\mathbb{P}_2^*$ as divisor of bidegree $(1,1)$. Then for $k\in\mathbb{Z}$ there is the exact sequence

$$0\to\mathcal{O}_\Pi(k-1,-1)\to\mathcal{O}_\Pi(k,0)\to\mathcal{O}_F(k,0)\to 0.$$

If $k<0$, the direct image under q_* will be

$$0\to q_{*1}\mathcal{O}_F(k,0)\to q_{*2}\mathcal{O}_\Pi(k-1,-1)\to q_{*2}\mathcal{O}_\Pi(k,0)\to 0.$$

Now

$$q_{*2}\mathcal{O}_\Pi(k,0)=h^0(\mathcal{O}_{\mathbb{P}^2}(-k-3))\cdot\mathcal{O}_{\mathbb{P}_2^*}$$

has trivial Chern classes. Since the first Chern class of $q_{*2}\mathcal{O}_\Pi(k-1,-1)$ equals

$$-h^2(\mathcal{O}_{\mathbb{P}^2}(k-1))=-h^0(\mathcal{O}_{\mathbb{P}^2}(-k-2))=-\tfrac{1}{2}(k^2+k)$$

one finds for $k<0$:

$$c_1(q_{*1}\mathcal{O}_F(k,0))=-\tfrac{1}{2}(k^2+k).$$

Now one can add over all k_i and l_i to obtain

$$\begin{aligned}c_1(L)-c_1(K)&=-\tfrac{1}{2}\Sigma(l_i^2+l_i)+\tfrac{1}{2}\Sigma(k_i^2+k_i)\\&=(c_2-2)+1\\&=c_2(V)-1=\Delta_0(V).\end{aligned}$$

Proof of ii). This is the inequality (5).

Proof of iii). Take a variety X and a family of semi-stable bundles over $X\times\mathbb{P}_n$, i.e., a rank-2 bundle $\mathcal{V}$ over $X\times\mathbb{P}_n$ such that all bundles $\mathcal{V}_x:=\mathcal{V}|\{x\}\times\mathbb{P}_n$ are semi-stable with $c_1(\mathcal{V}_x)=-2$. Locally (w.r. to X) one can find resolutions

$$0\to\mathcal{U}\to\oplus\mathcal{O}(k_i)\to\mathcal{V}\to 0$$

where $\mathcal{O}(k_i)$ means the pull-back to $X\times\mathbb{P}_n$ of $\mathcal{O}_{\mathbb{P}}(k_i)$.

Forming $q_{*1}p^*$ simultaneously for all $x\in X$, one obtains over $X\times\mathrm{Gr}(1,n)$ an exact sequence

$$0\to q_{*1}p^*\mathcal{U}\xrightarrow{\Lambda}\oplus q_{*1}p^*\mathcal{O}(k_i)\to q_{*1}p^*\mathcal{V}\to 0.$$

Now the divisor of zeroes of $\det(\Lambda)$ on $X\times\mathrm{Gr}(1,n)$ restricts on $\{x\}\times\mathrm{Gr}(1,n)$ to the divisor of $\det(\Lambda)|\{x\}\times\mathrm{Gr}(1,n)$, which by 6.1. is the divisor $S(\mathcal{V}_x)$. This means that the divisors $S(\mathcal{V}_x)$, $x\in X$, form an analytic, resp. algebraic family.

7. The Null-Correlation Bundle V_0 on $\mathbb{P}_3$

Here will be given some equivalent definitions of V_0. This bundle on $\mathbb{P}_3$ belonged to the first known examples of indecomposable 2-bundles on $\mathbb{P}_3$ and its properties are well-known. I want to discuss them here in some detail, mainly because one particular result (Lemma 11) is needed in the proof of Theorem 3.

7.1. *Some Old Terminology*

Put in this paragraph

$$P = \mathbb{P}_3, \qquad G = \mathrm{Gr}(1,3).$$

Using Plücker-coordinates one may embed $G \subset \mathbb{P}_5$ as nonsingular quadric hypersurface.

A *line complex* is a hypersurface $S \subset G$. The degree of this complex S is its degree as divisor on G. It coincides with the degree of the cone described by those lines through a general point $x \in P$ which are parametrized by S. A *linear complex* is a complex of degree one, i.e., the intersection of the Plücker quadric G with some hyperplane $\mathbb{P}_4 \subset \mathbb{P}_5$. There are the two species of linear complexes S:

a) S is *special*, if the hyperplane touches G at *one* point e. Let $L \subset P$ be the line corresponding to this point e, then S parametrizes precisely the lines in P intersecting L.

b) S is *general*, if the hyperplane intersects G transversally in every point of S. Then S itself is a nonsingular variety.

