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The Plancherel Formula for the Universal Covering Group of $SL(R,2)$*

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Introduction

The problem of finding a Plancherel formula in the case of a large class of groups, in particular of semi-simple Lie groups, can be formulated as follows. One has to construct a family F of irreducible unitary representations in such a fashion, that by taking the traces of the integrals of a fixed sufficiently regular (e.g. C^∞ with a compact support) function with respect to members of F[1]) (the traces being assumed to exist) and by forming the integral of these traces by aid of a suitable measure on the space parametrizing the members of F, one should obtain the value assumed by our function at the unity. Often one can set up a basis in a canonical fashion for any irreducible representation in the resp. representation space, and the matrix elements with respect to this basis, eigenfunctions of the differential operators invariant under left and right translations, turn out to be expressible in terms of certain special functions. In these cases the Plancherel theorem is essentially equivalent to a set of completeness relations involving these functions, and thus the former is often capable of giving simple interpretation of seemingly unrelated facts of the classical analysis. Methodologically, one can sometimes obtain the Plancherel formula through such completeness relations and conversely, though the transition may be not quite easy.

Often the trace of the integral, formed with respect to a fixed member of F, considered as a linear functional, is a distribution generated by a function, which is locally integrable with respect to the invariant measure. In analogy with the case of the compact groups, one calls this function the character of the representation, and the Plancherel formula assumes the meaning of a completeness relation of these characters.

For any complex semi-simple group the Plancherel formula has already been found some time ago (cf. [5]). The real case, because of the existence of nonconjugate Cartan subalgebras is more difficult, and as far as now only

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[1]) If $\{T(a)\}$ is the given representation, this means the operator $\int_G f(a)\, T(a)\, d\mu(a)$, where $d\mu(a)$ is the element of a fixed left invariant measure on the group G.

partial results are available. The case of the group $SL(R, 2)$ (group of all 2×2 real matrices with a determinant 1), which is particularly interesting from the point of view of the special functions, was taken up first by BARGMANN ([1]) through the discussion of the matrix coefficients, determined by him as eigenfunctions of the Laplacian. A detailed proof, with much emphasis on the group theoretic meaning, has recently been given by TAKAHASHI ([10]). An invariant approach was outlined by HARISH-CHANDRA in [3].

The purpose of the present paper is to give a detailed discussion of the Plancherel formula for the universal covering group G of $SL(R, 2)$. One of the new features of this case comes from the fact, that no proper covering group of $SL(R, 2)$ possesses a faithful linear representation. G is also universal covering group of the tree-dimensional Lorentz group, and among the Lorentz groups of dimension $\geqq 3$ it is only in this case, that the center is infinite. We intend to give two different proofs, and though one of them is going to be an extension of Harish Chandra's method, our emphasis will always be on the connection with classical analysis. The Plancherel formula of any other covering group of $SL(R, 2)$ can be obtained by an easy modification of the reasonings employed in any of these proofs.

The formula to be derived is as follows. The family F of irreducible representations (cf. above) consists of three subfamilies $C_q^{(\tau)}\left(0 \leqq \tau < 1, q > \frac{1}{4}\right)$, D_l^+ and D_l^- $\left(l > \frac{1}{2}\right)$. The notation has been chosen in conformity with those of BARGMANN; we obtain representations of $SL(R, 2)$ by putting $\tau = 0, \frac{1}{2}$ in the first case, $l = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ in the second and third case resp. Assume now, that $f(a)$ $(a \in G)$ is indefinitely differentiable and has a compact support. We denote the trace of its integral with respect to a representation of type $C_q^{(\tau)}$ by $T_\sigma^{(\tau)}(f)$ $\left(\sigma = \sqrt{q - \frac{1}{4}} > 0\right)$. We consider also the trace of its integral with respect to a direct sum of two representations of type D_l^+ and D_l^- resp.; we denote this by $T_l(f)$. The Plancherel formula, normalizing the Haar measure on G appropriately, is given by

$$f(e) = \int_0^\infty \int_0^1 \sigma[\operatorname{Re} \tanh \pi(\sigma + i\tau)]\, T_\sigma^{(\tau)}(f)\, d\tau\, d\sigma + \int_{\frac{1}{2}}^\infty \left(l - \frac{1}{2}\right) T_l(f)\, dl\,.$$

It is instructive to compare this with the Plancherel formula for $SL(R, 2)$

$$f(l) = \int_0^\infty \sigma(\tanh \pi\sigma)\, T_\sigma^{(0)}(f)\, d\sigma + \int_0^\infty \sigma(\coth \pi\sigma)\, T_\sigma^{\left(\frac{1}{2}\right)}(f)\, d\sigma + \sum_{k=1}^\infty \frac{k-1}{2} T_{\frac{k}{2}}(f)\,.$$

Similarly, as in the case of $SL(R, 2)$, F does not contain representations from all equivalence classes of irreducible unitary representations of G. Among others, the representations of the "exceptional domain" $C_q^{(\tau)}(0 \leqq \tau < 1, \tau\left(1 - \tau\right) < q \leqq \frac{1}{4}\right)$ are missing (for details cf. Part I).

The paper consists of three Parts. The first Part gives the classification and realization of the irreducible representations of G. Though, as mentioned before, not all representations are needed for the Plancherel formula, to put things in the proper perspective, a description of all representations is included. What concerns the classification we follow closely BARGMANN's discussion of the same problem for $SL(R, 2)$. — Part II gives the first proof based on a detailed discussion of the matrix elements. The idea of the proof, specialized to $SL(R,2)$, was previously outlined in [6]. Instead of the Laplacian of G, we start with the realizations of Part I to obtain detailed description of the matrix elements, which are expressible in terms of hypergeometric functions. During the course of the discussion we get for them analogues of the integral representations of Laplace and Dirichlet-Mehler for Legendre polynomials. The main result then follows from a special completeness relation involving these functions (cf. (2.19)). The second proof, contained in the last Part, is modeled after HARISH-CHANDRA's proof for $SL(R, 2)$ ([3]). Among the essential modifications needed we mention, in particular, the computation of the characters of the representations $D_l^{\pm}$ $\left(l > \frac{1}{2}\right)$, since the method of [3] makes an essential use of the existence of a faithful linear representation. Our computations are based on the integral formulas for the matrix coefficients obtained in Part II.

