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Differential Operators of Principal Type

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1. Introduction

In the study of a differential operator $P(x, D)$[1]) a fundamental question is of course whether the equation

$$P(x, D)u = f \tag{1.1}$$

can always be solved at least locally when f has a high degree of local regularity. The first example where the answer is negative was given by H. LEWY [5]. For the operator

$$P(x, D) = -iD_1 + D_2 - 2(x^1 + ix^2)D_3$$

he proved that for some $f \in C^\infty$ there is not in any open set a solution which has Hölder continuous derivatives of the first order. (In fact, there does not even exist a distribution solution of (1.1) for every $f \in C^\infty$.)

On the other hand, various sufficient conditions for the local existence of solutions of (1.1) are known. First of all, the existence of solutions has been proved for every equation with constant coefficients (MALGRANGE, EHRENPREIS). It is also well known that (1.1) can be solved locally when P is elliptic. Another class of operators for which this is true was introduced by HÖRMANDER [3], Chap. IV; it was defined by the conditions

A) The differential operator $P(x, D)$ is of order m and, if $p(x, \xi)$ is the homogeneous part of $P(x, \xi)$ of order m, we have

$$\sum_1^n |\partial p(x, \xi)/\partial \xi_j|^2 \neq 0, \; 0 \neq \xi \in R^n . \tag{1.2}$$

(This means that the characteristic surface has no real singular point.)

B) $p(x, \xi)$ has real coefficients.

Under mild smoothness assumptions it was proved for an operator P in this class that if Ω_δ is the sphere with radius δ and centre at a fixed point where (1.2) is valid, then we have with L^2 norms

$$\sum_{|\alpha| < m} \delta^{2(\alpha - m)} \|D_\alpha u\|^2 \leqq C_0 \|P(x, D)u\|^2 , \quad u \in C_0^\infty(\Omega_\delta), \; \delta \leqq \delta_0 . \tag{1.3}$$

From this inequality it was concluded that the equation $P^t u = f$, where P^t is the formal adjoint of P, has a solution $u \in L^2(\Omega_\delta)$ for every $f \in L^2(\Omega_\delta)$. The

[1]) For notations see [3], particularly p. 176.

main purpose of this paper is to make a more detailed investigation of the conditions under which (1.3) is valid and to prove stronger consequences of that inequality. Before stating the results we introduce a definition.

Definition 1.1. A differential operator $P(x, D)$ defined in an open set Ω is said to be of principal type in Ω if (1.3) is valid, with Ω_δ defined as $\{x; x \in \Omega, |x - x_0| < \delta\}$, where x_0 is an arbitrary fixed point in Ω, when $\delta \leqq \delta_0(x_0)$.

This definition agrees with Definition 2.1, p. 186, in HÖRMANDER [3] when the coefficients are constant.

Section 2 will be devoted to proving conditions which are necessary for a differential operator to be of principal type. It is easily proved that (1.2) is necessary (Theorem 2.2). This is not the case for condition B, however. It was already pointed out in [3] that with no essential change of proof it might be replaced by the weaker condition, where $\bar{p}(x, \bar{\xi}) = \overline{p(x, \xi)}$,

B') $\qquad \bar{p}(x, D)\, p(x, D) - p(x, D)\, \bar{p}(x, D) \quad$ is of order $< 2m - 1$.

In general this commutator may be of order $2m - 1$ even if P is elliptic, in which case (1.1) is also known to hold (see e.g. [2] or [4]). B' is therefore not a necessary condition either. However, using arguments developed from an analysis of Lewy's example we shall prove in Theorem 2.1 that if (1.3) is valid then

B'') $\qquad C_{2m-1}(x, \xi) = 0 \quad \text{if} \quad p(x, \xi) = 0,\ x \in \Omega,\ \xi \in R^n$,

where C_{2m-1} is the homogeneous part of order $2m - 1$ of the commutator in B'. (C_{2m-1} involves only first order derivatives of the coefficients of p and is thus defined when these coefficients are in C^1 even if the commutator itself does not have a sense.) When P is of the first order and the coefficients are analytic we also prove a stronger result in section 3. In fact, we prove that the equation $Pu = f$ does not have a solution $u \in \mathscr{D}'$ for every $f \in \mathscr{D} = C_0^\infty$ unless B'' is valid. When B'' is not fulfilled for all x in an open non void subset of Ω we also show that for some $f \in C^\infty(\Omega)$ the equation $Pu = f$ cannot be solved anywhere in Ω. This extends Lewy's result mentioned above.

In section 4, Theorem 4.3, we prove sufficient conditions for a differential operator to be of principal type. We then have to use a condition which is stronger than B'' but weaker than B'. In the first order case it is very close to B'', however.

After a preparatory discussion in section 5 of some normed spaces of distributions, we show in section 6 that the L^2 norms in (1.3) may be replaced by such norms if P is of principal type with coefficients in C^∞. In section 7 this leads to the result that every point in Ω has a neighbourhood Ω_δ, depending on the integer $k \geqq 0$, such that there exists a solution of (1.1) with all derivatives of order $\leqq k + m - 1$ in $L^2(\Omega_\delta)$ if the derivatives of f of order $\leqq k$ are in $L^2(\Omega_\delta)$. The method is an improvement of that used in [4] to study distribution solutions of formally hypoelliptic equations.

2. Necessary conditions for an operator to be of principal type

In this section we shall first prove that B″ is a necessary condition for (1.3) to hold. In order to simplify the notations we always assume that the point x which occurs in B″ is the origin. Thus we assume from now on that Ω is a neighbourhood of 0. For the sake of brevity we also introduce the notation

$$\|u\|_k = \Big(\sum_{|\alpha|\leqq k} \|D_\alpha u\|^2\Big)^{1/2}.$$

Lemma 2.1. Suppose that there exists a function $u \in C^\infty(\Omega)$ such that

$$p(x, \operatorname{grad} u) = o(|x|^2),\ x \to 0 . \tag{2.1}$$

Assume further that u has the Taylor expansion

$$u(x) = i\sum_1^n x^j \xi_j + \frac{1}{2}\sum_1^n \sum_1^n x^j x^k \alpha_{jk} + O(|x|^3) \tag{2.2}$$

where ξ_j are real and satisfy

$$\sum_1^n |\partial p(0, \xi)/\partial \xi_j|^2 \neq 0 , \tag{2.3}$$

the matrix α_{jk} is symmetric and the matrix $\operatorname{Re}\alpha_{jk}$ is negative definite. If P has continuous coefficients we then have when $v \in C_0^\infty(\Omega)$

$$\sup \|v\|_{m-1}/\|P(x, D)v\| = \infty . \tag{2.4}$$

Hence (1.3) does not hold for any constant C even for fixed δ when $x_0 = 0$.

Proof. First note that since $\operatorname{Re}\alpha_{jk}$ is a negative definite matrix, it follows from (2.2) that

$$\operatorname{Re} u(x) = \sum_1^n \sum_1^n x^j x^k \operatorname{Re}\alpha_{jk} + O(|x|^3) \leqq -2a|x|^2 + O(|x|^3) ,$$

where $a < 0$. Hence

$$\operatorname{Re} u(x) \leqq -a|x|^2 \tag{2.5}$$

for sufficiently small $|x|$. Replacing if necessary Ω by a smaller neighbourhood of 0, we may assume that (2.5) is valid in the whole of Ω. Then the function

$$v_t = \varphi e^{tu} ,$$

where $\varphi \in C_0^\infty(\Omega)$, $t > 0$, is in $C_0^\infty(\Omega)$ and (2.5) gives

$$|v_t| \leqq |\varphi| e^{-at|x|^2} .$$

Our aim is to show that

$$\|v_t\|_{m-1}/\|P(x, D)v_t\| \to \infty \quad \text{as} \quad t \to \infty .$$

This will prove (2.4).

We can write

$$P(x, D)v_t = e^{tu}\sum_0^m a_j t^j$$

where a_j are functions of x. We only need to compute a_m and a_{m-1}. Leibniz' formula gives

$$P(x, D)(\varphi e^{tu}) = \sum_{\alpha} \frac{D_\alpha \varphi}{|\alpha|!} P^{(\alpha)}(x, D) e^{tu},$$

where

$$P^{(\alpha)}(x, \zeta) = \partial^k P(x, \zeta)/\partial \zeta_{\alpha_1} \ldots \partial \zeta_{\alpha_k},\ k = |\alpha|.$$

Introduce the decomposition

$$P(x, D) = p(x, D) + q(x, D) + r(x, D)$$

with p and q homogeneous of order m and $m-1$, respectively, and r of order $< m-1$. Then the coefficients of $e^{tu}t^m$ and $e^{tu}t^{m-1}$ in

$$\varphi p(x, D) e^{tu} + \left(\sum_1^n D_j \varphi p^{(j)}(x, D) + \varphi q(x, D) \right) e^{tu} \tag{2.6}$$

are also a_m and a_{m-1}. (We use the notation j for the multi-index of length 1 consisting only of the index j.) We therefore get

$$a_m = i^{-m} \varphi p(x, \operatorname{grad} u), \tag{2.7}$$

and

$$a_{m-1} = i^{-m+1} \left\{ \sum_1^n D_j \varphi p^{(j)}(x, \operatorname{grad} u) + \varphi(q(x, \operatorname{grad} u) + i^{m-1} a) \right\}, \tag{2.8}$$

where a is a continuous function depending on the coefficients of p and on u. The hypothesis (2.1) gives in view of (2.7)

$$a_m(x) = o(|x|^2), \quad x \to 0. \tag{2.9}$$

We now choose φ so that $\varphi(0) = 1$ and

$$\sum_1^n D_j \varphi(0) p^{(j)}(0, \xi) + \varphi(0)(q(0, \xi) + a(0)) = 0.$$

This is possible in view of (2.3). The continuity of a_{m-1} then gives

$$a_{m-1}(x) = o(1), \quad x \to 0. \tag{2.10}$$

Take a positive number ε and choose a neighbourhood U of 0 such that

$$|a_m(x)| < \varepsilon |x|^2, \quad |a_{m-1}(x)| < \varepsilon, \quad x \in U.$$