A *null-correlation* is an isomorphism $N: P \to P^*$, $x \to E_x$, with $x \in E_x$ for all x. E_x is called the *null-plane* of x and the uniquely determined point $x_E \in E$ with $E = E_{x_E}$ is called the *null-point* of E. N determines an involution $L \mapsto L^*$ of G. $L^* \subset P$ is the line having the two equivalent properties

i) the points of L^* are the null-points of the planes through L;

ii) the planes through L^* are the null-planes of the points on L.

Either $L \cap L^* = \emptyset$, or $L = L^*$. The lines $L = L^*$ fixed under the involution form a general linear complex.

If S and S' are two different linear complexes, then $S \cap S'$ is always the intersection of two special linear complexes, i.e., there are two lines L and $L' \subset P$ such that $S \cap S'$ parametrizes the lines intersecting both L and L'. The lines L and L' are uniquely determined and are called the *directrices* of $S \cap S'$.

For details see the classic [6].

7.2. *First Description of* V_0

Fix a null-correlation N. After the choice of coordinates $x_0, \ldots, x_3$ for P and dual coordinates $\xi_0, \ldots, \xi_3$ for P^*, N can be defined by a nonsingular alternating 4×4 matrix A: E_x is the plane with dual coordinates $\xi = A \cdot x$. After the right choice of coordinates, A will be in normal form

$$A = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}$$

and E_x is the plane with coordinates $\xi = (x_1, -x_0, x_3, -x_2)$. Use the standard diagramm (D_2) putting $P^* = \mathrm{Gr}(2,3)$. The flag-variety $F = F(0,2,3)$ is a divisor on $P \times P^*$ with equation $\Sigma x_i \xi_i = 0$. Use the letter N also to denote the graph in $P \times P^*$ of the null-correlation. It satisfies the equations $\xi = A \cdot x$. Let $\mathcal{I} \subset \mathcal{O}_F$ be the ideal sheaf of N. Use the notation $\mathcal{O}_F(k_1, k_2)$ as explained in 2.1 and define analogously $\mathcal{O}(k_1, k_2) = \mathcal{O}_{P\times P^*}(k_1, k_2)$. Then we have exact sequences

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathcal{J}(0,1) & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-1,0) & \longrightarrow & \mathcal{O}(0,1) & \longrightarrow & \mathcal{O}_F(0,1) & \longrightarrow & 0 \\
 & & & & & \searrow & \downarrow & & \\
 & & & & & & \mathcal{O}_N(0,1) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

where the first horizontal morphism is multiplication by $\Sigma x_i \cdot \xi_i$. The dotted arrow is restriction, but after using p_* to identify $\mathcal{O}_N(0,1) \to \mathcal{O}_{\mathbb{P}}(1)$, this arrow maps $\xi_0, \ldots, \xi_4$ onto $x_1, -x_0, x_3, -x_2$ respectively. Now apply p_* to these exact sequences to obtain:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & p_*\mathcal{J}(0,1) & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_P(-1) & \xrightarrow[(x_0,\ldots,x_4)]{} & 4\mathcal{O}_P & \longrightarrow & p_*\mathcal{O}_F(0,1) & \longrightarrow & 0. \\
 & & & & & \underset{(x_1,-x_0,x_3,-x_2)}{\searrow} & \downarrow & & \\
 & & & & & & \mathcal{O}_P(1) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

The horizontal sequence became the well-known representation of $T_P(-1) \simeq p_*\mathcal{O}_F(0,1)$. We put $V_0 := p_*\mathcal{J}(0,1)$ and find that we can identify it with the subbundle of $T_P(-1)$ consisting of elements $\Sigma a_i \partial/\partial x_i$ satisfying

$$a_0 x_1 - a_1 x_0 + a_2 x_3 - a_3 x_2 = 0.$$

From the exact sequence

$$0 \to V_0 \to T_P(-1) \to \mathcal{O}_P(1) \to 0$$

one computes the Chern classes c_1, c_2 of V_0 to obtain

$$c_1 = 0, \quad c_2 = 1.$$

Proposition 5. $h^0(V_0)=0$, *i.e.*, V_0 *is stable, but for every plane* $E\subset P$, $h^0(V_0|E)=1$, *i.e.*, $V_0|E$ *is semi-stable and not stable.*

Proof. The direct image sheaf $q_*\mathscr{I}(0,1)$ vanishes, therefore

$$h^0(V_0)=h^0(p_*\mathscr{I}(0,1))=h^0(\mathscr{I}(0,1))=h^0(q_*\mathscr{I}(0,1))=0.$$