Many computations connected with G can be reduced to the consideration of the analogous problems with $SL(R, 2)$. Since this group has already been discussed in detail, in particular by BARGMAN [1] and TAKAHASHI [10], we are going to make use of certain parts, to be specified later, of these papers. On the other hand, we thought to help the reader by giving short proofs, specialized to G, of certain facts, available in a much more general context (cf. for instance the proof, in Part I, for the existence of the trace, several integral relations in Part III. etc.).

Part I. The irreducible representations of G

A. Classification of the representations

1. Preliminaries. In what follows we summarise certain facts concerning unitary representations and properties of the group G. Since most of these are either standard or easily verifiable, we shall indicate but a few proofs.

Let G be a Lie group, and $a \to T(a)$ $(a \in G)$ a continuous unitary representation of G acting on a separable unitary space $\mathfrak{H}$. Let $\mathfrak{L}$ be the Lie algebra of G; for $l \in \mathfrak{L}$ we denote by H_l the self-adjoint operator uniquely determined by the condition $T(\exp lt) = \exp(-iH_l t)$; finally, let D_l be its domain of definition. Then one has the following situation ([7] Theorem 3.1): There exists a dense submanifold $B \subset \mathfrak{H}$ such that a. $B \subset D_l$ for any $l \in \mathfrak{L}$, and the minimal closed extension of the restriction H_l' of H_l to B is H_l. b. We have $H_l B \subseteq B$ and $T(a) B \subseteq B$ $(a \in G)$ c. the map $l \to -iH_l'$ $(l \in \mathfrak{L})$ gives a representation of $\mathfrak{L}$ by linear transformations of B into itself.

From now on we assume, that G is the universal covering group of $SL(R, 2)$, for which we put G_1. We identify the Lie algebra $\mathfrak{L}$ of G with that of G_1. Hence

$\mathfrak{L}$ can be identified with the collection of all 2×2 real matrices with trace 0. The elements

$$l_0 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$(1.1) \qquad l_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$l_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

form a basis in $\mathfrak{L}$ and satisfy the following relations

$$(1.2) \qquad [l_0, l_1] = l_2, \quad [l_1, l_2] = -l_0, \quad [l_2, l_0] = l_1 .$$

We denote the canonical homomorphism from G onto G_1 by Φ. The adjoint representation of G will then be given by $\mathrm{Ad}(a)(x) = \Phi(a)x[\Phi(a)]^{-1}(a \in G, x \in \mathfrak{L})$. If $x = x_0 l_0 + x_1 l_1 + x_2 l_2$, $\mathrm{Ad}(a)$ leaves the quadratic form $x_0^2 - x_1^2 - x_2^2$ invariant, and the map $a \to \mathrm{Ad}(a)$ gives a homomorphism from G onto the connected component of the identity in the three dimensional Lorentz group; in what follows it will be denoted by G.

Assume now, that $T(a)$ is an *irreducible* unitary representation of G. We write H_j for H_{l_j} $(j = 0, 1, 2)$ and form the operator $Q' = H_1^2 + H_2^2 - H_0^2$. It is densely defined, since it is certainly defined on B, and symmetric. Using the fact, that the operators of the adjoint representation commute with the matrix $l_0^2 - l_1^2 - l_0^2$, one easily shows (cf. the reasoning in [1] 5e, p. 601), that the minimal closed extension Q of Q' is of the form qI, where q is a real number and I the identity operator. Even without the assumption of irreducibility Q turns out to be self-adjoint (cf. Theorem in [8]); it is called the Casimir operator belonging to our representations.

It is known, that the group manifold of G_0 is homeomorphic to the product of the Euclidean plane with the one-dimensional torus, hence its Poincaré group, the center of G, is infinite cyclic (cf. [1] § 4). We put $o_\varphi = \exp(l_0 \varphi) \in G$ $(-\infty < < \varphi < \infty)$, and $\gamma = o_{2\pi}$; the center of G is generated by γ. The subgroup $\{o_\varphi\}$, denoted by O in the following, is a closed subgroup of G isomorphic to R^1.

We put $U_\varphi = T(o_\varphi) = \exp(-H_0 \varphi)$. Since $U_{2\pi} = T(\gamma)$ commutes with any $T(a)$ $(a \in G)$, it is necessarily of the form $e^{-i 2\pi \tau} I$. From this we conclude, that $e^{i\tau\varphi} U_\varphi$ is a unitary representation, periodic with 2π, of R^1, hence it is completely reducible. Hence H_0 possesses a complete system of eigenelements.

Since a unitary representation is uniquely determined by the operators H_j $(j = 0, 1, 2)$, we can start the classification of the irreducible representations by characterizing irreducible triples, satisfying commutation relations corresponding to those of (1.2). We shall do this by describing their action on eigenelements of H_0. The possibility of this procedure is garanteed by the following two statements:

a. Putting D_j for the domain of H_j $(j = 0, 1, 2)$ we have $D_0 \subset D_1 \cap D_2$,
b. Denoting by D the linear manifold consisting of all finite linear combinations of eigenelements of H_0, H_j $(j = 0, 1, 2)$ is the closure of its restriction to D.
To prove a. observe, that for any $f \in D_0$ we can find a sequence $\{f_n;\ f_n \in B\}$,

such that $f_n \to f$ and $H_0 f_n \to H_0 f$. But for any $h \in B$ we have $qh + H_0^2 h = H_1^2 h + H_2^2 h$; replacing h by $f_n - f_m$ and forming the scalar product of both sides with the same element, we see, that the sequences $\{H_j f_n\}$ $(j = 1, 2)$ converge too, implying $f \in D_1 \cap D_2$. Concerning b. we observe, that it is certainly true for H_0. Next repeating the previous reasonings with D in place of B, we see, that the domains of the closures of the restrictions of H_j $(j = 1, 2)$ to D contain D_0. But $B \subset D_0$, and hence by property a. of B these closures coincide with H_j $(j = 1, 2)$ resp.