Using (2.5) we then obtain with a constant C

$$|P(x, D) v_t| \leq t^{m-1}(\varepsilon t |x|^2 + \varepsilon + C/t) e^{-ta|x|^2}, \quad x \in U.$$

Hence

$$\int_U |P(x, D) v_t|^2 dx \leq t^{2m-2} \int (\varepsilon t |x|^2 + \varepsilon + C/t)^2 e^{-2ta|x|^2} dx$$
$$= t^{2m-2-n/2} \varepsilon^2 \int (|x|^2 + 1 + C/t\varepsilon)^2 e^{-2a|x|^2} dx.$$

The integral converges to

$$B^2 = \int (|x|^2 + 1)^2 e^{-2a|x|^2} dx$$

when $t \to \infty$. Hence we have for large t

$$\int_U |P(x, D) v_t|^2 dx \leq t^{2m-2-n/2} 2 B^2 \varepsilon^2.$$

Since (2.5) also gives

$$\int_{CU} |P(x, D)v_t|^2 dx = O(t^{2m} e^{-2ct})$$

for some $c > 0$, it follows for sufficiently large t that

$$\int_{\Omega} |P(x, D)v_t|^2 dx \leqq 4B^2 t^{2m-2-n/2} \varepsilon^2 . \tag{2.11}$$

Next we estimate $\|v_t\|_{m-1}$ from below, which is much easier. Note that it follows from (2.3) that $\xi \neq 0$ if $m > 1$. In that case we may thus assume for example that $\xi_1 \neq 0$. Consider

$$D_1^{m-1} v_t = (\varphi (D_1 u)^{m-1} t^{m-1} + \cdots) e^{tu} .$$

(When $m = 1$ we should read $0^0 = 1$ here.) Since $\varphi(0)\,(D_1 u(0))^{m-1} = \xi_1^{m-1} \neq 0$ we have

$$|\varphi|\;|D_1 u|^{m-1} \geqq 2c > 0$$

in a neighbourhood of 0. Since $\operatorname{Re} u(x) = O(|x|^2)$ we have $\operatorname{Re} u(x) \geqq -A|x|^2$ for some $A > 0$ so we get for sufficiently large t when x is in this neighbourhood

$$|D_1^{m-1} v_t| \geqq t^{m-1} c e^{-tA|x|^2} .$$

Hence it follows for large t that

$$\|v_t\|_{m-1}^2 \geqq \|D_1^{m-1} v_t\|^2 \geqq \int_{t|x|^2 < 1} t^{2m-2} c^2 e^{-2tA|x|^2} dx = B_1^2 t^{2m-2-n/2} , \tag{2.12}$$

where B_1 is another constant $\neq 0$.

Combining (2.11) and (2.12) we obtain

$$\varliminf_{t \to \infty} \|v_t\|_{m-1} / \|P(x, D) v_t\| \geqq B_1 / 2B\varepsilon .$$

Since ε is an arbitrary positive number, it follows that

$$\lim_{t \to \infty} \|v_t\|_{m-1} / \|P(x, D) v_t\| = \infty ,$$

which completes the proof.

We shall now study the condition (2.1) further.

Lemma 2.2. Assume that the coefficients of p are in C^2 at 0 and that (2.3) holds. Then (2.1) can be fulfilled by a function $u \in C^\infty(\Omega)$ with the Taylor expansion (2.2) if and only if

$$p(0, \xi) = 0 , \tag{2.13}$$

$$\sum_1^n \alpha_{jk} p^{(k)}(0, \xi) = -i p_j(0, \xi) , \quad j = 1, \ldots, n , \tag{2.14}$$

where

$$p_j(x, \xi) = \partial p(x, \xi) / \partial x^j . \tag{2.15}$$

Proof. If $u \in C^\infty(\Omega)$ it is clear that (2.1) is fulfilled if and only if $p(x, \operatorname{grad} u)$ and its derivatives of order $\leqq 2$ vanish at 0. The equations (2.13) and (2.14) express the vanishing of $p(x, \operatorname{grad} u)$ and its first derivatives at 0. Hence it only remains to prove that the second derivatives of $p(x, \operatorname{grad} u)$ will vanish for a suitable choice of the third derivatives of u, if (2.13) and (2.14) are valid. However, in view of (2.3) which shows the existence of non characteristic planes through 0 this follows trivially from the Cauchy-Kovalevsky theorem if we replace the coefficients of p by their Taylor expansions of order 2 at 0, and this does not change the condition on u.

The existence of a matrix α_{jk} satisfying the requirements of Lemmas 2.1 and 2.2 is examined in the next lemma.

Lemma 2.3. Given two vectors $(a_1, \ldots, a_n)$ and $(f_1, \ldots, f_n)$ with complex components and some $a_j \neq 0$, there is a symmetric matrix α_{jk} with negative definite real part satisfying

$$(2.16) \qquad \sum_1^n \alpha_{kj} a_j = f_k\,, \quad k = 1, \ldots, n\,,$$

if and only if

$$(2.17) \qquad \mathrm{Re} \sum_1^n f_k \bar{a}_k < 0\,.$$

Proof. a) (2.17) is necessary. In fact, multiplying (2.16) by $\bar{a}_k$ and adding, we get by using the symmetry of α_{jk} if $a_j = b_j + i c_j$

$$(f, a) = \sum_1^n f_k \bar{a}_k = \sum_1^n \sum_1^n \alpha_{kj} a_j \bar{a}_k = \sum_1^n \sum_1^n \alpha_{kj} b_j b_k + \sum_1^n \sum_1^n \alpha_{kj} c_j c_k\,.$$

Since $\mathrm{Re}\,\alpha_{kj}$ is negative definite and the real vectors $(b_1, \ldots, b_n)$ and $(c_1, \ldots, c_n)$ are not both zero, we get (2.17).

b) (2.17) is a sufficient condition. We have to separate two cases.

b$_1$) a is proportional to a real vector. Multiplying a and f by the same complex number we may assume that a is real. Writing $\alpha = \beta + i\gamma$, $f = g + ih$ with real β, γ, g, h, (2.16) becomes in matrix notation

$$\beta a = g\,, \quad \gamma a = h\,.$$

It is obvious that there is a real symmetric matrix γ with $\gamma a = h$. Write $g = g_1 + a(g, a)/2(a, a)$. We then have $(g_1, a) = (g, a)/2 < 0$, hence the matrix β defined by

$$\beta x = \frac{(g, a)}{2(a, a)}\, x + \frac{(x, g_1)}{(a, g_1)}\, g_1$$

is immediately seen to be negative definite, and since it is obviously symmetric it has the required properties.

b$_2$) a is not proportional to a real vector. We shall prove that

$$\alpha = \frac{\mathrm{Re}(f, a)}{(a, a)}\, I + i\gamma$$

for some real γ has the required properties. Here I is the identity matrix. The condition on γ is

$$(2.18) \qquad i\gamma a = f_1$$

where

$$f_1 = f - \frac{\mathrm{Re}(f, a)}{(a, a)}\, a$$

has the property

$$\mathrm{Re}\,(f_1, a) = 0\,. \tag{2.19}$$

To prove that such a matrix γ exists we note that the set of vectors in C^n which can be written $i\gamma a$ with some real symmetric γ is a linear set (with respect to real scalars). The equation of a plane containing this set can be written

$$\mathrm{Re}\,(z, g) = 0$$

with some $g \in C^n$. For every $\xi \in R^n$ the matrix defined by $\gamma x = \xi(x, \xi)$ is real and symmetric, and $\gamma a = \xi(a, \xi)$. Hence we must have

$$\mathrm{Re}\, i(\xi, g)\,(a, \xi) = 0$$

so that $(\xi, g)\,(a, \xi)$ is always real. Since a is not proportional to any real vector it follows that g is a real multiple of a. Hence the equation $\mathrm{Re}\,(z,g)=0$ is a consequence of the equation $\mathrm{Re}\,(z, a) = 0$. In view of (2.19) we can thus find a real symmetric matrix γ so that (2.18) is valid. The proof is complete.

We shall now combine Lemmas 2.1, 2.2 and 2.3. Set

$$\begin{aligned} C_{2m-1}(x, \xi) &= 2\,\mathrm{Re}\left(\sum_1^n -i p_j(x, \xi)\, \overline{p}^{(j)}(x, \xi)\right) \\ &= \sum_1^n i\left(p^{(j)}(x, \xi)\, \overline{p}_j(x, \xi) - p_j(x, \xi)\, \overline{p}^{(j)}(x, \xi)\right), \quad \xi \in R^n\,. \end{aligned} \tag{2.20}$$

Theorem 2.1. Let the coefficients of $P(x, D)$ be continuous and those of p be in C^2. Suppose that

$$\|u\|_{m-1} \leqq C\,\|P(x, D)u\|\,, \quad u \in C_0^\infty(\Omega)\,. \tag{2.21}$$

Then we have

$$C_{2m-1}(x, \xi) = 0 \quad \text{if} \quad p(x, \xi) = 0,\ x \in \Omega,\ \xi \in R^n\,. \tag{2.22}$$

Proof. We may assume that $x = 0$. In proving (2.22) we may also assume that (2.3) holds, for (2.22) is trivially satisfied otherwise. Lemmas 2.1 and 2.2 then show that the equations (2.14) cannot be fulfilled by a symmetric matrix with negative definite real part, hence $C_{2m-1}(x, \xi) \geqq 0$ in view of Lemma 2.3. Replacing ξ by $-\xi$ we get $C_{2m-1}(x, -\xi) \geqq 0$ and since C_{2m-1} is an odd function of ξ it follows that $C_{2m-1}(x, \xi) = 0$. The proof is complete.