For every fibre $p^{-1}x$, $x\in P$, the restriction $\mathscr{I}(0,1)|p^{-1}x$ is generated by its two global sections. The canonical morphism

$$p^*V_0=p^*p_*\mathscr{I}(0,1)\to\mathscr{I}(0,1)$$

therefore is surjective and fits into an exact sequence

$$0\to\mathcal{O}_F(0,-1)\xrightarrow{s}p^*V_0\to\mathscr{I}(0,1)\to 0$$

with s vanishing only along N. Restricting this exact sequence to any fibre $q^{-1}e$, $e\in P^*$, one finds that $V_0|E\simeq p^*V|q^{-1}e$ has a nontrivial section vanishing simply at the null-point $x_E=p(N\cap q^{-1}e)$ of E. Since $c_1(V_0)=0$, this section generates $\Gamma(V|E)$, hence $V|E$ is semi-stable.

7.3. *Second Description of* V_0

By Lemma 5, the jumping lines of V_0 through an arbitrary point $x\in P$ are just the lines through x contained in the null-plane E_x. The jumping lines of V_0 therefore are parametrized by the general linear complex $S\subset \mathrm{Gr}(1,3)$ determined by the null-correlation N. For any jumping line L

$$V_0|L\simeq\mathcal{O}_L(1)\oplus\mathcal{O}_L(-1),$$

because x_E was a simple zero of the unique section in $V_0|E$.

Next use the standard diagram (D_1) and put

$$\tilde{S}=q^{-1}S\subset F(0,1,3)$$
$$\alpha:=p|\tilde{S}:\tilde{S}\to P$$
$$\beta:=q|\tilde{S}:\tilde{S}\to S$$
$$\mathcal{O}_{\tilde{S}}(k_1,k_2)=\alpha^*\mathcal{O}_P(k_1)\otimes\beta^*\mathcal{O}_S(k_2).$$

By base-change, the sheaf $\beta_*\alpha^*V_0(-1)$ is a line bundle on S and $\beta^*(\beta_*\alpha^*V_0(-1))$ is a line-subbundle of $\alpha^*V_0(-1)$. Using the Chern classes of V_0 and the fact that x_E is a simple zero of the non-trivial section in $\Gamma(V_0|E)$, one even finds $\beta_*\alpha^*V_0(-1)\simeq\mathcal{O}_S(-1)$. On $\tilde{S}$ one therefore has an exact sequence

$$0\to\mathcal{O}_{\tilde{S}}(1,-1)\to\alpha^*V_0\to\mathcal{O}_{\tilde{S}}(-1,1)\to 0.$$

This shows that one might also define V_0 as $\alpha_*\mathcal{O}_{\tilde{S}}(-1,1)$.

7.4. For later application we have to know some direct-image sheaves.

Lemma 11. a) $\quad\alpha_*\mathcal{O}_{\tilde{S}}(0,l)=(S^lV_0)(l)\qquad(l>0)$

b) $\alpha_{*1}\mathcal{O}_{\tilde{S}}(0,-l)=(S^{l-2}V_0)(-l)\qquad(l>0)$.

Proof. a) The case $l=1$ is our second description of V_0. If $l>1$, on every open $U\subset P$ we have a natural morphism

$$S^l(\Gamma(\alpha^{-1}U,\mathcal{O}_{\tilde{S}}(0,1)))\to\Gamma(\alpha^{-1}U,\mathcal{O}_{\tilde{S}}(0,l))$$

commuting with restrictions and bijective on the α-fibres. Sheafifying and using base-change for the α-fibres, one obtains a sheaf isomorphism

$$(S^l V_0)(l)=S^l(V_0(1))\to\alpha_*\mathcal{O}_{\tilde{S}}(0,l).$$

b) The case $l=1$ is trivial. Let us prove the case $l=2$: Choose another linear complex $S'\subset \mathrm{Gr}(1,3)$ and put $D=S\cap S'$ and $\tilde{D}=\beta^{-1}D$. We may choose S' such that D is non-singular and such that the directrices $L, L'\subset P$ of D are skew lines. Then there is the exact sequence

$$0\to\mathcal{O}_{\tilde{S}}(0,-2)\xrightarrow{S'}\mathcal{O}_{\tilde{S}}(0,-1)\to\mathcal{O}_{\tilde{D}}(0,-1)\to 0.$$