Next we form the operators $H_+ = H_1 + iH_2$, $H_- = H_1 - iH_2$. Assuming $H_0 f = \lambda f$, and observing, that the image of O in the adjoint group is the group of rotations leaving the x_0 axis fixed, a repetition of the reasonings in [1] 5f, p. 601, (replace F by H_+ and G by H_- resp.) leads to the following relations

$$H_0 H_+ f = (\lambda + 1) H_+ f, \quad H_0 H_- f = (\lambda - 1) H_- f\,.$$

Note, that this in particular implies, that $H_j D \subseteq D$ $(j = 0, 1, 2;$ for the definition of D cf. b. in the previous paragraph). By virtue of the second relation in (1.2) we have also $[H_1, H_2] f = (H_1 H_2 - H_2 H_1) f = -iH_0 f$, or $[H_+, H_-] f = -2H_0 f$ for every $f \in B$, hence for any element for which the left hand side is defined; hence, in particular for any f in D.

Summing up all, if $H_0 f = \lambda f$ we have the following relations

$$\begin{aligned} H_0 H_+ f &= (\lambda + 1) H_+ f \\ H_0 H_- f &= (\lambda - 1) H_- f \\ (H_+ H_- - H_- H_+) f &= -2H_0 f = -2\lambda f \\ (H_+ H_- + H_- H_+) f &= 2(qI + H_0^2) f = 2(qI + \lambda^2) f \end{aligned} \tag{1.3}$$

$\left(\text{for the last equation observe, that } qI = Q = H_1^2 + H_2^2 - H_0^2 = \frac{1}{2}(H_+ H_- + H_- H_+) - H_0^2 \text{ on } D\right)$.

2. Description of the infinitesimal operators. The following discussion is very analogous to that of [1], in particular 5g (p. 605), for which the reader is referred for further details of some computations.

Forming the sum and difference of the last two relations in (1,3), we obtain

$$\begin{aligned} H_- H_+ f &= [q + \lambda(\lambda + 1)] f \\ H_+ H_- f &= [q + \lambda(\lambda - 1)] f\,. \end{aligned}$$

Replacing now f by $H_+^{j-1} f$ and $H_-^{j-1} f$ $(j = 1, 2, \ldots)$ resp. in these equations, and taking into account the first two relations in (1.3) we get

$$\begin{aligned} H_- H_+^j f &= \alpha_j H_+^{j-1} f \\ H_+ H_-^j f &= \beta_j H_-^{j-1} f\,, \end{aligned} \tag{1.4}$$

where $\alpha_j = q + (\lambda + j - 1)(\lambda + j)$ and $\beta_j = q + (\lambda - j)(\lambda - j + 1)$.

The above relations imply

(1.5) $\|H_+^{j+1}f\|^2 = (H_+^{j+1}f, H_+^{j+1}f) = (H_-H_+^j f, H_+^j f) = \alpha_{j+1}\|H_+^j f\|^2$

and

(1.6) $\|H_-^{j+1}f\|^2 = (H_-^{j+1}, H_-^{j+1}f) = (H_+H_-^{j+1}f, H_-^j f) = \beta_{j+1}\|H_-^j f\|^2$.

From which, in particular, we conclude, that $\alpha_j, \beta_j \geqq 0$.

Now we turn to the description of all possible irreducible triples H_j $(j = 0, 1, 2)$; we achieve this by doing the same for H_0, H_+, H_-. During the course of the discussion it will turn out, that the difference between two eigenvalues of H_0 is a integer, a result, which, incidentally, follows from the discussion of the spectral properties of H_0 given above in 1.

I. Suppose first, that no member of the sequences $\{H_+^j f\}$ and $\{H_-^j f\}$ $(j = 0, 1, 2, \ldots)$ vanishes $(H_0 f = \lambda f)$. In this case, by virtue of the first two relations in (1.3), a number $0 \leqq \tau < 1$ must occur among the corresponding eigenvalues of H_0. Hence we shall immediately assume that $H_0 f = \tau f$ and, in addition, $\|f\| = 1$. Because of (1.5) and (1.6) the constants α_j, β_j must be positive, which happens if and only if $q > \tau(1 - \tau)$. Putting

$$a_s = \prod_{j=1}^{s} (\alpha_j)^{\frac{1}{2}}, \quad b_s = \prod_{j=1}^{s} (\beta_j)^{\frac{1}{2}}$$

and

$$f_\tau = f,\ f_{\tau+s} = \frac{H_+^s f}{a_s}, \quad f_{\tau-s} = \frac{H_-^s f}{b_s} \quad (s = 1, 2, \ldots)$$

we have $\|f_m\| = 1$, $H_0 f_m = m f_m$ $(m = \tau \pm t, t = 0, 1, 2, \ldots)$. Furthermore, using (1.4) an easy computation shows, that

$$\begin{aligned} H_+ f_m &= (q + m(m+1))^{\frac{1}{2}} f_{m+1} \\ H_- f_m &= (q + m(m-1))^{\frac{1}{2}} f_{m-1}\,. \end{aligned} \tag{1.7}$$

Assume now, that $|\varepsilon_m| = 1$; putting $g_m = \varepsilon_m f_m$, $\omega_m = \frac{\varepsilon_m}{\varepsilon_{m-1}}$ we finally obtain

$$\begin{aligned} H_0 g_m &= m g_m \\ H_+ g_m &= \omega_m (q + m(m+1))^{\frac{1}{2}} g_{m+1} \\ H_- g_m &= \frac{1}{\omega_{m-1}} (q + m(m-1))^{\frac{1}{2}} g_{m-1}\,. \end{aligned} \tag{1.8}$$

Observe, that using the notations of 1., we have $T(\gamma) = e^{-2\pi i \tau} I$; in order to have a representation of $G_1 = SL(R, 2)$, we must evidently have $e^{-2\pi i \tau} = \pm 1$, implying $\tau = 0$, or $\frac{1}{2}$.

