GUNNING, R. C., and R. NARASIMHAN
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Immersion of Open Riemann Surfaces

R. C. GUNNING and RAGHAVAN NARASIMHAN

The object of this paper is to prove the following result.[1]

Theorem. *Any (connected) open Riemann surface X admits a holomorphic immersion into the complex plane; that is, there is a holomorphic mapping $F: X \to \mathbb{C}$ which is a local homeomorphism.*

Before starting on the proof, we shall set out a few preliminaries. Let X be an open Riemann surface. An open subset U of X is called Runge in X if any holomorphic function on U can be approximated, uniformly on compact subsets of U, by holomorphic functions on X. For any compact subset K of X, set

$$\hat{K} = \{x \in X \mid |f(x)| \leqq \sup_{y \in K} |f(y)| \quad \text{for all } f \text{ holomorphic on} X\}.$$

The set $\hat{K}$ is the union of K with the relatively compact connected components of $X - K$, and $\hat{K}$ is again compact. An open subset U of X is Runge in X if and only if $\hat{K} \subset U$ for any compact subset $K \subset U$. (Theorem of BEHNKE-STEIN [1]; for a proof of this result in the above form, see MALGRANGE [4], pp. 344—345.)

For later use, we make the following remark.

Lemma 1. *Let K be a compact subset of X such that $K = \hat{K}$, and $\{x_1, \ldots, x_r\}$ be a finite set of points in $X - K$. Then, given $\varepsilon > 0$, there is a holomorphic function h on X such that $h(x_i) = 0$, $i = 1, \ldots, r$, and $|h(x) - 1| < \varepsilon$ for $x \in K$.*

Proof. For $1 \leqq i \leqq r$ and $\delta > 0$ there is a holomorphic function h_i on X such that $|h_i(x) - 1| < \delta$, $x \in K$, and $|h_i(x_i)| < \delta$. We have only to set

$$h(x) = \prod_{1 \leqq i \leqq r} (h_i(x) - h_i(x_i))$$

with a small enough δ.

Proposition 1. *Let K be a compact subset of X with $K = \hat{K}$. Then any continuous complex-valued function on K which is holomorphic in the interior of K can be approximated, uniformly on K, by holomorphic functions on X.*

When $X = \mathbb{C}$ this result is due to S. N. MERGELIAN; the general case is due to E. BISHOP [2].

[1] An interesting special case of the theorem has been proved by H. OELJEKLAUS (Unverzweigte Konkretisierung von Riemannschen Flächen. Bayr. Akad. Wiss. 1966).

Lemma 2. *Let D, D' be relatively compact, connected open subsets of X with smooth boundaries (that is, such that their closures are Riemann surfaces with boundary); suppose that $\bar{D} \subset D'$, and that D and D' are Runge in X. Then there are simple closed piecewise differentiable curves $\gamma_1, \ldots, \gamma_p$ in D, such that the images of these curves in the relative homology group $H_1(D', D)$ form a basis for that group, and that $K = \hat{K}$ where $K = \bar{D} \cup \gamma_1 \cup \ldots \cup \gamma_p$.*

Proof. In this case, the exact homology sequence of the pair (D', D) has the form

$$0 \to H_1(D) \to H_1(D') \to H_1(D', D) \to 0\,;$$

the singular cycles γ_i represent elements of $H_1(D')$, and they are required to be such that their images form a basis of $H_1(D', D)$. Note that these homology