The direct image sheaves of $\mathcal{O}_{\tilde{S}}(0,-1)$ under α vanish. Hence

$$\alpha_{*1}\mathcal{O}_{\tilde{S}}(0,-2)\simeq\alpha_*\mathcal{O}_{\tilde{D}}(0,-1).$$

To compute the line bundle $\alpha_*\mathcal{O}_{\tilde{D}}(0,-1)$, fix a line $M\subset P$ interesting neither L nor L'. Then

$$\alpha|\alpha^{-1}M\cap\tilde{D}\to M$$

is an isomorphism and $\beta|\alpha^{-1}M\cap\tilde{D}$ identifies this curve with a conic $C\subset \mathrm{Gr}(1,3)$ parametrizing the regulus of lines intersecting L, L', and M. So the degree of $\mathcal{O}_{\tilde{S}}(-1)$ on C is -2 and

$$\alpha_*\mathcal{O}_D(0,-1)\simeq\mathcal{O}_P(-2).$$

If $l>0$, over every open $U\subset P$ we have the cup-product pairing

$$\Gamma(\alpha^{-1}U,\mathcal{O}(0,l-2))\otimes H^1(\alpha^{-1}U,\mathcal{O}(0,-l))$$
$$\to H^1(\alpha^{-1}U,\mathcal{O}(0,-2)).$$

It commutes with restrictions and on the α-fibres becomes the perfect pairing of Serre-duality. We obtain a perfect pairing of sheaves

$$\alpha_*\mathcal{O}_{\tilde{S}}(0,l-2)\otimes_{\mathcal{O}_P}\alpha_{*1}\mathcal{O}_{\tilde{S}}(0,-l)\to\mathcal{O}_P(-2)$$

showing

$$\alpha_{*1}\mathcal{O}_{\tilde{S}}(0,-l)\simeq S^{l-2}(V_0(1))^*\otimes\mathcal{O}_P(-2)$$
$$\simeq(S^{l-2}V_0^*)(-l).$$

Because of $V_0\simeq V_0^*$ this proves our assertion.

7.5. *Third Description of V_0*

Via α_* the spaces $\Gamma(\mathcal{O}_{\tilde{S}}(0,1)=\Gamma(\mathcal{O}_S(1))$ and $\Gamma(V_0(1))$ can be identified. Any nontrivial section in $\Gamma(\mathcal{O}_S(1))$ defines another linear complex S' and a set D as above with two skew directrices L, $L'\subset P$. The corresponding section in $\Gamma(V(1))$ vanishes at $L\cup L'$

only and there of order one (as $c_2(V(1))=2$ tells us). We therefore might also define V_0 by an extension

$$0\to\mathcal{O}\to V_0(1)\to\mathcal{I}_{L\cup L'}(2)\to 0$$

(compare Lemma 3).

8. Restricting Stable Bundles to Hyperplanes

8.1. Here will be given the proof of Theorem 3 from the Introduction. So fix some stable rank-2 bundle V over $\mathbb{P}_n$, $n\geqq 3$. If $V|E$ is stable for one hyperplane E, Proposition 1 tells that it is stable on almost all hyperplanes E. So assume that $V|E$ is unstable on all hyperplanes $E\subset\mathbb{P}_n$.

After replacing V by V_{norm}, one may assume that either $c_1=c_1(V)=0$ or $c_1=-1$. Stability of V then is equivalent with $h^0(V)=0$ and instability of $V|E$ with $h^0(V|E)>0$. Lemma 6 shows that $V|E$ can never be trivial.

One has $d(V)=0$ or 1 by the Theorem of Grauert-Mülich (Theorem 1), i.e., $d(V|L)=0$ or 1 for the general line $L\subset\mathbb{P}_n$. Every hyperplane $E\subset\mathbb{P}_n$ containing such a line must satisfy $d(V|E)=0$ or 1 too. The Lemmas 2′ and 5 then can be applied to show that $h^0(V|E)=1$ for such an E. Next one can apply the standard construction in the case $m=n-1$ (putting $\mathbb{P}_n^*=\mathrm{Gr}(n-1,n)$). In particular one obtains from

4.1. an open set $U\subset\mathbb{P}_n^*$ parametrizing the hyperplanes E with $h^0(V|E)=1$. The complement of U in $\mathbb{P}_n^*$ even is a subvariety A of codimension $\geqq 2$.

4.2. a morphism $s:q^*\mathcal{O}_{\mathbb{P}^*}(-l)\to p^*V$, $l>0$, vanishing at a subvariety $Z\subset F$ *of codimension* 2. (*If* Z were empty, then V would be decomposable, hence unstable.) Whenever $e\in U$, then $Z\cap q^{-1}e$ is of codimension 2 in $q^{-1}e$ too, and is mapped under p on the zero-set of the unique non-trivial section $s_E\in\Gamma(V|E)$, $E\subset\mathbb{P}_n$ the hyperplane corresponding to e.