II. a. Assume next, that for some positive integer j we have $H_-^j f = 0$. Replacing f by $H_-^{j-1} f$ we get $H_- f = 0$ and $H_0 f = l f$ with an appropriately chosen l. Hence $\beta_1 f = H_+ H_- f = 0$, or $\beta_1 = (q + l(l-1)) = 0$, and $q = l(1-l)$. If $\alpha_1 = 0$, we have $H_+ f = 0$ in addition to $H_- f = 0$. But then $H_j f = 0$ $(j = 1,2)$ which implies that f is invariant under $T(a)$; or the latter is the trivial representation. Excluding this case we have $\alpha_1 = 2l > 0$, or $l > 0$, and also $\alpha_j = j(2l + j - 1) > 0$ $(j = 1, 2, \ldots)$. Putting as before

$$a_s = \prod_{j=1}^{s} (\alpha_j)^{\frac{1}{2}},$$

and

$$f_l = f, \quad f_{l+s} = \frac{H_+^s f}{a_1} \quad (s = 1, 2, \ldots)$$

assuming $\|f\| = 1$, we get $\|f_s\| \equiv 1$ and relations identical with (1.7). Replacing f_m by $g_m = \varepsilon_m f_m$, $|\varepsilon_m| = 1$, $(m = l, l + 1, \ldots,)$, defining ω_m as before, we arrive at a situation similar to (1.8).

b. The case of a nontrivial representation, where for some j $H_+^j f = 0$ is very very similar, and we restrict ourselves to list the final formulas. Here H_0 is going to have a greatest eigenvalue, which is necessarily negative. Assuming it in the form $-l$, $l > 0$, we get $q = l(1 - l)$, and we can construct a sequence of eigenvectors g_m $(m = -l, -l + 1, \ldots,)$ satisfying $H_0 g_m = m g_m$ and once more relations analogous to those of (1.8).

Observe, that in case a. (b. resp.) we have $T(\gamma) = e^{-2\pi i l}$ ($e^{2\pi i l}$ resp.). Hence in order to obtain a representation of G_1 we must have $2l$ integer.

Note, that in each of the preceding cases the sequence $\{g_m\}$ forms a complete orthonormal system in the representation space $\mathfrak{H}$, and the range of m is just the spectrum of the operator H_0. The former statement is an immediate consequence of the fact, that this sequence goes into itself under the action of the operators H_0, H_+ and H_-, and hence under the action of H_j $(j = 0, 1, 2)$. This implies, that the closed subspace $\mathfrak{H}'$ of $\mathfrak{H}$ generated by the vectors $\{g_m\}$ is invariant under any operator $T(a)$ $(a \in G)$ of our representation; hence in view of the irreducibility of the latter, $\mathfrak{H}' = \mathfrak{H}$. This finishes the characterization of the infinitesimal operators of the irreducible unitary representations of G, these being assumed to exist. That this is indeed so will be proved in the next section.

For later reference, we summarize the result of the previous discussion introducing at the same time some notations. We have the following classes of irreducible representations (we omit the trivial representation).

I. $C_q^{(\tau)} : Q = qI \left(q > \frac{1}{4}, \ 0 \leqq \tau < 1\right)$. The spectrum of H_0 consists of the numbers $\{\tau \pm j, j = 0, 1, 2, \ldots\}$.

II. The series D_l^+ and D_l^-: $Q = l(1 - l)I$ $(l > 0)$ and the spectrum of H_0 consists of $\{l + j\}$ and $\{-l - j\}$ $(j = 0, 1, 2, \ldots)$ resp.

(1.9)

III. $E_q^{(\tau)}$: $\quad Q = qI \quad \left(0 \leqq \tau < 1, \quad \tau(1 - \tau) < q \leqq \frac{1}{4}\right)$ Spectrum of H_0 $= \{\tau \pm j, j = 0, 1, 2, \ldots)$.

We shall see later, that for the Plancherel formula we need classes I and II only $\left(\text{and even from the latter only those with } l > \frac{1}{2}\right)$. These are going to make up the family F of the Introduction.

B. Realization of the irreducible representations

First we introduce several subgroups of G, which will be useful in the following.

a. Consider the subgroup $S_0 \subset G_1$ of all triangular matrices having 0 in the lower left corner, and positive elements in the diagonal; a typical element of S_0 has the form

$$\begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix}.$$

Consider now the complete inverse image S of S_0 in G. It is known (and easily verified directly), that S is the direct product of the center Z of G with the component of the identity.

The canonical map from the latter onto S_0 is an isomorphism; in what follows it, too, will be denoted by S_0. Furthermore O (cf. A.1) is the complete inverse image of the subgroup of rotations in G_1; every $a \in G$ can be written in the form so ($s \in S$, $o \in O$), and $so = s'o'$ implies $s' = sz$, $o' = oz^{-1}(z \in Z)$. The map $\psi: S_0 \times O \to G$ defined by $\psi(s, o) = so$ is a diffeomorphism between the corresponding manifolds.

b. We denote the Cartan subgroup of G_1 consisting of all diagonals $\in S_0$ by H_0. We use the same letter to denote the corresponding subgroup $\subset S_0 \subset G$, and H for its complete inverse image in G.

c. Finally, we write N for the subgroup $\subset S_0 \subset G$ corresponding to elements of S_0 having 1 in the diagonal.