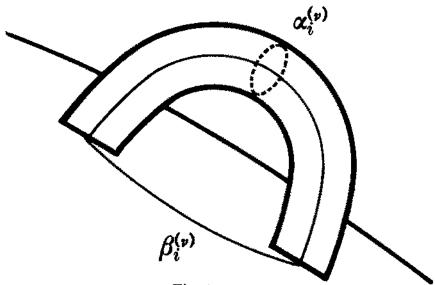

Fig. 1

groups are free abelian groups, so that the above exact sequence splits; generators for $H_1(D)$ together with the cycles γ_i are generators for the homology group $H_1(D')$. To find the desired curves, we use the familiar classification theorem for compact orientable surfaces with boundary (see SEIFERT-THRELFALL [5]); any such surface is homeomorphic to a sphere with a finite number of handles attached and a finite number of open discs deleted. In particular, each connected component of the bordered Riemann surface $\bar{D}' - D$ has this form; a typical connected component $W^{(\nu)}$ will thus be homeomorphic to a sphere with h_ν handles attached and d_ν discs $\Delta_i^{(\nu)}$ deleted. For each handle select a pair $\alpha_i^{(\nu)}$, $\beta_i^{(\nu)}$ of simple closed curves forming a canonical pair of cuts on the handle, as in the figure; for different handles the cuts will be taken to be disjoint. Suppose that for the first $c_\nu \leqq d_\nu$ of the discs $\Delta_j^{(\nu)}$ the boundaries $\partial \Delta_j^{(\nu)}$, $1 \leqq j \leqq c_\nu$, are disjoint from the subset $\partial \bar{D} \subset D'$. Let $\sigma_j^{(\nu)}$ be the boundary of a slightly larger disc containing $\Delta_j^{(\nu)}$. For the remaining discs $\Delta_k^{(\nu)}$, $c_\nu < k \leqq d_\nu$, select simple arcs $\tau_k^{(\nu)}$ such that $\tau_k^{(\nu)}$ joins $\Delta_k^{(\nu)}$ to $\Delta_{k+1}^{(\nu)}$, $(c_\nu < k < d_\nu)$, such that except for the end points the arc $\tau_k^{(\nu)}$ lies in the interior of $W^{(\nu)}$, and such that the arcs $\tau_k^{(\nu)}$ are disjoint from one another and from the curves $\alpha_i^{(\nu)}$, $\beta_i^{(\nu)}$. Then these curves $\alpha_i^{(\nu)}$, $\beta_i^{(\nu)}$, $\sigma_j^{(\nu)}$, $\tau_k^{(\nu)}$ represent generators for the relative homology group $H_1(D', D)$, and contain a subset which form a basis (for each ν for which $c_\nu \geqq 1$, we must drop one of the $\sigma_j^{(\nu)}$). The arcs $\tau_k^{(\nu)}$ can be extended through $\bar{D}$ to form simple closed curves. The basis so constructed will be taken as the curves $\gamma_1, \ldots, \gamma_p$. Note that these curves have the property that, with respect to any fixed triangulation, any cycle γ on the complex $K=\bar{D}\cup\gamma_1\cup\ldots\cup\gamma_p$ is of the form $\gamma=\Sigma n_i\gamma_i+\lambda$ where $n_i\in\mathbb{Z}$ and λ is a cycle in $\bar{D}$; this is actually an equality of cycles, not just a homology relation. This follows from the fact that if we contract $\bar{D}$ to a point, K becomes a disjoint union of wedges of circles.

Now to prove that $K=\hat{K}$, it suffices to show that $D'-K$ has no relatively compact connected components, since D' is Runge in X. If there were such a component U, its boundary $\partial U=\gamma$ would be a cycle in K, hence would be of the form $\gamma=\Sigma n_i\gamma_i+\lambda$ as above. Passing to the relative homology group $H_1(D', D)$ (denoted by putting a tilde over the cycles), it follows that $0=\Sigma n_i\tilde{\gamma}_i$ since $\tilde{\gamma}=\partial\tilde{U}=0$; thus $n_i=0$ and $\gamma\subset\bar{D}$. However this would mean that U is a relatively compact connected component of $X-\bar{D}$, contradicting the hypothesis that D is Runge in X.

Lemma 3. *Let $[a, b]$ be a closed interval on the real line, f be a continuous complex-valued function on $[a, b]$, and c be any complex constant. Then there exists a continuous complex-valued function g with compact support in the open interval (a, b) such that*

$$(1)\qquad \int_a^b e^{f(x)+g(x)}\,dx=c\,,\qquad \int_a^b g(x)\,e^{f(x)+g(x)}\,dx\neq 0\,.$$

Proof. It is clear that there exists a step function $u(x)$ with compact support in (a, b) and satisfying conditions (1). (In fact, if $c\neq 0$ we can choose the step function to be zero outside any subinterval, and if $c=0$ we can choose the function to be zero outside two subintervals.) Select a uniformly bounded sequence $\{g_\nu\}$ of continuous functions with compact support in (a, b) converging to the step function u, uniformly on compact subsets of the complement of a finite subset of $[a, b]$, and consider the complex analytic functions of a complex variable s defined by

$$\varphi(s)=\int_a^b e^{f(x)+su(x)}\,dx\,,\qquad \varphi_\nu(s)=\int_a^b e^{f(x)+sg_\nu(x)}\,dx\,.$$

The functions $\varphi_\nu(s)$ converge to $\varphi(s)$ uniformly on compact subsets of the s-plane. Since $\varphi(1)=c$ and $\varphi'(1)\neq 0$, it is evident that for a sufficiently large value of ν there will be a point $s_0\in\mathbb{C}$, $s_0\neq 0$, such that $\varphi_\nu(s_0)=c$ and $\varphi'_\nu(s_0)\neq 0$; the function $g(x)=s_0g_\nu(x)$ then has the desired properties.