Corollary. Given a homogeneous differential operator $p(D)$ of order m with constant coefficients, the inequality (1.3) holds for all homogeneous operators $p(x, D)$ of order m with $p(x_0, D) = p(D)$ and coefficients in C^1, if and only if $p(D)$ is elliptic.

Proof. That ellipticity is sufficient follows for example from FRIEDRICHS [2] (or HÖRMANDER [4]). We can then even sum for $|\alpha| \leqq m$ in the left hand side of (1.3). To prove the necessity we note that (1.2) must be valid in view of Theorem 2.3 in HÖRMANDER [3] (see also Theorem 2.2 below). The condition in Theorem 2.1 can then only be void if $p(\xi)$ does not have any real zero.

We shall now show that C_{2m-1} is in fact the principal part of the commutator $\bar{p}p - p\bar{p}$. Thus we now assume that the coefficients of p are in C^m so that this commutator is defined. Write

$$p(x, D) = \sum_{|\alpha|=m} a_\alpha(x) D_\alpha , \quad \bar{p}(x, D) = \sum_{|\alpha|=m} \bar{a}_\alpha(x) D_\alpha .$$

Leibniz' formula gives

$$\bar{p}(x, D)\, p(x, D) = \sum_{\alpha,\beta} D_\beta a_\alpha(x)/|\beta|! \cdot \bar{p}^{(\beta)}(x, D) D_\alpha .$$

Terms of order $2m$ occur only when $|\beta| = 0$, their characteristic polynomial is $\bar{p}(x, \xi)\, p(x, \xi)$. Terms of order $2m-1$ occur when $|\beta| = 1$; their characteristic polynomial is thus

$$\sum_{|\alpha|=m} \sum_{j=1}^{n} (D_j a_\alpha(x))\, \bar{p}^{(j)}(x, \xi)\, \xi_\alpha = -i \sum_{1}^{n} p_j(x, \xi)\, \bar{p}^{(j)}(x, \xi) .$$

Repeating the argument with p and $\bar{p}$ interchanged and subtracting the results, we find that $\bar{p}(x, D)\, p(x, D) - p(x, D)\, \bar{p}(x, D)$ is of order $2m-1$ and that its principal part has the characteristic polynomial $C_{2m-1}(x, \xi)$.

Even if the coefficients of p are only continuously differentiable we can show that C_{2m-1} is the principal part of the commutator in a weak sense, namely

$$(p(x, D)u, p(x, D)v) - (\bar{p}(x, D)u, \bar{p}(x, D)v) = \sum_{\alpha} \sum_{\beta} (c_{\alpha\beta}(x) D_\alpha u, D_\beta v) , \tag{2.23}$$

when $u, v \in C_0^\infty(\Omega)$. The indices in the sum satisfy $|\alpha| + |\beta| = 2m - 1$, $|\alpha| \leq m$, $|\beta| \leq m$; $c_{\alpha\beta}$ are continuous and we have

$$C_{2m-1}(x, \xi) = \sum_{\alpha,\beta} c_{\alpha\beta}(x)\, \xi_\alpha \xi_\beta . \tag{2.24}$$

To prove this we start from the formula

$$(p(x, D)u, p(x, D)v) = \sum_{\alpha} \sum_{\beta} (a_\alpha D_\alpha u, a_\beta D_\beta v) .$$

We integrate by parts, first shifting one of the derivatives in D_β from v to u, then one of the derivatives in D_α from u to v and so on. In doing so, we will of course also differentiative a coefficient sometimes. As soon as we get a term where a coefficient is differentiated, however, we do not perform any more integrations by parts in that term. It is clear that this procedure will give

$$(p(x, D)u, p(x, D)v) = \sum_{\alpha,\beta} (a_\alpha D_\beta u, a_\beta D_\alpha v) + \sum (c_{\alpha\beta} D_\alpha u, D_\beta v) ,$$

where one of the multi-indices α and β in the last sum has length m and the other length $m-1$. The first term in the right hand side is obviously $(\bar{p}(x, D)u, \bar{p}(x, D)v)$. The coefficients $c_{\alpha\beta}$ are linear combinations of products of a coefficient a_γ or $\bar{a}_\gamma$ and a first derivative of another. To prove (2.24) we first notice that we have in fact already proved this formula when the coefficients a_γ are in C^∞. Since (2.24) is an identity involving the coefficients and their first derivatives it must thus be valid in general.

Remark. The coefficients $c_{\alpha\beta}$ are by no means intrinsically defined. The proof above may also lead to different values depending on the order in which the integrations by parts are performed.

We finally verify the necessity of condition A.

Theorem 2.2. Suppose that $P(x, D)$ has continuous coefficients and that (1.3) is valid. Then

$$\sum_{1}^{n} |\partial p(x_0, \xi)/\partial \xi_j|^2 \neq 0\,, \quad 0 \neq \xi \in R^n. \tag{2.25}$$

Proof. We may assume that $x_0 = 0$. Let $U \in C_0^\infty(\Omega_1)$ and put

$$u_\delta(x) = \delta^{m-1-n/2}\, U(x/\delta)$$

into (1.3). We have $u_\delta \in C_0^\infty(\Omega_\delta)$ and, if $|\alpha| = m - 1$,

$$D_\alpha u_\delta(x) = \delta^{-n/2}(D_\alpha U)\,(x/\delta)\,.$$

Hence a substitution of variables gives that $\|D_\alpha u_\delta\| \to \|D_\alpha U\|$ when $\delta \to 0$. Further we have

$$\delta P(x, D)\, u_\delta = \sum_\alpha a_\alpha(x)\, \delta^{m-|\alpha|}\, \delta^{-n/2}(D_\alpha U)\,(x/\delta)$$

and after a substitution of variables we obtain

$$\|\delta P(x, D)\, u_\delta\| \to \|p(0, D)\, U\|\,, \quad \delta \to 0\,.$$

Hence

$$\sum_{|\alpha| = m-1} \|D_\alpha U\|^2 \leqq C_0 \|p(0, D)\, U\|^2, \quad U \in C_0^\infty(\Omega_1)\,,$$

and the proof of Theorem 2.3 in HÖRMANDER [3] thus shows that (2.25) holds.

3. The first order case

When $m = 1$ we shall now establish an improved version of Theorem 2.1. The nature of the improvement is that we disprove a property similar to (1.3) but with a weaker norm in the left hand side and a stronger in the right hand side. This will give us a generalization of the result of LEWY [5]. In analogy with Lemma 2.1 we first prove

Lemma 3.1. Suppose that the coefficients of the first order operator $P(x, D)$ are analytic and that there is a solution u of the equation

$$p(x, D)\, u = 0\,, \quad x \in \Omega\,, \tag{3.1}$$

which satisfies at 0 the same assumptions as those made in Lemma 2.1. Then we can find functions $v_t \in \mathscr{D}(\Omega)$, depending on a real parameter $t > 0$ so that

$$P(x, D)\, v_t \to 0 \quad \text{in} \quad \mathscr{D}(\Omega) \quad \text{as} \quad t \to \infty \tag{3.2}$$

but for some $f \in \mathscr{D}(\Omega)$

$$\overline{\lim_{t \to \infty}}\; \left|\int f v_t\, dx\right| = \infty\,. \tag{3.3}$$

Hence v_t does not converge to 0 in $\mathscr{D}'(\Omega)$ as $t \to \infty$ [2]).

[2]) In this section we use the notation $\mathscr{D}(\Omega)$ for the space $C_0^\infty(\Omega)$ with the pseudotopology of SCHWARTZ and $\mathscr{D}'(\Omega)$ for the space of distributions in Ω, that is, the linear forms on $\mathscr{D}(\Omega)$ which are continuous for the pseudotopology. We refer to SCHWARTZ [6] for the basic definitions and results.

Proof. Writing

$$P(x, D) = \sum_1^n a^j D_j + q$$

we choose a function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi(0) = 1$ and

$$P(x, D) = \sum_1^n a^j D_j \varphi + \varphi q = 0 \tag{3.4}$$

in a neighbourhood U of 0. (In the proof of Lemma 2.1 we only needed (3.4) when $x = 0$.) This is possible since the equation (3.4) can be solved in a neighbourhood of 0 in virtue of Cauchy-Kovalevsky's theorem and (2.3); multiplication by a function which is in C_0^∞ in that neighbourhood and equals 1 in another neighbourhood of 0 gives a function with the desired properties. (The assumption that the coefficients are analytic and not only infinitely differentiable is only used here.) (2.5) gives with a constant $c > 0$

$$\operatorname{Re} u(x) < -2c\,, \quad x \in \complement U\,. \tag{3.5}$$

Now set

$$v_t = \varphi e^{t(u+c)}. \tag{3.6}$$

Using (3.1) and Leibniz' formula we obtain

$$P(x, D) v_t = (P(x, D)\varphi) e^{t(u+c)}.$$

This function vanishes in U and its support is always contained in that of φ. Since $u + c \leqq -c$ in $\complement U$ it follows immediately that $P(x, D) v_t$ and all its derivatives tend to 0 uniformly when $t \to \infty$. Hence (3.2) is valid.