We denote the left and right regular representation on G by L_a and R_a resp. We form the subgroups $g_j(t) = \exp(l_j t)$ ($j = 0, 1, 2$; for l_j cf. (1.1)) of G, and for $f \in C^\infty$ we put $(H_j f)(a) = i \frac{d}{dt} f(a g_j(t))\Big|_{t=0}$, and as in A.1 we introduce the operators $H_+ = H_1 + iH_2$, $H_- = H_1 - iH_2$ and $Q = \frac{1}{2}(H_+H_- + H_-H_+) - H_0^2$.

To obtain realizations of the irreducible unitary representations, listed in A.2 as a priori possible ones, we shall specify linear subspaces of C^∞, invariant under R_a. Then introducing an invariant metric we form the completion, and show, that the actions of H_0, H_+, H_- and members of a suitably chosen complete orthonormal system $\in C^\infty$ are given by (1.8). In this fashion it turns out in a natural fashion, that our representations are (proper or improper) subrepresentations of representations induced by certain characters of the maximal solvable subgroup S.

Now consider the linear family $F \subset C^\infty$ defined by the conditions a. $L_n f \equiv f$ ($n \in N$) b. $Qf = qf$, c. $R_\gamma f \equiv e^{-2\pi i\tau} f$ ($\gamma = g(2\pi) \in Z$, cf. A.1); here g and τ are real constants to be specified later. In view of properties of G considered above we can use $\lambda > 0$, $-\infty < \mu$, $\varphi < +\infty$ as global coordinates on G, and accordingly write $f \in C^\infty$ as an indefinitely differentiable function $f(\lambda, \mu, \varphi)$. One easily verifies, that a. implies $\frac{\partial f}{\partial \mu} \equiv 0$, and c. $f(\lambda, \mu, \varphi + 2\pi) \equiv e^{-2\pi i\tau} f(\lambda, \mu, \varphi)$. In order to consider b and for later use we observe, that the expression of H_0, H_+, H_- and Q in terms of λ, μ, φ is as follows

$$(1.10)\qquad \begin{aligned} H_0 &= i\frac{\partial}{\partial\varphi} \\ H_+ &= -e^{-i\varphi}\left(\frac{i\lambda}{2}\frac{\partial}{\partial\lambda}+\frac{\partial}{\partial\varphi}+\cdots\right) \\ H_- &= e^{i\varphi}\left(-\frac{i\lambda}{2}\frac{\partial}{\partial\lambda}+\frac{\partial}{\partial\varphi}+\cdots\right) \\ Q &= -\left[\frac{\lambda^2}{4}\frac{\partial^2}{\partial\lambda^2}+\frac{3\lambda}{4}\frac{\partial}{\partial\lambda}+\cdots\right]. \end{aligned}$$

We did not write out terms containing derivation according to μ. To obtain (1.10) it evidently suffices to verify these relations on G_1 (that is for functions periodic with 4π in φ on G), where it can be done by straightforward computations (cf. the similar computations in [10] Lemma 3, p. 62). Writing out $Qf = qf\ \left(q \neq \frac{1}{4}\right)$ for f satisfying a. and c. we get $f(\lambda, \varphi) \equiv \lambda^{\varrho_+}f_+(\varphi) + \lambda^{\varrho_-}f_-(\varphi)$, where $\varrho_\pm = -1 \mp \sqrt{1-4q}$, $f_\pm(\varphi + 2\pi) \equiv e^{-2\pi i\tau}f_\pm(\varphi)$. Let now $\lambda^\varrho f$ be any of these two summands; we rewrite it in an invariant form as follows. First we define a function $f(o)$ on O through $f(o_\varphi) \equiv f(\varphi)$ (we recall that $o_\varphi = g_0(\varphi)$); we have $f(o\gamma) = e^{-2i\pi\tau}f(o)$. Next we define (non-unitary) characters of S by putting $\chi_\pm(s\gamma^j) = \lambda^{\varrho_\pm}e^{-2\pi i\tau j}$ $(j = 0, \pm 1, \pm 2, \ldots;\ s \in S_0)$; we have again $\chi_\pm(s\gamma) = e^{-2\pi i\tau}\chi_\pm(s)$. Hence the function $f(g)$ $(g \in G)$ corresponding to $\lambda^{\varrho_\pm}f(\varphi)$ is of the form $\chi_\pm(s)f(o)$, where $s \in S$, $o \in O$ is *any* pair satisfying $so = g$, and F is a direct sum of the subspaces $F_\pm = \{\chi_\pm(s)\,f(o)\}$. Any of these subspaces is invariant under R_a; for $f(g) \equiv \chi(s)\,f(o)$ (we assume to have fixed a sign, not indicated in what follows) implies $(R_af)\,(g) = \chi(s)\,f_1(o)$, where $f_1(o) \equiv \chi(s(oa))\,f(o\bar{a})$, and $s(oa) \in S$, $o\bar{a} \in O$ any pair giving $s(oa)o\bar{a} = oa$; evidently $f_1(o\gamma) \equiv f_1(o)\cdot e^{-2\pi i\tau}$. Any function $f \in F_\pm$ is uniquely determined by its restriction to O, which can be any C^∞ function satisfying condition c. above. We can define a representation $\overline{R}_a$ of G on them by requiring

$$(R_af)\,(g) \equiv \chi(s)\,(\overline{R}_af)\,(o)\ (g = so)\,.$$

Now we proceed to give the explicit description of the irreducible representations, by defining an inner product and specifying the constants q and τ of the above considerations, according to (1.9). In what follows we shall use $\varrho_+ \cdot \varrho_-$ leads to unitarily equivalent representations, as it can be verified easily; we shall later need this fact for the family $C_q^{(\tau)}$ (cf. (1.9) and I below).