Lemma 4. *Let D, D' be relatively compact, connected open subsets of X with smooth boundaries; suppose that $\bar{D}\subset D'$ and that both are Runge in X. Let ω be a nowhere vanishing holomorphic 1-form on X such that $\int_\gamma\omega=0$ for any closed piecewise differentiable curve γ in D. Then for any $\varepsilon>0$ there exists a holomorphic function w on X such that $|w(x)|<\varepsilon$ for every $x\in D$, and that $\int_\gamma\omega e^w=0$ for all closed piecewise differentiable curves γ in D'.*

Proof. Let $\gamma_1, \ldots, \gamma_p$ be the curves whose existence is assured by Lemma 2; and applying Lemma 2 to the pair (D, θ) let $\gamma_{p+1}, \ldots, \gamma_q$ be simple closed piecewise differentiable curves in D forming a basis for $H_1(D)$ and such that $L = \hat{L}$ where $L = \gamma_{p+1} \cup \ldots \cup \gamma_q$. To prove the lemma, it is sufficient to find a function w holomorphic on X and such that $|w(x)| < \varepsilon$ for all $x \in D$ and $\int_{\gamma_i} \omega e^w = 0$ for $i = 1, \ldots, q$.

By Lemma 3 there are continuous functions u_i on the set $\gamma_1 \cup \ldots \cup \gamma_q$ such that the supports of the u_i are disjoint, u_i is identically zero on γ_j for $i \neq j$, u_i is zero on $\bar{D}$ for $1 \leqq i \leqq p$, and

$$(2) \qquad \begin{cases} \int_{\gamma_i} \omega e^{u_i} = 0, \ \int_{\gamma_i} u_i \omega e^{u_i} \neq 0 \quad \text{for} \quad 1 \leqq i \leqq p, \\ \int_{\gamma_i} u_i \omega \neq 0 \quad \text{for} \quad p < i \leqq q. \end{cases}$$

For any $s = (s_1, \ldots, s_q) \in \mathbb{C}^q$ put

$$\varphi_i(s) = \int_{\gamma_i} \omega e^{s_1 u_1 + \cdots + s_q u_q}.$$

This function is an entire function of the q complex variables $(s_1, \ldots, s_q)$; and for the point $a = (1, \ldots, 1, 0, \ldots, 0)$, with the first p coordinates having the value 1, we have

$$(3) \qquad \begin{cases} \varphi_i(a) = 0 \quad \text{for} \quad i = 1, \ldots, q, \\ \dfrac{\partial \varphi_i}{\partial s_j}(a) = 0 \quad \text{for} \quad i \neq j,\ i, j = 1, \ldots, q, \\ \dfrac{\partial \varphi_i}{\partial s_i}(a) \neq 0 \quad \text{for} \quad i = 1, \ldots, q. \end{cases}$$

(For since $u_i | \gamma_j = 0$ whenever $i \neq j$, it is evident that $\dfrac{\partial \varphi_i}{\partial s_j}(a) = 0$ for $i \neq j$. If $1 \leqq i \leqq p$ we have $\varphi_i(a) = \int_{\gamma_i} \omega e^{u_i} = 0$ and $\dfrac{\partial \varphi_i}{\partial s_i}(a) = \int_{\gamma_i} u_i \omega e^{u_i} \neq 0$ by (2); while if $p < i \leqq q$ we have $\varphi_i(a) = \int_{\gamma_i} \omega = 0$ by assumption on ω, since $\gamma_i \subset D$, and $\dfrac{\partial \varphi_i}{\partial s_i}(a) = \int_{\gamma_i} u_i \omega \neq 0$ by (2).) Thus as a complex analytic mapping

$$\varphi = (\varphi_1, \ldots, \varphi_q) : \mathbb{C}^q \to \mathbb{C}^q,$$

the image $\varphi(a) = 0 \in \mathbb{C}^q$, and the Jacobian matrix $\dfrac{\partial(\varphi)}{\partial(s)}(a)$ is nonsingular.