It is sufficient to prove that there exists a function $f \in C^\infty(\Omega)$ such that (3.3) is valid. In fact, if $\chi \in C_0^\infty(\Omega)$ equals 1 in the support of φ we then only have to note that $\chi f \in C_0^\infty(\Omega)$ and that

$$\int f v_t\, dx = \int (\chi f) v_t\, dx$$

so that (3.3) holds with f replaced by χf. We shall construct a function for which (3.3) holds and which is contained in the space F of all $f \in C^\infty(\Omega)$ such that

$$\sup_\Omega |D_\alpha f| < \infty \tag{3.7}$$

for every α. The topology in F is given by the semi-norms on the left hand side in (3.7). It is obvious that F is a complete metrizable space, hence an «espace tonnelé» in the terminology of Bourbaki [1], pp. 1—2.

Now suppose that the assertion of the theorem were false so that the integrals

$$\int f \varphi e^{t(u+c)}\, dx \tag{3.8}$$

are bounded as $t \to \infty$ for all $f \in F$. According to Banach-Steinhaus' theorem (see Bourbaki [1], pp. 64—65, Proposition 1 and Théorème 1) this implies that the functionals (3.8) on F form an equicontinuous set in F', hence are all $\leqq 1$ for every f in a neighbourhood V of 0 in F. In other words, there exists an integer N and an $\varepsilon > 0$ such that

$$|\int f \varphi e^{t(u+c)} dx| \leqq 1 \quad \text{if} \quad \sup_{\Omega} |D_\alpha f| \leqq \varepsilon , \quad |\alpha| \leqq N . \tag{3.9}$$

Hence

$$|\int f \varphi e^{t(u+c)} dx| \leqq \varepsilon^{-1} \sup_{x \in \Omega, |\alpha| \leqq N} |D_\alpha f(x)| , \quad f \in F , \tag{3.10}$$

in view of the homogenity of the inequality.

Now take a function $F \in C_0^\infty(R^n)$ and put

$$f_t(x) = t^n F(tx) .$$

It is obvious that

$$\varepsilon^{-1} \sup_{\Omega} |D_\alpha f_t| \leqq C t^{n+N}, \ |\alpha| \leqq N, t \geqq 1 . \tag{3.11}$$

Further a substitution gives for large t

$$\int f_t \varphi e^{t(u+c)} dx = \int F(x) \ \varphi(x/t) e^{t(u(x/t)+c)} dx .$$

When $t \to \infty$ we have $\varphi(x/t) \to 1$ and $tu(x/t) \to i\langle x, \xi \rangle$ uniformly in the support of F. Hence

$$\int f_t \varphi e^{t(u+c)} dx = e^{tc}(\int F(x) e^{i\langle x, \xi\rangle} dx + o(1)) . \tag{3.12}$$

If we choose F so that $\int F(x) e^{i\langle x, \xi \rangle} dx \neq 0$, (3.12) and (3.11) give a contradiction when combined with (3.10). Hence (3.3) is valid for some $f \in F$ and thus for some $f \in \mathscr{D}(\Omega)$. The proof is complete.

As in section 2 we denote the first order part of the commutator of $\bar{p}$ and p by C_1.

Theorem 3.1. Suppose that the coefficients of P are analytic and that the equation

$$P(x, D) u = f \tag{3.13}$$

has a solution $u \in \mathscr{D}'(\Omega)$ for every $f \in \mathscr{D}(\Omega)$. Then

$$C_1(x, \xi) = 0 \quad \text{if} \quad p(x, \xi) = 0 , \quad x \in \Omega, \xi \in R^n . \tag{3.14}$$

Proof. Assuming that (3.14) is not valid we shall find a function $f \in \mathscr{D}(\Omega)$ such that (3.13) has no solution in $\mathscr{D}'(\Omega)$. If (3.14) is not valid we may assume that it fails to hold when $x = 0 \in \Omega$, and since C_1 is an odd function of ξ we can then find $\xi \in R^n$ such that

$$p(0, \xi) = 0 , \quad C_1(0, \xi) < 0 .$$

In virtue of Lemma 2.3 this shows that the equations (2.13) and (2.14) can be satisfied with a symmetric matrix α_{jk} with negative definite real part. Since $C_1(0, \xi) < 0$ the coefficients a^j do not all vanish at 0. Hence the Cauchy-Kovalevsky theorem proves the existence of a solution of (3.1) satisfying the assumptions of Lemma 3.1 at least in a neighbourhood $\Omega_1 \subset \Omega$ of 0. (See the proof of Lemma 2.2.)

Let P^t be the formal adjoint of P defined by

$$\int (Pu)v\,dx = \int u(P^t v)\,dx,\ u, v \in \mathcal{D}(\Omega)\,.$$

We have

$$P^t = -\sum_1^n D_j a^j + q$$

so that $-P^t$ has the principal part p. Hence Lemma 3.1 shows that for $\tau > 0$ there are functions v_τ in $\mathcal{D}(\Omega_1)$, such that

$$P^t(x, D)v_\tau \to 0 \quad \text{in} \quad \mathcal{D}(\Omega_1), \tau \to \infty\,,$$

but

$$\overline{\lim_{\tau\to\infty}} \left|\int f v_\tau\,dx\right| = \infty$$

for some $f \in \mathcal{D}(\Omega_1)$. We claim that the equation $Pu = f$ has no solution $u \in \mathcal{D}'(\Omega)$. For suppose it had. Then

$$\int f v_\tau\,dx = (Pu)\,(v_\tau) = u(P^t v_\tau) \to 0\,, \quad \tau \to \infty\,,$$

which is a contradiction. The proof is complete.

Our next aim is to show as Lewy did in his example that the equation (3.13) may fail to have a solution anywhere.

Theorem 3.2. Suppose that the coefficients of P are analytic and that (3.14) is not valid for any open non void set $\omega \subset \Omega$ when x is restricted to ω. Then there exist functions $f \in \dot{B}(\Omega)$ such that the equation (3.13) does not have a solution $u \in \mathcal{D}'(\omega)$ for any open non void set $\omega \subset \Omega$. The set of such f is of the second category[3]).

Proof. a) If ω is a fixed open non void subset of Ω we shall first prove that the set M of functions $f \in \dot{B}(\Omega)$ such that (3.13) has a solution in $\mathcal{D}'(\omega)$ is of the first category. Let $\omega_1 \subset \omega$ be open, non void and have compact closure contained in ω. Every distribution u in ω then satisfies for some integer N the estimate (Schwartz [6], Chap. III, p. 83)

$$|u(\varphi)| \leqq N \sum_{|\alpha| \leqq N} \sup|D_\alpha \varphi|,\ \varphi \in \mathcal{D}(\omega_1)\,. \tag{3.15}$$

The set M_N of functions f in $\dot{B}(\Omega)$ such that $P(x, D)u = f$ in ω_1 for some $u \in \mathcal{D}'(\omega_1)$ satisfying (3.15) is closed, convex and symmetric. That M_N is convex and symmetric is obvious. To see that M_N is closed we only have to note that the set of distributions satisfying (3.15) is compact for the weak topology in $\mathcal{D}'(\omega_1)$. In fact, let $f_j \in M_N$, that is, $f_j = Pu_j$ in ω_1 for some $u_j \in \mathcal{D}'(\omega_1)$ satisfying (3.15). We can find a weak limit u of u_j and u also satisfies (3.15). If $f_j \to f$ in $\dot{B}(\Omega)$ we get $f = Pu$ in ω_1 so that $f \in M_N$.

M_N cannot have any interior point. For Theorem 3.1 with Ω replaced by ω_1 shows that there is a function $g \in \mathcal{D}(\omega_1)$ such that $tg \notin M_N$ when $t \neq 0$. If f were an interior point of M_N we would have $f + tg \in M_N$ for small t. Since

[3]) $\dot{B}(\Omega)$ denotes the set of all infinitely differentiable f in Ω such that to every α and ε there is a compact set $K \subset \Omega$ so that $|D_\alpha f| < \varepsilon$ in $\complement K$. This is a complete metrizable space with the topology defined by the semi-norms $\sup|D_\alpha f|$.

$-f \in M_N$ and M_N is convex this implies that $(f + tg - f)/2 = tg/2 \in M_N$, which is a contradiction. Since M_N is closed and has no interior point, the set $\bigcup M_N$ is by definition of the first category and since $M \subset \bigcup M_N$ it follows that M is also of the first category.

b) Let ω_j be a countable basis for open subsets of Ω, none of them void. For example we may take all open spheres with rational radius and centre contained in Ω. Denote by $M^{(j)}$ the set of functions $f \in \dot{B}(\Omega)$ such that (3.13) has a solution in $\mathscr{D}'(\omega_j)$. From a) it follows that $M^{(j)}$ is of the first category. Hence $\bigcup M^{(j)}$ is also of the first category. Take $f \notin \bigcup M^{(j)}$. Then the equations (3.13) cannot be solved in any ω_j. If ω is an arbitrary open non void subset of ω, we have $\omega_j \subset \omega$ for some j. Thus the equation (3.13) cannot be solved in ω. This completes the proof.