We define the operators H_0, H_+ and H_- for $\overline{R}_a$; denoting them by $\overline{H}_0$, $\overline{H}_{0+}$ and $\overline{H}_-$ resp., we have the following relations $(f \in F_+)$

$$(1.11)\qquad (H_0f)\,(g) = \chi(s)\,(\overline{H}_0f)\,(o),\ (H_\pm f)\,(g) = \chi(s)\,(\overline{H}_\pm f)\,(o)\ (g = so)\,.$$

I. *Case of* $C_q^{(\tau)}$. We fix $0 \leqq \tau < 1$, $q > \frac{1}{4}$ and put $\sigma = \sqrt{q - \frac{1}{4}} > 0$; we have $\varrho = -1 + 2i\sigma$. Next we consider the functions $g_m \equiv e^{-im\varphi}$ $(m = \tau \pm j$,

$j = 0, 1, 2, \ldots$). Using relations (1.10) and (1.11) we obtain

$$\begin{aligned} &\overline{H}_0 g_m = m g_m \\ (1.12) \qquad &\overline{H}_+ g_m = -i\left[m + \frac{1}{2} - i\sigma\right] g_{m+1} \\ &\overline{H}_- g_m = -i\left[m - \frac{1}{2} + i\sigma\right] g_{m-1}\,. \end{aligned}$$

Next we form the Hilbert space $L^2_\tau(O)$ of all functions on O, satisfying

$$f(o\,\gamma) = e^{-2\pi i \tau} f(o)\,, \int_0^{2\pi} |f(o_\varphi)|^2\, d\varphi < +\infty\,,$$

with an inner product

$$(f, g) = \frac{1}{2\pi}\int_0^{2\pi} f(o_\varphi)\,\overline{g(o_\varphi) d_\varphi}\,.$$

The sequence $\{g_m\}$ forms a complete orthonormal system in $L^2_\tau(O)$. Observing, that $\left|\left(m + \frac{1}{2}\right) \pm i\sigma\right|^2 = \left(m + \frac{1}{2}\right)^2 + \sigma^2 = m(m+1) + q$, putting

$$\omega_m = \frac{1}{i}\,\frac{\left(m + \frac{1}{2}\right) - i\sigma}{(q + m(m+1))^{\frac{1}{2}}}\,,$$

we can rewrite relations (1.12) in the form

$$\begin{aligned} &\overline{H}_0 g_m = m g_m \\ (1.13) \qquad &\overline{H}_+ g_m = \omega_m (q + m(m+1))^{\frac{1}{2}} g_{m+1} \\ &\overline{H}_- g_m = \frac{1}{\omega_{m-1}} (q + m(m-1))^{\frac{1}{2}} g_{m-1}, \ |\omega_m| = 1 \\ &\qquad (m = \tau \pm j, j = 0, 1, 2, \ldots)\,. \end{aligned}$$

A comparison of (1.13) with (1.8) shows, that extending the representation $\overline{R}_a$, starting with a dense submanifold of sufficiently regular elements, by continuity to $L^2_\tau(O)$ we obtain an irreducible unitary representation of type $C_q^{(\tau)}$.

II. a. *Case of* D_l^+. We put $\tau = l > 0$, $q = l(1 - l)$, giving $\varrho = -2l$. We consider the system of functions $g_m = \gamma_m e^{-im\varphi}$, where

$$\gamma_m = \left[\frac{\Gamma(l+m)}{\Gamma(m - l + 1)\,\Gamma(2l)}\right]^{\frac{1}{2}}$$

($m = l + j$, $j = 0, 1, 2, \ldots$), and form a Hilbert space H_l^+ by requiring, that $\{g_m\}$ should form a complete orthonormal system. The collection of all C^∞ functions, representable as a series in terms of the system $\{e^{-im\varphi}\}$, can be identified with a dense submanifold of H_l^+. A straightforward computation, similar to that of I above, using (1.10) and (1.11) shows, that putting $\omega_m \equiv i$, the system $\{g_m\}$ satisfies relations identical with those of (1.8), and $\overline{H}_- g_l = 0$. Hence, as above, $\overline{R}_a$ again extends to an irreducible unitary representation, now of type D_l^+ (cf. (1.9)).

II. b. *Case of* D_l^-. Now we put $\tau = -l$, $(l > 0)$, $q = l(1 - l)$; we have again $\varrho = -2l$. Proceeding as above, we obtain our representation forming a Hilbert space H_l^- by aid of the system $g_m = \gamma_m e^{-im\varphi}$, where

$$\gamma_m = \left[\frac{\Gamma(l + |m|)}{\Gamma(|m| - l + 1)\,\Gamma(2l)}\right]^{\frac{1}{2}}$$

$(m = -l - j, j = 0, 1, 2, \ldots)$. In order to obtain relations (1.8), we have to put $\omega_m = -i$.

III. *Case of* $E_q^{(\tau)}$. (As stated at the end of A.2, this family will not be needed for the Plancherel formula). We fix $0 \leqq \tau < 1$, $\tau(1-\tau) < q \leqq \frac{1}{4}$, which gives $\varrho = -1 + 2\sigma$, where $\sigma = \sqrt{\frac{1}{4} - q} > 0$. Now the representation, proceeding as in II. a above, can be obtained by constructing a Hilbert space $H_\sigma^{(\tau)}$ by aid of the system $g_m = \gamma_m e^{-im\varphi}$, where

$$\gamma_m = \left[\frac{\Gamma\left(\tau + \frac{1}{2} + \sigma\right)\;\Gamma\left(m + \frac{1}{2} - \sigma\right)}{\Gamma\left(\tau + \frac{1}{2} - \sigma\right)\;\Gamma\left(m + \frac{1}{2} + \sigma\right)}\right]^{\frac{1}{2}}$$

$(m = \tau + j, j = 0, 1, 2, \ldots)$. We use again $\omega_m \equiv i$.