4.3. a rational cross-section $\Gamma\subset p^*P(V)$ degenerating over Z only and a surjective morphism $\tilde{p}:\Gamma\to P(V)$. If $e\in U$ with E the corresponding hyperplane, $x\in E$, and if $(x,E)\notin Z$, then the point of Γ over (x,E) is mapped onto the point in $P(V(x))$ determined by the subspace $\mathbb{C}\cdot s_E(x)$. In particular the direction $\mathbb{C}\cdot s_E(x)$ in the fibre $V(x)$ changes, if E varies.

For all E parametrized by points $e\in U$, the jumping lines $L\subset E$ of V appear in codimension one (Lemma 5). The jumping lines therefore form a hypersurface $S\subset\mathrm{Gr}(1,n)$. Whenever L is a jumping line, then $V|L$ contains a unique positive sub-line bundle $(V|L)^+\subset V|L$. If E is a hyperplane corresponding to $e\in U$, then $s_E|L$ completely lies in $(V|L)^+$. This shows in particular:

(7) Whenever $e_1,e_2\in U$ correspond to hyperplanes E_1,E_2 having in common a jumping line L, and if $x\in L$ is a point with $(x,E_1)\notin Z$, $(x,E_2)\notin Z$, then the subspaces $\mathbb{C}\cdot s_{E_1}(x)$ and $\mathbb{C}\cdot s_{E_2}(x)$ in $V(x)$ coincide.

8.2. *The Proof of Theorem 3 in the Cases $n>3$*

The points $x\in\mathbb{P}_n$, such that all lines through x are jumping lines, form a proper subvariety $X\subset\mathbb{P}_n$. Fix a point $x\in\mathbb{P}_n\setminus X$ and two hyperplanes E_1,E_2 through x with corresponding points $e_1,e_2\in U$. Since $n\geqq 4$, E_1 and E_2 will have in common at least a plane, and through every point of this plane, in particular through x, there will pass at least one jumping line $L \subset E_1 \cap E_2$. Whenever $(x, E_1) \notin Z$ and $(x, E_2) \notin Z$, then $s_{E_1}(x)$ will be a multiple of $s_{E_2}(x)$ by (7) above. This means that $\tilde{p}$ is constant on the part of Γ lying over $p^{-1}(x) \setminus (q^{-1}A \cup Z)$. Over the complement in F of $p^{-1}(X) \cup q^{-1}(A) \cup Z$, the morphism $\tilde{p}|\Gamma$ then has rank n only, a contradiction with 4.3.

8.3. *Proof that S is a General Linear Complex*

Let now be $n=3$. Assume that the hypersurface $S \subset \mathrm{Gr}(1,3)$ has degree >1. Then the cone described by the jumping lines through a general point $x \in \mathbb{P}_3$ is not linear. Fix a point $x \in \mathbb{P}_3$ not on

the set X described in 8.2;
the degeneration set of $p|Z$;
the degeneration set of $p|q^{-1}A$.

Then all planes E through x – except perhaps finitely many, say $E_1, \ldots, E_k$ – are parametrized by points in U and $s_E(x) \neq 0$. Whenever L_1, L_2 are two jumping lines through x not contained in the same plane E_i, $i=1, \ldots, k$, then they span such a general plane E and determine in $V(x)$ the same subspace $(V|L_1)^+(x) = (V|L_2)^+(x)$. For general x, not all jumping lines through x can be contained in the same plane E_i. So all of them, except perhaps $L_0 = E_1 \cap \ldots \cap E_k$ determine the same direction in $V(x)$. This will then be the direction $\mathbb{C} \cdot s_E(x)$ for all $E \neq E_1, \ldots, E_k$ not containing L_0. Again we arrive at the contradiction with 4.3: For general $x \in \mathbb{P}_n$, $\tilde{p}|\Gamma$ is constant over an open set of $p^{-1}x$.