Lewy's example corresponds to $(n = 3)$

$$p(x, \xi) = -i\xi_1 + \xi_2 - 2(x^1 + i x^2)\xi_3 .$$

We get $C_1(x, \xi) = -8\xi_3$. Since $p(x, \xi) = 0$ if $\xi_3 = 1$, $\xi_1 = -2x^2$, $\xi_2 = 2x^1$, the condition (3.14) is not valid for any $x \in R^n$. Hence the hypotheses of Theorem 3.2 are fulfilled with $\Omega = R^n$. We may notice that Theorem 3.2 gives a stronger result than LEWY [5] who only gave a function $f \in C^\infty(R^n)$ for which (3.14) does not have any solution with Hölder continuous first derivatives in any open set.

4. Sufficient conditions for an operator to be of principal type

We first prove an estimate which follows from the methods used in Chapter IV in [3]. However, in the proof we shall use a simplified form of the arguments of [3] which has been given by TRÈVES [7]. The simplification consists in a direct proof of the inequalities found in [3] by systematic use of the energy integral method.

Theorem 4.1. Let the coefficients of $p(x, D)$ be in C^1 and assume that (1.2) is fulfilled. Let Ω_δ be the sphere $\{x; |x - x_0| < \delta\}$. Then there exist constants C_0 and $\delta_0 > 0$ such that if $\delta < \delta_0$

$$\sum_{|\alpha| < m} \delta^{2(|\alpha| - m)} \|D_\alpha u\|^2 \leqq C_0(\|p(x, D)u\|^2 + \|\bar{p}(x, D)u\|^2),\ u \in C_0^\infty(\Omega_\delta) . \tag{4.1}$$

Proof. We start by noting that Leibniz' formula gives

$$p(x, D)(i x^k u) = i x^k p(x, D)u + p^{(k)}(x, D)u .$$

Hence, writing $(f, g) = \int f\bar{g}\,dx$,

$$(p^{(k)}(x, D)u, p^{(k)}(x, D)u) = (p(x, D)(i x^k u), p^{(k)}(x, D)u) - \\ - (i x^k p(x, D)u, p^{(k)}(x, D)u),\ u \in C_0^\infty(\Omega_\delta) .$$

Let $x_0 = 0$. Using Cauchy-Schwarz' inequality and the inequality $|x^k| < \delta$ in Ω_δ, we obtain for $u \in C_0^\infty(\Omega_\delta)$

$$\|p^{(k)}(x,D)u\|^2 \leqq \mathrm{Re}(p(x,D)(i x^k u), p^{(k)}(x,D)u) + \delta\|p(x,D)u\|\|p^{(k)}(x,D)u\|. \tag{4.2}$$

In order to study the first term on the right we have to shift the operators p and $p^{(k)}$. To do so we need the following lemma.

Lemma 4.1. Let p and q be homogeneous differential operators of order m and $m-1$ respectively, both with coefficients in $C^1(\Omega)$. Then we have when $u, v \in C_0^\infty(\Omega)$

$$(4.3)\quad (p(x, D)v, q(x, D)u) = (\bar{q}(x, D)v, \bar{p}(x, D)u) + \sum_{|\alpha| = |\beta| = m-1} (c_{\alpha\beta} D_\alpha v, D_\beta u),$$

where $c_{\alpha\beta}$ are continuous functions.

Proof. Writing $p(x, D) = \sum_{|\alpha| = m} a_\alpha(x) D_\alpha$ and $q(x, D) = \sum_{|\beta| = m-1} b_\beta(x) D_\beta$ we get

$$(p(x, D)v, q(x, D)u) = \sum (a_\alpha D_\alpha v, b_\beta D_\beta u).$$

We integrate by parts here, first shifting one of the derivatives in D_α from left to right, then one of the derivatives in D_β from right to left and so on. In doing so we will of course also differentiate a coefficient sometimes. However, the term which then appears contains derivatives of u and of v of order $m-1$ and we do not operate again on such a term. It is clear that this procedure will give

$$(p(x, D)v, q(x, D)u) = \sum (a_\alpha D_\beta v, b_\beta D_\alpha u) + \sum_{|\alpha| = |\beta| = m-1} (c_{\alpha\beta} D_\alpha v, D_\beta u)$$

The functions $c_{\alpha\beta}$ are linear combinations of products of coefficients in p and in $\bar{q}$ and their derivatives, hence continuous. Since the first sum on the right is obviously equal to $(\bar{q}(x, D)v, \bar{p}(x, D)u)$, the lemma is proved.

It should be noticed that the proof is only a less precise form of the arguments concerning C_{2m-1} given after the corollary to Theorem 2.1.

Completion of the proof of Theorem 4.1. We can now study the first term in the right hand side of (4.2) using Lemma 4.1 with $q = p^{(k)}$. With the notation

$$|u|_k^2 = \sum_{|\alpha| = k} \|D_\alpha u\|^2$$

this gives with constants C_1 and C_2

$$(4.4)\quad \operatorname{Re}(p(x, D)(i x^k u), p^{(k)}(x, D)u) \leqq \operatorname{Re}(\bar{p}^{(k)}(x, D)(i x^k u), \bar{p}(x, D)u) +$$
$$+ C_1 |u|_{m-1} |x^k u|_{m-1} \leqq C_2(\|\bar{p}(x, D)u\| + |u|_{m-1})(\delta |u|_{m-1} + |u|_{m-2}),\ u \in C_0^\infty(\Omega_\delta)$$

since $\bar{p}^{(k)}(x, D)(i x^k u) = i x^k \bar{p}^{(k)}(x, D)u + \bar{p}^{(kk)}(x, D)u$. (We suppose in (4.4) that δ is bounded from above, for example $\delta < 1$, in order to possess a bound for the coefficients of p and of the coefficients $c_{\alpha\beta}$ in Lemma 4.1.) Put (4.4) into (4.2) and add for all k. This gives with a constant C (from now on C will denote different constants at different occasions)

$$(4.5)\quad \sum_1^n \|p^{(k)}(x, D)u\|^2 \leqq$$
$$\leqq C(\|p(x, D)u\| + \|\bar{p}(x, D)u\| + |u|_{m-1})(\delta |u|_{m-1} + |u|_{m-2}),\ u \in C_0^\infty(\Omega_\delta).$$

We now use the classical inequality (see [3], p. 246)

$$(4.6)\quad |u|_k \leqq C\delta |u|_{k+1}.$$

This gives when combined with (4.5)

$$\text{(4.7)} \qquad \sum_1^n \|p^{(k)}(x, D)u\|^2 \leqq$$
$$\leqq C\delta(\|p(x, D)u\| + \|\bar{p}(x, D)u\| + |u|_{m-1})\, |u|_{m-1},\ u \in C_0^\infty(\Omega_\delta)\,.$$

To prove (4.1) it only remains to show that

$$\text{(4.8)} \qquad |u|_{m-1}^2 \leqq C \sum_1^n \|p^{(k)}(x, D)u\|^2, \quad u \in \mathscr{D}(\Omega_\delta)$$

if $\delta \leqq \delta_0'$. For combined with (4.7) this gives

$$|u|_{m-1} \leqq C\delta(\|p(x, D)u\| + \|\bar{p}(x, D)u\| + |u|_{m-1})$$

and together with (4.6) this proves (4.1) when $\delta \leqq \min(\delta_0', 1/2C)$.

The inequality (4.8) follows from the fact that (1.2) holds when $x_0 = 0$. In fact, for homogeneity reasons (1.2) implies that

$$|\xi|^{2(m-1)} \leqq C \sum_1^n |p^{(k)}(0, \xi)|^2, \quad \xi \in R^n.$$

Multiplying by $|\hat{u}(\xi)|^2$ and integrating we get from Parseval's formula

$$|u|_{m-1}^2 \leqq C \sum_1^n \|p^{(k)}(0, D)u\|^2, \quad u \in C_0^\infty(R^n)\,.$$

Hence

$$|u|_{m-1}^2 \leqq C \sum_1^n \|p^{(k)}(x, D)u + (p^{(k)}(0, D) - p^{(k)}(x, D))u\|^2 \leqq$$
$$\leqq 2C\left(\sum_1^n \|p^{(k)}(x, D)u\|^2 + \sum_1^n \|(p^{(k)}(0, D) - p^{(k)}(x, D))u\|^2\right).$$

Since the coefficients of $p^{(k)}(0, D) - p^{(k)}(x, D)$ are $O(|x|)$, this gives with another C

$$|u|_{m-1}^2 \leqq C\left(\sum_1^n \|p^{(k)}(x, D)u\|^2 + \delta^2 |u|_{m-1}^2\right), \quad u \in C_0^\infty(\Omega_\delta)\,.$$

When $C\delta^2 < 1/2$ we get with the same C

$$|u|_{m-1}^2 \leqq 2C \sum_1^n \|p^{(k)}(x, D)u\|^2, \quad u \in C_0^\infty(\Omega_\delta)\,,$$

and (4.8) is proved. This also completes the proof of Theorem 4.1.

Theorem 4.1 shows that p will be of principal type as soon as it is possible to estimate $\bar{p}u$ in terms of pu. Before proving such results we give a simple auxiliary theorem.

Theorem 4.2. Let $P(x, D)$ be of principal type and have continuous coefficients. Then any other operator with the same principal part and continuous coefficients is also of principal type.