C. In what follows, we prove for the following statement, which will be essential when setting up the Plancherel formulas.

Suppose $f(a)$ $(a \in G)$ *is* C^∞ *and has a compact support, and let* $T(a)$ *be an irreducible representation. Then the operator* $T_f = \int\limits_G f(a)\, T(a)\, d\mu(a)$ *is of trace class.* ($d\mu(a)$ is the element of the Haar measure on G).

This is known to be valid for any semi-simple group; but in the special case of G we are concerned with the proof is very simple. $\mathrm{Tr}(T_f)$ as linear functional in f is a distribution generated by a locally integrable function, called the character of our representation. We shall obtain its exact form, for the representations of the Plancherel formula, in III.A.

We recall, that an operator A acting on a Hilbert space $\mathfrak{H}$ is of the trace class, if it can be represented as the product of two Hilbert-Schmidt operators. Then, if $\{e_j\}$ is a complete orthonormal system in $\mathfrak{H}$, the series $\sum\limits_{i=1}^{\infty} (A e_j, e_j)$ is absolutely convergent, and its sum, denoted by $\mathrm{Tr}(A)$, is the same for any basis.

Now we observe, that

$$\int\limits_G f(a)\, d\mu(a) = \int\limits_{S_0 \times O} f(so)\, d\mu_l(s)\, d\mu(o) \tag{1.14}$$

where (using our usual parametrization of O) $d\mu(o) = d\varphi$, and $d\mu_l(s)$ is the appropriately normalized left invariant Haar measure on G. Indeed, by virtue of the discussion of B we certainly have $d\mu(a) = f(s, o)\, d\mu_l(s)\, d\mu(o)$, where $f(s, o)$ is continuous on $S_0 \times O$; using the fact, that $d\mu(a)$ is invariant under both left and right translations, the result follows.

If f is C^∞ and of a compact support on G, then $f(so)$ is C^∞ on $S_0 \times O$ vanishing outside a set $C \times I$, where C and I are compact subsets of S_0 and O

resp. Using (1.14) we get

$$T_f = \int_{S_0} T(s) \left(\int_{-\infty}^{\infty} f(s o_\varphi)\, T(o_\varphi)\, d\varphi \right) d\mu_l(s);$$

since $T(o_\varphi) = \sum_{(m)} e^{im\varphi} P_m$, where m runs over the spectrum of H_0 (cf. B), the integral according to φ can be written as $\sum_{(m)} F_m(s) P_m$ with

$$F_m(s) = \int_{-\infty}^{\infty} f(s o_\varphi) e^{-im\varphi} d\varphi,$$

the series converging strongly. Observe, incidentally, that for a fixed integer $r > 0$ and $m \neq 0$ we have $|F_m(s)| \leqq \frac{B^{(r)}}{|m|^r}$, with $B^{(r)} = K \sup_{\varphi \in I} \left| \frac{\partial^r f}{\partial \varphi^r}(s o_\varphi) \right|$, K depending on the support of f only. Hence $T_f = \sum_m F_m P_m$, where $F_m = \int_{S_0} F_m(s)\, T(s)\, d\mu_l(s)$; we have for any fixed integer $r > 0$ $\|F_m\| \leqq \frac{\overline{B}^{(r)}}{|m|^r}$, $\overline{B}^{(r)} = \mu_l(C) B^{(r)}$.

Putting finally

$$A = \sum_m (1 + |m|) F_m P_m$$

$$B = \sum_m \frac{1}{1 + |m|} P_m$$

the series on the right hand sides converge strongly, and represent operators of class Hilbert-Schmidt (observe that $\dim P_m = 1$). Moreover $T_f = A B$, proving our statement.

If the functions $\{f_n\}$ are C^∞, have the same support in G, and if for each integer $r > 0$, their r^{th} derivatives in φ tend to 0 uniformly, by virtue of the above estimates for $\|F_m\|$ we can conclude, that $\mathrm{Tr}(T_{f_n}) \to 0$.

Part II. The Plancherel formula and special functions

A. Matrix coefficients

For our first proof of the Plancherel formula of G we need explicit description of certain of the matrix coefficients of the irreducible representations taking part in the formula. While deriving these we shall obtain a group theoretic interpretation of integral representations of some special functions.

1. In what follows we put $g_u = \exp(l_2 u) \subset G$ and as before $o_\varphi = \exp(l_0 \varphi) \subset G$ (cf. (11), $-\infty < u,\ \varphi < +\infty$). For sake of brevity we use the same notation for the subgroups corresponding to l_0 and l_2 in G_0 and G_1 (cf. I.A.1), the context giving the correct interpretation. Also, we put $U = \{g_u\}$ and $O = \{o_\varphi\}$, $\subset G$.

One immediately verifies, that the restriction of the adjoint representation of G to U induces an isomorphism with its image in G_0, which is a closed subgroup. On the other hand, it is known ([10] Lemma 2, p. 60), that every element a in G_0 can be written as $o_{\varphi_1} g_u o_{\varphi_2}$ ($u \geqq 0$, $0 \leqq \varphi_1, \varphi_2 < 2\pi$), and if $a \notin O$ this representation is unique. From this one shows at once, that any $a \in G$ can be written in the form $o_{\varphi_1} g_u o_{\varphi_2}$ ($-\infty < \varphi_1 < \infty$, $u \geqq 0$, $0 \leqq \varphi_2 < 2\pi$) and for $a \notin O$ this representation is again unique. More exactly, putting R_+ for the open positive half line, the map from $R^1 \times R_+ \times T^1$ onto $G - O$, carrying $(\varphi_1, u, \varphi_2)$ into $o_{\varphi_1} g_u o_{\varphi_2}$, is a diffeomorphism between these two manifolds. The variables φ_1, u, φ_2 will sometimes be referred to as Eulerian coordinates.