Assume next that $\deg S = 1$, but that the linear complex S be special. Let $L_0 \subset \mathbb{P}_3$ be the line determining S. Whenever E is a plane not through L_0, the jumping lines in E form the pencil of lines through $x_E = E \cap L_0$. In particular s_E vanishes in this point x_E only. Whenever E_0 contains L_0, then all lines in E_0 are jumping lines and $V|E_0$ is properly unstable. In particular there is some $k>0$ with $h^0(V(-k)|E_0)=1$ by Lemma 2′. The unique non-trivial section in $V(-k)|E_0$ has finitely many zeroes only. In all points $x \in E_0$ except these finitely many, the jumping lines L determine the same subspace $(V|L)^+(x) \subset V(x)$. Varying E_0 through L_0, one obtains a Zariski-open subset of those points $x \in \mathbb{P}_3 \setminus L_0$. Whenever E is a plane not through L_0, then again $\mathbb{C} \cdot s_E(x) \subset V(x)$ is independent of E. Once more we arrive at the same contradiction with 4.3.

8.4. *Proof that d(V|L) is Constant for all Jumping Lines L*

So far we found that the jumping lines form a general linear complex S. In particular there is no plane containing jumping lines only. Therefore $h^0(V|E)=1$ for all planes E, and each restricted bundle $V|E$ admits a section s_E (unique up to multiplication by constants) vanishing in a single point $x_E \in E$ only, the null-point of E. x_E is the intersection of all the jumping lines in E. Every jumping line L carries the bundle $V|L \simeq \mathcal{O}_L(k) \oplus \mathcal{O}_L(-k+c_1)$ with $k=k(L)>0$ and $c_1=0$ or $=-1$.

$$k(L) = l \text{ for all jumping lines } L. \tag{8}$$

To prove this, fix an arbitrary plane $E \subset \mathbb{P}_3$ and some line $M \not\subset E$ intersecting E in its null-point x_E. The planes through M are parametrized by a line $R \subset \mathbb{P}_3^*$. Put $\tilde{R} := q^{-1}R \subset F$ and $\sigma := p|\tilde{R}$. This map σ is the converse of dilatating $\mathbb{P}_3$ along M. Also, if $\tilde{E}:=\sigma^{-1}E\subset\tilde{R}$, then $\sigma|\tilde{E}$ is the converse of dilatating the plane E in x_E. The situation is this

$$\begin{array}{ccccccc} R & \xleftarrow{q} & \tilde{R} & \supset & \tilde{E} & \supset & \sigma^{-1}(x_E) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \mathbb{P}_3 & \supset & E & \ni & x_E. \end{array}$$

Let M^* be the line associated with M by the null-correlation (see 7.1). Since M is not a jumping line, $M\cap M^*=\emptyset$. Furthermore, M^* is contained in E and does not pass through x_E. The restriction $\tilde{s}$ of $s:q^*\mathcal{O}_{\mathbb{P}^*}(-l)\to p^*V$ to $\tilde{R}$ vanishes only in $\tilde{M}:=\sigma^{-1}M^*$. Whenever $L\subset E$ is a jumping line in E, hence containing x_E, and $\tilde{L}\subset\tilde{E}$ its proper transform, then

$$\sigma^*V|\tilde{L}\simeq V|L\simeq\mathcal{O}_L(k)\oplus\mathcal{O}_L(c_1-k), \qquad k=k(L).$$

Near $\tilde{L}$, the morphism $\tilde{s}$ is just an ordinary section in p^*V. The image of $\tilde{s}|\tilde{L}$ therefore lies in the positive subbundle of $\sigma^*V|\tilde{L}$. The point $\tilde{M}\cap\tilde{L}$ is the only zero of $\tilde{s}|\tilde{L}$, hence $\tilde{s}|\tilde{L}$ must vanish there of order $k(L)$.

Now recall that s_E had its only zero at x_E. Over $\tilde{E}\setminus\sigma^{-1}(x_E)$, σ^*s_E therefore generates a trivial line subbundle $\mathbb{C}\cdot\sigma^*s_E$ of σ^*V, which on every line $\tilde{L}$ coincides with the positive subbundle of $\sigma^*V|\tilde{L}$. Therefore over $\tilde{E}\setminus\sigma^{-1}(x_E)$, the morphism $\tilde{s}:q^*\mathcal{O}_R(-l)\to\sigma^*V$ really has its image in this subbundle $\mathbb{C}\cdot\sigma^*s_E$, and one may view that morphism there as a section in $q^*\mathcal{O}_R(l)$ vanishing only in the points $\tilde{L}\cap\tilde{M}$ of $\tilde{M}$ and there of order $k(L)$. First of all this means that $k=k(L)$ is constant, independently of L. Secondly this means that over $\tilde{E}\setminus\sigma^{-1}(x_E)$ the two bundles $q^*\cdot\mathcal{O}_R(l)$ and $[\tilde{M}]^k$ are isomorphic. Intersecting with the inverse image of a line in E different from M^* and not passing through x_E, one finds indeed $k=l$.