For $i = 1, \ldots, p$ extend the function u_i to be identically zero on $\bar{D}$; then u_i is continuous on K and holomorphic in the interior of K. Since $K = \hat{K}$, it follows from Proposition 1 that there exist sequences of functions $w_i^{(\nu)}$ holomorphic on X and converging uniformly to u_i (as extended) on all of K; in particular, the functions $w_i^{(\nu)}$ converge uniformly to zero on $\bar{D}$. For $i = p + 1, \ldots, q$

the functions u_i can by Proposition 1 be approximated uniformly on L by functions v_i holomorphic in all of X; as an immediate consequence of Lemma 1 the functions v_i can be chosen so that they all vanish on the finite point set $\partial D \cap (\gamma_1 \cup \ldots \cup \gamma_p)$. Let v'_i be the functions defined on K by setting $v'_i(x) = v_i(x)$ whenever $x \in \bar{D}$ and $v'_i(x) = 0$ whenever $x \in \bar{D} - K$; these functions are continuous on K and holomorphic on the interior of K, so by another application of Proposition 1 they also can be approximated uniformly on K by functions w_i holomorphic on X. Thus there exist sequences of functions $w_i^{(\nu)}$ holomorphic on X and converging uniformly to the functions u_i on $\gamma_1 \cup \ldots \cup \gamma_q$ for $i = p+1, \ldots, q$ as well. Select a fixed value $\mu = (\mu_{p+1}, \ldots, \mu_q)$ and consider the complex analytic mapping $\psi = (\psi_1, \ldots, \psi_q) : \mathbb{C}^q \to \mathbb{C}^q$ where

$$\psi_i(s) = \int_{\gamma_i} \omega e^{s_1 u_1 + \cdots + s_p u_p + s_{p+1} w_{p+1}^{(\mu)} + \cdots + s_q w_q^{(\mu)}} \quad ;$$

if μ has been chosen large enough, $\psi(a) = 0$ and the Jacobian matrix $\frac{\partial(\psi)}{\partial(s)}(a)$ is nonsingular. Now introduce the functions

$$\psi_i^{(\nu)}(s) = \int_{\gamma_i} \omega e^{s_1 w_1^{(\nu)} + \cdots + s_p w_p^{(\nu)} + s_{p+1} w_{p+1}^{(\mu)} + \cdots + s_q w_q^{(\mu)}} \, ;$$

these are entire functions of s which converge uniformly on compact subsets of $\mathbb{C}^q$ to the functions ψ_i as $\nu \to \infty$. Thus for any $\delta > 0$ and any value of ν sufficiently large, there is a point $s^0 = (s_1^0, \ldots, s_q^0)$ with $|s^0 - a| < \delta$ such that $\psi_i^{(\nu)}(s^0) = 0$ for $i = 1, \ldots, q$. The function

$$w = s_1^0 w_1^{(\nu)} + \cdots + s_p^0 w_p^{(\nu)} + s_{p+1}^0 w_{p+1}^{(\mu)} + \cdots s_q^0 w_q^{(\mu)}$$

then is holomorphic on X, satisfies $\int_{\gamma_i} \omega e^w = 0$ for all i, and satisfies $|w(x)| < \varepsilon$ for all $x \in \bar{D}$, provided that δ is chosen sufficiently small and ν sufficiently large. That completes the proof of the lemma.

With these preparations made, the proof of the theorem is quite easy. The Riemann surface X can be exhausted by a sequence $\{D_k\}$ of relatively compact connected subsets which are Runge in X, have smooth boundaries, and satisfy $\bar{D}_k \subset D_{k+1}$, [1]. It is well-known that there exists a holomorphic differential 1-form ω on X which has no zeros; for since $H^2(X, \mathbb{Z}) = 0$ and X is a Stein manifold, its analytic tangent line bundle is trivial, [3]. By applying the obvious induction argument to Lemma 4, it is apparent that there exists a holomorphic function w on X such that $\int_{\gamma} \omega e^w = 0$ for all piecewise differentiable closed curves γ in X. Setting

$$F(x) = \int_{x_0}^{x} \omega e^w ,$$

the function F is clearly holomorphic and single valued on the entire surface X, and $dF = \omega e^w \neq 0$; that concludes the proof.

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R. C. GUNNING
Department of Mathematics
Princeton University

Prof. RAGHAVAN NARASIMHAN
The Institute for Advanced Study
Princeton, New Jersey

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