Proof. If (1.3) is valid and r is of order $< m$ and has continuous coefficients, we get for $\delta < \delta_0$

$$\sum_{|\alpha| < m} \delta^{2(|\alpha| - m)} \|D_\alpha u\|^2 \leqq C_0 \|P(x, D)u\|^2 \leqq 2C_0 \big(\|(P(x, D) + r(x, D))u\|^2 + \|r(x, D)u\|^2\big) \leqq 2C_0 \|(P(x, D) + r(x, D))u\|^2 + C \sum_{|\alpha| < m} \|D_\alpha u\|^2, \quad u \in C_0^\infty(\Omega_\delta).$$

When δ is so small that $\delta^2 C < 1/2$ and $\delta < \min(1, \delta_0)$, we get

$$\sum_{|\alpha| < m} \delta^{2(|\alpha| - m)} \|D_\alpha u\|^2 \leqq 4C_0 \|(P(x, D) + r(x, D))u\|^2, \quad u \in C_0^\infty(\Omega_\delta).$$

Hence $P + r$ is also of principal type.

Theorem 4.3. Let $P(x, D)$ be an operator with continuous coefficients and the coefficients of the principal part $p(x, D)$ in C^2. Assume that (1.2) is valid and that

(4.9) $$C_{2m-1}(x, \xi) = \bar{q}(x, \xi)\, p(x, \xi) + \bar{q}(x, \xi)\, p(x, \xi)$$

where $q(x, \xi)$ is a polynomial in ξ of order $m - 1$ with coefficients in C^1. Then P is of principal type[4]). We also have with a constant C when $\delta < \delta_0$

(4.10) $$C^{-1}\|p(x, D)u\| \leqq \|\bar{p}(x, D)u\| \leqq C\|p(x, D)u\|, \quad u \in C_0^\infty(\Omega_\delta).$$

Proof. Theorem 4.2 shows that it is sufficient to prove the theorem when $P = p$. Also note that the inequality (4.10) implies (1.3) in view of Theorem 4.1.

To prove (4.10) we use the identity

(2.23) $$\|p(x, D)u\|^2 - \|\bar{p}(x, D)u\|^2 = \sum_\alpha \sum_\beta (c_{\alpha\beta} D_\alpha u, D_\beta u), \quad u \in C_0^\infty(\Omega_\delta)$$

where $c_{\alpha\beta}$ are continuously differentiable and

(2.24) $$\sum_\alpha \sum_\beta c_{\alpha\beta}(x)\, \xi_\alpha \xi_\beta = C_{2m-1}(x, \xi).$$

Using this identity and (4.9) one can prove that

(4.11) $$\left|\sum (c_{\alpha\beta} D_\alpha u, D_\beta u) - (p(x, D)u, q(x, D)u) - (\bar{q}(x, D)u, \bar{p}(x, D)u)\right| \leqq \leqq C|u|_{m-1}^2, \quad u \in C_0^\infty(\Omega_\delta),$$

where C is a constant. We postpone the proof for a moment in order to prove first that (4.10) follows from (4.11). Using (4.11), (2.23) and the inequality between geometric and arithmetic means, we obtain

$$\big|\|p(x, D)u\|^2 - \|\bar{p}(x, D)u\|^2\big| \leqq C|u|_{m-1}^2 + \tfrac{1}{2}\big(\|p(x, D)u\|^2 + \|\bar{p}(x, D)u\|^2 + \|q(x, D)u\|^2 + \|\bar{q}(x, D)u\|^2\big), \quad u \in C_0^\infty.$$

Since q is of order $m - 1$ this gives with another constant C for $u \in C_0^\infty$

(4.12) $$\|p(x, D)u\|^2 \leqq C|u|_{m-1}^2 + 3\|\bar{p}(x, D)u\|^2,$$

(4.13) $$\|\bar{p}(x, D)u\|^2 \leqq C|u|_{m-1}^2 + 3\|p(x, D)u\|^2.$$

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[4]) Since (1.3) is valid for elliptic operators, with summation over all α with $|\alpha| \leqq m$, it is easy to show that P is also of principal type if p is the product of an elliptic polynomial with continuous coefficients and a polynomial with "sufficiently smooth" coefficients satisfying the hypotheses of Theorem 4.3. The proof may be left to the reader.

Now (4.1) gives that

$$|u|_{m-1}^2 \leqq C_0 \delta^2 (\|p(x, D)u\|^2 + \|\bar{p}(x, D)u\|^2), \quad u \in C_0^\infty(\Omega_\delta).$$

Combining this inequality with the inequalities (4.12) and (4.13) and taking δ so small that $CC_0\delta^2 < 1/2$, we obtain (4.10) with $C = 7$.

It remains to prove the estimate (4.11). The proof is somewhat complicated by the fact that we have only assumed that the coefficients of q as well as $c_{\alpha\beta}$ are in C^1. If they had been in C^∞ we could have integrated by parts in (4.11), shifting all derivatives to the left hand side. The fact that

$$(4.14) \qquad \sum_\alpha \sum_\beta c_{\alpha\beta} \xi_\alpha \xi_\beta = \bar{q}(x, \xi)\, p(x, \xi) + q(x, \xi)\, \bar{p}(x, \xi) = C_{2m-1}(x, \xi)$$

would then show that the terms of order $2m - 1$ cancel each other. Shifting $m - 1$ derivatives back again we would get an expression for the quantity to estimate which involves only the derivatives of u of order $< m$.

Our weaker smoothness assumptions make a slight modification necessary. By successive integrations by parts we shall prove that

$$(4.15) \sum \sum (c_{\alpha\beta} D_\alpha u, D_\beta u) = \sum_{|\alpha| = m, |\beta| = m-1} \sum (c^*_{\alpha\beta} D_\alpha u, D_\beta u) + \sum_{|\alpha| = |\beta| = m-1} \sum (d_{\alpha\beta} D_\alpha u, D_\beta u).$$

Here $c^*_{\alpha\beta} = 0$ unless $\alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_m \leqq \beta_1 \leqq \cdots \leqq \beta_{m-1}$, and the coefficients $c^*_{\alpha\beta}$, $d_{\alpha\beta}$ are all continuous. Further, we have

$$(4.16) \qquad \sum \sum c^*_{\alpha\beta}(x) \xi_\alpha \xi_\beta = C_{2m-1}(x, \xi).$$

To prove this we consider one of the terms $(c_{\alpha\beta} D_\alpha u, D_\beta u)$. One of the multi-indices α and β is of length m and the other of length $m - 1$. We shift one differentiation from the side involving a differentiation of order m to the other side. The term which then appears when a coefficient is differentiated is immediately included in the last sum in (4.15). With the other term we repeat the same procedure. After a finite number of steps we can of course arrive at a sum of the same form as the last sum in (4.15) added to $(c_{\alpha\beta} D_{\alpha^*} u, D_{\beta^*} u)$, where $|\alpha^*| = m$, $|\beta^*| = m - 1$, $\alpha_1^* \leqq \alpha_2^* \leqq \cdots \leqq \alpha_m^* \leqq \beta_1^* \leqq \cdots \leqq \beta_{m-1}^*$ and (α^*, β^*) is a rearrangement of (α, β). This gives (4.15) and (4.16), by using (2.24). In view of the normalization of the coefficients $c^*_{\alpha\beta}$ it is obvious that *they are uniquely determined by* $C_{2m-1}(x, \xi)$. From this fact and (4.14) it follows that the same procedure must give

$$(4.17) \qquad (p(x, D)u, q(x, D)u) + (\bar{q}(x, D)u, \bar{p}(x, D)u) = \sum (c^*_{\alpha\beta} D_\alpha u, D_\beta u) + \\ + \sum_{|\alpha| = |\beta| = m-1} \sum (e_{\alpha\beta} D_\alpha u, D_\beta u)$$

with the same coefficients $c^*_{\alpha\beta}$ as in (4.15) and continuous $e_{\alpha\beta}$. Subtracting (4.17) from (4.15) we obtain (4.11). The proof is complete.

Remark. The estimates given in this chapter are only local. In fact, an example showing that global estimates are not possible has been given by TRÈVES [7], p. 8.

5. Some spaces of distributions

By $\mathscr{H}^s$, $-\infty < s < \infty$, we shall denote the space of temperate distributions u such that $\hat{u}$ is a function satisfying

$$\|u\|_s^2 = \int |\hat{u}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi < \infty \, . \tag{5.1}$$

Clearly $\mathscr{H}^s$ and $\mathscr{H}^{-s}$ are dual spaces with respect to a bilinear form extending the form $\int uv\,dx$. If s is a positive integer, $\mathscr{H}^s$ consists of all $u \in L^2$ with $D_\alpha u \in L^2$, $|\alpha| \leqq s$, and the notation $\|u\|_s$ coincides with that used in the preceding sections. In particular, $\mathscr{H}^0 = L^2$. As before we shall write $\|u\|$ instead of $\|u\|_0$ for the L^2 norm. Below we shall chiefly use another norm, equivalent to $\|u\|_s$, namely

$$\|u\|_{s,\varepsilon}^2 = \int |\hat{u}(\xi)|^2 (|\xi|^2 + \varepsilon^{-2})^s \, d\xi \, . \tag{5.1)'$$

We have

$$\sum_1^n \|D_j u\|_{s,\varepsilon}^2 + \varepsilon^{-2} \|u\|_{s,\varepsilon}^2 = \|u\|_{s+1,\varepsilon}^2 \, . \tag{5.2}$$

This follows immediately by computation.