8.5. At this point we can forget the standard-construction based on the planes in $\mathbb{P}_3$. Rather we have to use now the standard diagram (D_1) and the $\mathbb{P}_1$-bundles

$$S \xleftarrow{\beta} \tilde{S} \xrightarrow{\alpha} \mathbb{P}_3$$

as in 7.3. In 8.4 we proved

$$\alpha^*V|\beta^{-1}y\simeq\mathcal{O}_{\mathbb{P}_1}(l)\oplus\mathcal{O}_{\mathbb{P}_1}(c_1-l), \quad c_1=c_1(V)=0 \text{ or } -1,$$

for all points $y\in S$. The base-change principle is applied again, to show that $\beta_*\alpha^*V(-l)$ is a line bundle on S. We need that

$$\beta_*\alpha^*V(-l)\simeq\mathcal{O}_S(-l). \tag{9}$$

To prove this, fix some point $x\in\mathbb{P}_3$. The jumping lines $L\subset E_x$ through x are parametrized by a curve R,

$$R\subset S\subset \mathrm{Gr}(1,3)\subset\mathbb{P}_5$$

of degree one. Put again

$$\tilde{E}=\beta^{-1}R\subset\tilde{S},\ \sigma:=\alpha|\tilde{E}:\tilde{E}\to E_x.$$

By base-change from S to R we only have to verify

$$(\beta|\tilde{E})_*\sigma^*V\simeq\mathcal{O}_R(-l).$$

Let $C := \sigma^{-1}(x_E) \subset \tilde{E}$ be the exceptional curve. The section $\sigma^* s_E$ in $\sigma^* V$ vanishes only along C, and there of order l, i.e., the bundle $\sigma^* V \otimes [C]^{-l}$ admits a section without zeroes. This proves

$$(\beta|\tilde{E})_*(\sigma^* V \otimes [C]^{-l}) \simeq \mathcal{O}_R .$$

Formula (9) then follows readily from the isomorphism

$$\sigma^* V(-l) \simeq \sigma^* V \otimes [C]^{-l} \otimes \beta^* \mathcal{O}_R(-l).$$

8.6. *The End of the Proof.* The image of the canonical morphism

$$\beta^* \mathcal{O}_S(-l) = \beta^* \beta_* \alpha^* V(-l) \to \alpha^* V(-l)$$

is a line subbundle of $\alpha^* V(-l)$ and gives rise to an exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(0, -l) \to \alpha^* V(-l) \to \mathcal{O}_{\tilde{S}}(c_1 - 2l, l) \to 0 . \tag{10}$$

Since $l > 0$ and since $V(-l)$ is trivial on the fibres $\alpha^{-1} x$, $x \in \mathbb{P}_3$ this sequence cannot split, hence it defines a non-trivial class in $H^1(\mathcal{O}_{\tilde{S}}(2l - c_1, -2l))$. From Lemma 11b) and a), one finds using Leray's theorem

$$\begin{aligned} h^1(\mathcal{O}_{\tilde{S}}(2l - c_1, -2l)) &= h^0(\alpha_{*1} \mathcal{O}_{\tilde{S}}(2l - c_1, -2l)) \\ &= h^0((S^{l-2} V)(-c_1)) \\ &= h^0(\mathcal{O}_{\tilde{S}}(2 - 2l - c_1, 2l - 2)) \\ &= 0 \end{aligned}$$

if $2 - c_1 - 2l$ is negative. This shows that necessarily $l = 1$. Now apply α_* to the sequence (10) to obtain

$$V \simeq \alpha_*(\mathcal{O}_{\tilde{S}}(1, c_1 - 1)) .$$

Indeed, up to tensoring with $\mathcal{O}_{\mathbb{P}}(c_1)$, the bundle V coincides with the null-correlation bundle V_0 (compare 7.3). But the case $c_1 \neq 0$ can finally be excluded, since it leads to the contradiction $c_1 = 2c_1$. This proves Theorem 3.

9. Corollaries of Theorem 3

9.1. Schwarzenberger showed in [11, Theorem 10] that any rank-2 bundle V over the projective plane with discriminant

$$\Delta(V) = c_1(V)^2 - 4c_2(V) \geqq 0$$

is unstable. This is now easily generalized to $\mathbb{P}_n$, $n \geqq 2$.

Corollary 1. *Let V be some* rank-2 *bundle over $\mathbb{P}_n$ $n \geqq 2$. If $\Delta(V) \geqq 0$, then V must be unstable.*

Proof. Assume that V is stable. If $\Delta(V) \geqq 0$, then V cannot be the null-correlation bundle V_0, and by repeated application of Theorem 3 one finds that $V|E$ is stable for the general plane $E \subset \mathbb{P}_n$. Since $\Delta(V|E) = \Delta(V)$, this would contradict Schwarzenberger's result.