Next we shall prove two lemmas concerning the regularization of elements in $\mathscr{H}^{-s}$, $s > 0$. Both are partly contained in [4]. (Note that $\mathscr{H}^{-s}$ was denoted by $\mathscr{D}'_s$ in [4].) Let $\varphi \in C_0^\infty$ satisfy the condition

$$\int \varphi \, dx \neq 0 \, . \tag{5.3}$$

φ will be held fixed in the argument that follows. Set

$$\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(x/\varepsilon) \, .$$

Lemma 5.1. For every $s > 0$ there are positive constants C_1 and C_2 such that

$$C_1 \|u\|_{-s,\varepsilon_0}^2 \leqq s \int_0^{\varepsilon_0} \|u * \varphi_\varepsilon\|^2 \varepsilon^{2s-1} \, d\varepsilon \leqq C_2 \|u\|_{-s,\varepsilon_0}^2 \, , \quad u \in \mathscr{H}^{-s}, \ \varepsilon_0 > 0 \, . \tag{5.4}$$

$1/C_1$ and C_2 are bounded when s is bounded.

Proof. Parseval's formula gives

$$s \int_0^{\varepsilon_0} \|u * \varphi_\varepsilon\|^2 \varepsilon^{2s-1} d\varepsilon = \int |\hat{u}(\xi)|^2 d\xi \; s \int_0^{\varepsilon_0} |\hat{\varphi}(\varepsilon\xi)|^2 \varepsilon^{2s-1} d\varepsilon \, .$$

Therefore (5.4) is equivalent to the inequality

$$C_1 \leqq (|\xi|^2 + \varepsilon_0^{-2})^s \, s \int_0^{\varepsilon_0} |\hat{\varphi}(\varepsilon\xi)|^2 \varepsilon^{2s-1} \, d\varepsilon \leqq C_2 \, . \tag{5.5}$$

To prove this inequality we first note that if M is the maximum of $|\hat{\varphi}|$ we have

$$\varepsilon_0^{-2s} \, s \int_0^{\varepsilon_0} |\hat{\varphi}(\varepsilon\xi)|^2 \varepsilon^{2s-1} d\varepsilon \leqq M^2/2 \, .$$

Further, a substitution in the integral gives

$$|\xi|^{2s} s\int_0^{\varepsilon_0} |\hat{\varphi}(\varepsilon\xi)|^2\varepsilon^{2s-1}d\varepsilon \leqq |\xi|^{2s} s\int_0^{\infty} |\hat{\varphi}(\varepsilon\xi)|^2\varepsilon^{2s-1}d\varepsilon$$

$$= s\int_0^{\infty} |\hat{\varphi}(\varepsilon\xi')|^2\varepsilon^{2s-1}d\varepsilon\,,$$

where $\xi' = \xi/|\xi|$ is a unit vector. The integral when ε goes from 0 to 1 is $\leqq M^2/2$ and the integral from 1 to ∞ is also bounded because $\hat{\varphi}(\varepsilon\xi')$ tends rapidly to 0 when $|\varepsilon\xi'| = \varepsilon\to\infty$. This proves the last inequality in (5.5).

To prove the first inequality in (5.5) we have to use the assumption (5.3), which means that $\hat{\varphi}(0) \neq 0$. This implies that there are positive constants a and c such that $|\hat{\varphi}(\xi)| > c$ when $|\xi| < a$. Writing $\delta = \min(\varepsilon_0, a/|\xi|)$ we get

$$s\int_0^{\varepsilon_0} |\hat{\varphi}(\varepsilon\xi)|^2\varepsilon^{2s-1}d\varepsilon \geqq s\,c^2\int_0^{\delta} \varepsilon^{2s-1}d\varepsilon = c^2\delta^{2s}/2\,.$$

The inequality (5.5) now follows with $C_1 = c^2\min(1, a^{2s})/2$.

This lemma is essentially contained in Theorem 7.1 and Lemma 7.1 in [4]. We next pass to an improvement of Theorem 7.2 there, which itself is an extension of Friedrichs' lemma [2]. It may be remarked that this improvement can be used to simplify the proofs in [4].

Lemma 5.2. Let $a\in C_0^\infty$, $s > 0$. Then there exists a constant C_3 such that

$$(5.6)\quad \int_0^{\varepsilon_0} \|a(u*\varphi_\varepsilon) - (au)*\varphi_\varepsilon\|^2\varepsilon^{2s-1}d\varepsilon \leqq C_3\|u\|^2_{-s-1,\varepsilon_0}\,,\quad u\in\mathscr{H}^{-s-1},\ \varepsilon_0 < 1\,.$$

When s is bounded, C_3 is also bounded.

Proof. The Fourier transform of $\varepsilon^s(a(u*\varphi_\varepsilon) - (au)*\varphi_\varepsilon)$ is the function

$$F_\varepsilon(\xi) = \int \varepsilon^s\hat{a}(\xi-\eta)\,(\hat{\varphi}(\varepsilon\eta) - \hat{\varphi}(\varepsilon\xi))\,\hat{u}(\eta)\,d\eta\,.$$

Write

$$(5.7)\qquad K(\varepsilon,\xi,\eta) = \varepsilon^s|\hat{a}(\xi-\eta)|\,|\hat{\varphi}(\varepsilon\eta) - \hat{\varphi}(\varepsilon\xi)|\,(|\eta|^{s+1} + \varepsilon_0^{-s-1})\,.$$

Using Cauchy-Schwarz' inequality we get

$$|F_\varepsilon(\xi)|^2 \leqq \int K(\varepsilon,\xi,\eta)\,d\eta\int K(\varepsilon,\xi,\eta)\,|\hat{u}(\eta)|^2/(|\eta|^{s+1} + \varepsilon_0^{-s-1})^2 d\eta\,.$$

We shall prove that

$$(5.8)\qquad \int K(\varepsilon,\xi,\eta)\,d\eta \leqq C_4,\ \varepsilon < \varepsilon_0\,,$$

and that

$$(5.9)\qquad \int\int_0^{\varepsilon_0} K(\varepsilon,\xi,\eta)\,d\xi\,d\varepsilon/\varepsilon \leqq C_5\,.$$

Noting that $(|\eta|^{s+1} + \varepsilon_0^{-s-1})^2 \geqq |\eta|^{2(s+1)} + \varepsilon_0^{-2(s+1)} \geqq 2^{-s}(|\eta|^2 + \varepsilon_0^{-2})^{s+1}$ and using (5.8) we get

$$|F_\varepsilon(\xi)|^2 \leqq C_4 2^s\int K(\varepsilon,\xi,\eta)\,|\hat{u}(\eta)|^2(|\eta|^2 + \varepsilon_0^{-2})^{-s-1}d\eta\,.$$

Integrating with respect to $d\xi\,d\varepsilon/\varepsilon$ and using (5.9) we now obtain

$$\int\int_0^{\varepsilon_0} |F_\varepsilon(\xi)|^2\,d\xi\,d\varepsilon/\varepsilon \leqq C_4C_5 2^s\|u\|^2_{-s-1,\varepsilon_0}\,,$$

which is precisely the inequality (5.6).

It thus only remains to prove (5.8) and (5.9). We then have to consider separately the cases when ξ and η are close and when they are far apart.

If M is the maximum of $|\operatorname{grad}\hat{\varphi}|$ we get

$$K(\varepsilon, \xi, \eta) \leqq M\varepsilon^{s+1}|\hat{a}(\xi-\eta)|\,|\xi-\eta|\,(2^{s+1}|\xi-\eta|^{s+1}+\varepsilon_0^{-s-1}) \tag{5.10}$$

if $|\eta| \leqq 2|\xi-\eta|$. Next let $|\eta| > 2|\xi-\eta|$. The line segment joining ξ and η then lies outside the sphere with radius $|\eta|/2$. If we denote by $\Phi(t)$ the maximum of $|\operatorname{grad}\hat{\varphi}|$ outside the sphere with radius $t/2$, we thus get

$$K(\varepsilon, \xi, \eta) \leqq \varepsilon^{s+1}|\hat{a}(\xi-\eta)|\,|\xi-\eta|\,\Phi(\varepsilon|\eta|)\,(|\eta|^{s+1}+\varepsilon_0^{-s-1})\,. \tag{5.11}$$

Since φ is rapidly decreasing at infinity, this is also true for Φ.

Noting that $\Phi(\varepsilon|\eta|)$ and $(\varepsilon|\eta|)^{s+1}\,\Phi(\varepsilon|\eta|)$ are bounded and that $\varepsilon \leqq \varepsilon_0 \leqq 1$ we get from (5.10) and (5.11)

$$K(\varepsilon, \xi, \eta) \leqq C|\hat{a}(\xi-\eta)|\,|\xi-\eta|\,(|\xi-\eta|^{s+1}+1)\,. \tag{5.12}$$

Since the integral of the right hand side with respect to η is finite and independent of ε and ξ, the inequality (5.8) follows.

To prove (5.9) we make the estimate

$$\int_0^{\varepsilon_0} \Phi(\varepsilon\eta)\,|\eta|^{s+1}\varepsilon^s d\varepsilon \leqq \int_0^{\infty} \Phi(\varepsilon\eta)\,|\eta|^{s+1}\varepsilon^s d\varepsilon = \int_0^{\infty} \Phi(\varepsilon\eta')\varepsilon^s d\varepsilon$$

where $\eta' = \eta/|\eta|$ is a unit vector. Since $\Phi(\varepsilon\eta')$ tends rapidly to 0 when $|\varepsilon\eta'| = \varepsilon \to \infty$, this integral is a bounded function of η'. Hence we obtain from (5.10) and (5.11), using the fact that Φ is bounded

$$\int_0^{\varepsilon_0} K(\varepsilon, \xi, \eta)\,d\varepsilon/\varepsilon \leqq C\,|\hat{a}(\xi-\eta)|\,|\xi-\eta|\,(|\xi-\eta|^{s+1}+1)\,, \tag{5.13}$$

and (5.9) now follows immediately.

6. A priori estimates in $\mathscr{H}^{-s}$

We shall now prove that the L^2 norms in the definition of operators of principal type may be replaced by the norms $\|u\|_{-s}$.

Theorem 6.1. Let $P(x, D)$ be of principal type and have coefficients in C^∞. Define Ω_δ as in Definition 1.1. Then for every $s > 0$ there are positive constants C_s and δ_s such that

$$\sum_{|\alpha|<m} \delta^{2(|\alpha|-m)}\|D_\alpha u\|^2_{-s,\delta} \leqq C_s\|P(x, D)u\|^2_{-s,\delta}\,, \quad u \in C_0^\infty(\Omega_\delta)\,, \tag{6.1}$$

provided that $\delta \leqq \delta_s$. The constants C_s and δ_s^{-1} are bounded when s is bounded.

Proof. Take a function φ satisfying (5.3) with support in the unit sphere with centre at 0. Write

$$u_\varepsilon = u * \varphi_\varepsilon\,. \tag{6.2}$$

If $\varepsilon < \delta$ and $u \in C_0^\infty(\Omega_\delta)$ we have $u_\varepsilon \in C_0^\infty(\Omega_{2\delta})$. When $2\delta < \delta_0$ we may thus apply (1.3) to u_ε and get

$$\sum_{|\alpha|<m} (2\delta)^{2(|\alpha|-m)}\|D_\alpha u_\varepsilon\|^2 \leqq C_0\|P(x, D)u_\varepsilon\|^2\,.$$

Hence

$$\sum_{|\alpha|<m}(2\delta)^{2(|\alpha|-m)}\, s\int_0^\delta \|D_\alpha u_\varepsilon\|^2\varepsilon^{2s-1}d\varepsilon \leqq C_0\, s\int_0^\delta \|P(x,D)u_\varepsilon\|^2\varepsilon^{2s-1}d\varepsilon .$$

Since $D_\alpha u_\varepsilon = (D_\alpha u) * \varphi_\varepsilon$ the left hand side can be estimated from below by means of Lemma 5.1. This gives

$$(6.3)\qquad \sum_{|\alpha|<m}(2\delta)^{2(|\alpha|-m)}\|D_\alpha u\|^2_{-s,\delta} \leqq C_0 s/C_1\int_0^\delta \|P(x,D)u_\varepsilon\|^2\varepsilon^{2s-1}d\varepsilon .$$

With $f = P(x, D)u$ we have

$$(6.4)\qquad P(x,D)u_\varepsilon = f_\varepsilon + \sum_{|\alpha|\leqq m}\left(a_\alpha((D_\alpha u)*\varphi_\varepsilon) - (a_\alpha D_\alpha u)*\varphi_\varepsilon\right).$$

From Lemma 5.1 we obtain

$$(6.5)\qquad s\int_0^\delta \|f_\varepsilon\|^2\varepsilon^{2s-1}d\varepsilon \leqq C_2\|f\|^2_{-s,\delta}$$

and from Lemma 5.2, assuming that $\delta < 1$,

$$(6.6)\qquad \int_0^\delta \|a_\alpha((D_\alpha u)*\varphi_\varepsilon) - (a_\alpha D_\alpha u)*\varphi_\varepsilon\|^2\varepsilon^{2s-1}d\varepsilon \leqq C_3\|D_\alpha u\|^2_{-s-1,\delta} .$$

We now introduce the expression (6.4) for $P(x, D)u_\varepsilon$ in the inequality (6.3). After using Cauchy's inequality in the right hand side we can apply (6.5) and (6.6). With a new constant C we get

$$(6.7)\qquad \sum_{|\alpha|<m}\delta^{2(|\alpha|-m)}\|D_\alpha u\|^2_{-s,\delta} \leqq C\left(\|f\|^2_{-s,\delta} + \sum_{|\alpha|\leqq m}\|D_\alpha u\|^2_{-s-1,\delta}\right).$$

In those terms on the right hand side of (6.7) where $|\alpha| < m$ we use the inequality

$$(6.8)\qquad \|D_\alpha u\|_{-s-1,\delta} \leqq \delta\|D_\alpha u\|_{-s,\delta}$$

which follows from (5.2). On the other hand, if $|\alpha| = m$ we can write $D_\alpha = D_j D_\beta$ where $|\beta| = m - 1$. In view of (5.2) again we have

$$(6.9)\qquad \|D_\alpha u\|_{-s-1,\delta} \leqq \|D_\beta u\|_{-s,\delta} .$$

We now get from (6.7) by estimating the terms on the right hand side by means of (6.8) or (6.9)

$$\sum_{|\alpha|<m}\delta^{2(|\alpha|-m)}\|D_\alpha u\|^2_{-s,\delta} \leqq C\left\{\|f\|^2_{-s,\delta} + (\delta^2+n)\sum_{|\alpha|<m}\|D_\alpha u\|^2_{-s,\delta}\right\}.$$

(n is the dimension.) When $\delta < 1$ and $C\delta^2(\delta^2+n) < 1/2$, we get

$$\sum_{|\alpha|<m}\delta^{2(|\alpha|-m)}\|D_\alpha u\|^2_{-s,\delta} \leqq 2C\|f\|^2_{-s,\delta} ,$$

which completes the proof of Theorem 6.1.

Remark. The arguments of the proof could also be applied if instead of (1.3) we only knew that

$$\|u\|_{m-1} \leqq C(\delta)\|P(x,D)u\|,\ u \in C_0^\infty(\Omega_\delta) ,$$

where $C(\delta) \to 0$ as $\delta \to 0$. We then get a similar estimate with the other norms. However, we do not know any example of an operator for which this estimate but not (1.3) holds.

7. The existence of smooth solutions

Let $P(x, D)$ satisfy the same assumptions as in Theorem 6.1. Denote by $P^t(x, D)$ the formal adjoint of $P(x, D)$, defined by

$$\int (P^t(x, D)\varphi)u\,dx = \int \varphi P(x, D)u\,dx\,, \quad u \in C^\infty, \quad \varphi \in C_0^\infty .$$

Explicitly the formal adjoint is defined by

$$P^t(x, D)\varphi = \sum (-D)_\alpha (a_\alpha \varphi) .$$

The principal part is $(-1)^m p$. Hence Theorem 4.2 shows that P^t also satisfies the assumptions in Theorem 6.1, so that we may replace P by P^t there. Let δ_s and C_s be the constants then associated with P^t.

Theorem 7.1. Let f be a function such that $D_\alpha f \in L^2(R^\nu)$, $|\alpha| \leqq k$ $(k \geqq 0)$. If $\delta \leqq \delta_{k+m-1}$ there then exists a solution in Ω_δ (in the distribution sense) of the equation $P(x, D)u = f$, such that $D_\alpha u \in L^2(\Omega_\delta)$ when $|\alpha| \leqq k + m - 1$.

Proof. The equation $P(x, D)u = f$ means by definition that

$$\int (P^t(x, D)\varphi)u\,dx = \int \varphi f\,dx, \quad \varphi \in C_0^\infty(\Omega_\delta) . \tag{7.1}$$

Thus consider the mapping

$$P^t(x, D)\varphi \to \int \varphi f\,dx\,, \tag{7.2}$$

defined for $\varphi \in C_0^\infty(\Omega_\delta)$. The mapping is linear and

$$\left|\int \varphi f\,dx\right| \leqq \|f\|_{k,\delta} \|\varphi\|_{-k,\delta} . \tag{7.3}$$

Now we have according to Theorem 6.1 with $s = k + m - 1$

$$\sum_{|\alpha| < m} \delta^{2(|\alpha| - m)} \|D_\alpha \varphi\|^{1}_{-s,\delta} \leqq C_s \|P^t(x, D)\varphi\|^2_{-s,\delta} . \tag{7.4}$$

In view of (5.2) we can estimate $\delta^{-2}\|\varphi\|^2_{m-1-s,\delta}$ by means of the left hand side in (7.4) and obtain

$$\|\varphi\|_{m-1-s,\delta} \leqq C'\delta \|P^t(x, D)\varphi\|_{-s,\delta} . \tag{7.5}$$

Since $m - 1 - s = -k$ this gives combined with (7.3)

$$\left|\int \varphi f\,dx\right| \leqq C'\delta \|f\|_{k,\delta} \|P^t(x, D)\varphi\|_{-s,\delta} .$$

In view of the Hahn-Banach theorem the mapping (7.2) may hence be extended to a continuous linear form on $\mathscr{H}^{-s}$. Thus there exists an element $u \in \mathscr{H}^s$ such that

$$\|u\|_{s,\delta} \leqq C'\delta \|f\|_{k,\delta}$$

and (7.1) holds. But this means that $P(x, D)u = f$ in Ω_δ. The proof is complete.

Every function in $L^2(\Omega_\delta)$ may be extended to a function in $L^2(R^\nu)$ by defining it to be 0 outside Ω_δ. (Since Ω_δ has a smooth boundary there is also an extension theorem for functions with all derivatives of order $\leqq k$ in L^2, but it is then less trivial.) We thus get the following improvement of the results of [3].

Corollary. If P is of principal type with coefficients in C^∞ and $\delta \leqq \delta_{m-1}$, the equation $P(x, D)u = f$ with $f \in L^2(\Omega_\delta)$ has a solution u such that $D_\alpha u \in L^2(\Omega_\delta)$ when $|\alpha| \leqq m - 1$.

Added in proof. Theorems 3.1 and 3.2 may be extended to operators of any order and coefficients in C^∞. This will be done in an article to appear in this journal with the title "Differential equations without solutions".

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