It should be mentioned here, that in [12, Theorem 8] Schwarzenberger constructed stable rank-2 bundles V over the projective plane with arbitrarily given discriminant

$$\Delta<0, \quad \Delta\equiv 0 \text{ or } 1 (\operatorname{mod} 4), \quad \Delta\neq -4.$$

That a bundle on $\mathbb{P}_2$ with $\Delta=-4$ necessarily is unstable, was noticed later by Maruyama [8, Lemma 4.5]. Now accidentally $\Delta(V_0)=-4$ and the null-correlation bundle is the only stable rank-2 bundle over any $\mathbb{P}_n$ with discriminant -4.

9.2. It is not hard to produce examples of non-isomorphic stable (or unstable) bundles V, W on $\mathbb{P}_2$, which become isomorphic when restricted to any line. For classification purposes it would be quite useful to solve the corresponding problem on $\mathbb{P}_n$, $n\geqq 3$: *Given two stable* rank-2 *bundles on* $\mathbb{P}_n$ *with* $V|E\simeq W|E$ *for every hyperplane* $E\subset\mathbb{P}_n$. *Does this imply* $V\simeq W$?

Unfortunately I do not know the answer in the most important case $n=3$, although Theorem 3 can be applied to settle this question affirmatively in dimensions $n\geqq 4$. One even obtains a little more:

Corollary 2. *Let* V, W *be two stable* rank-2 *bundles on* $\mathbb{P}_n$, $n\geqq 4$. *Assume that* $V|E\simeq W|E$ *for all hyperplanes* $E\subset\mathbb{P}_n$ *parametrized by some Zariski-open set* $U\subset\mathbb{P}_n^*$. *Then* $V\simeq W$.

Proof. The null-correlation bundle V_0 does not survive on $\mathbb{P}_4$. This can e.g. be deduced from the integrality condition [4, Theorem 22.4.1] on the Chern classes of bundles over $\mathbb{P}_4$. So if $V|\mathbb{P}_n$, $n\geqq 4$, is stable, repeated application of Theorem 3 shows, that $V|\mathbb{P}_2$ is stable for the general plane $\mathbb{P}_2\subset\mathbb{P}_n$. Because of Proposition 3, $V|E$ must be stable for all hyperplanes E containing such a plane. By Proposition 1, the hyperplanes $E\subset\mathbb{P}_n$ with $V|E$ unstable form a Zariski-closed subset $A\subset\mathbb{P}_n^*$. Since none of the hyperplanes E containing a general plane $\mathbb{P}_2$ belongs to A, the dimension of A cannot exceed 2. In the same way one defines a subvariety $B\subset\mathbb{P}_n^*$ for W. So the bundles V and W both are stable for all hyperplanes $E\subset\mathbb{P}_n$ parametrized by $U_0:=\mathbb{P}_n^*\setminus(A\cup B)$. From Proposition 4 one concludes that $U\supset U_0$.

Now apply the standard construction to the bundle $V^*\otimes W$ in the case $m=n-1$. On all hyperplanes E corresponding to points $e\in U_0$,

$$\Gamma(V^*\otimes W|E)=\operatorname{Hom}(V|E, W|E)=\operatorname{End}(V|E)\simeq\mathbb{C},$$

hence

$$q_*p^*(V^*\otimes W)\simeq\mathcal{O}_{\mathbb{P}^*}(-l), \quad l\geqq 0.$$

By (2) from 4.2, the canonical morphism

$$s: q^*\mathcal{O}_{\mathbb{P}^*}(-l)\to\mathcal{H}om(p^*V, p^*W)$$

can vanish identically only on fibres $q^{-1}e$, $e\in A\cup B$. But since $V|E\simeq W|E$ is stable for all hyperplanes corresponding to points $e\in U_0$, the restriction $s|q^{-1}e$ must be an isomorphism for all these e. One may view s as a morphism $p^*V\to p^*W\otimes q^*\mathcal{O}_{\mathbb{P}^*}(l)$ with $\det(s)$ vanishing on $q^{-1}(A\cup B)$ only. Since $q^{-1}(A\cup B)$ does not contain a hypersurface, $\det(s)$ cannot vanish at all. This implies $l=0$, and s becomes an isomorphism $p^*V\to p^*W$. Obviously this isomorphism descends to $\mathbb{P}_n$.

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