Assume now, that $f(a)$ is continuous and of a compact support on G. Let $d\nu(a)$ be the element of a fixed Haar measure on G_0 and Z the center of G. Then, as it is known, if $d\mu(a)$ is the element of an appropriately normalized Haar measure on G, we have the following relation

$$\int_G f(a)\, d\mu(a) = \int_{G_0} \left(\sum_{z \in Z} f(az) \right) d\nu(a) .$$

Since for $d\nu(a)$, when expressed in Eulerian coordinates, we can choose $\frac{1}{(2\pi)^2} \operatorname{sh} u \, d\varphi_1 \, du \, d\varphi_2$ ([10] Lemma 2, Corollaire p. 60), the above relation shows, that the same expression, with an appropriate modification of the range of φ_1, defines a Haar measure an G too (for all this cf. also III.D.3).

2. Next we proceed to find certain expressions in Eulerian coordinates for the matrix elements, standing in the diagonal, when referred to the canonical basis $\{g_m\}$ determined in I.B., of the representations $C_q^{(\tau)}$ and $D_l^{\pm}$ (cf. (1.9)). For this, it clearly suffices to consider their restrictions to U.

I. *Case of* $C_q^{(\tau)}$. Let $T(a)$ be an irreducible representation of this type realized in L_τ^2 (cf. I.B.); in what follows we put $h_m^{(\sigma)}(u) = (T(g_u) g_m, g_m)$ ($m = \tau \pm j$, $j = 0, 1, 2, \ldots$; $0 \leqq \tau < 1$, $\sigma = \sqrt{q - \frac{1}{4}} > 0$). Let us write $o_\varphi g_u = s o_\varphi \overline{g_u}$, with $s \in S_0$. Identifying again $S_0 \subset G$ with its image in G_1, we write $\lambda(o_\varphi g_u)$ for the element, standing in the lower right corner, of the corresponding matrix. Then if $f(o) \in L_\tau^2$ putting $o_\psi = o_\varphi \overline{g_u}$, we have $(T(g_u) f)(o_\varphi) \equiv \equiv [\lambda(o_\varphi g_u)]^{-1 + 2\sigma i} f(o_\psi)$.

For our purposes it suffices to consider $0 \leqq \varphi < 2\pi$. Since the restriction of the canonical homomorphism of G onto G_1 to $o_\varphi g_u$ is injective, to determine λ and ψ as functions of φ and u we can compute in G_1. For reasons of continuity we are going to have $0 \leqq \psi < 4\pi$.

Consider now the element

$$a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

of G_1. If $a = s o_\psi$ ($s \in S_0$), a straightforward computation ([10] Lemma 1, p. 58) shows, that $\lambda^2 = \gamma^2 + \delta^2$, $\lambda e^{i\frac{\psi}{2}} = \delta + i\gamma$. Putting $a = g_u o_\varphi$, this gives $\lambda^2 = \operatorname{ch} u + \sin\varphi \operatorname{sh} u$ and $\lambda e^{i\frac{\psi}{2}} = \operatorname{ch}\frac{u}{2} e^{i\frac{\varphi}{2}} + i \operatorname{sh}\frac{u}{2} e^{-i\frac{\varphi}{2}}$.

Supposing $f(o_\varphi) \equiv e^{-im\varphi} \equiv g_m$, we obtain

$$(T(g_u) f)(o_\varphi) \equiv (\operatorname{ch} u + \sin\varphi \operatorname{sh} u)^{-\frac{1}{2} + i\sigma + m} \left(\operatorname{ch}\frac{u}{2} e^{i\frac{\varphi}{2}} + i \operatorname{sh}\frac{u}{2} e^{-i\frac{\varphi}{2}} \right)^{-2m}$$

the sense, in which the powers have to be taken, being evident. Hence, putting $\mu = \operatorname{ch} u$,

$$h_m^{(\sigma)}(n) = \left(\frac{1 + \mu}{2} \right)^m \frac{1}{2\pi} \int_0^{2\pi} (\operatorname{ch} u + \sin\varphi \operatorname{sh} u)^{-\frac{1}{2} + n + i\sigma} \left(1 + i \tanh\frac{u}{2} e^{-i\varphi} \right)^{-2m} d\varphi .$$

The substitution $\varphi \to \varphi + \frac{\pi}{2}$ gives

$$h_m^{(\sigma)}(n) =$$
(2.1)
$$= \left(\frac{1+\mu}{2}\right)^m \frac{1}{\pi} \int_0^\pi (\mathrm{ch}\,u + \cos\varphi\,\mathrm{sh}\,u)^{-\frac{1}{2}+m+i\sigma} 2\,\mathrm{Re}\left(1 + i\tanh\frac{u}{2}\,e^{-i\varphi}\right)^{-2m} d\varphi\,.$$

We introduce a new variable v in (2.1) by putting $e^v = \mathrm{ch}\,u + \cos\varphi\,\mathrm{sh}\,u$. Assume first $u > 0$, then we get $\frac{d\varphi}{dv} = -\frac{e^{\frac{v}{2}}}{Z}$, with $Z = \sqrt{2(\mathrm{ch}\,u - \mathrm{ch}\,v)}$ $(-u \leqq v \leqq u)$ and

$$h_m^{(\sigma)}(u) = \left(\frac{1+\mu}{2}\right)^m \frac{2^{2m}}{\pi} \int_0^u \frac{\cos\sigma v}{Z}\, 2\,\mathrm{Re}\left[2\,\mathrm{ch}\frac{v}{2} + iZ\right]^{-2m} dv\,. \tag{2.2}$$

We evidently have $h_m^{(\sigma)}(u) \equiv h_m^{(\sigma)}(-u)$, and have $h_m^{(\sigma)}(0) = 1$. Observe, that (2.2) is invariant under the substitution $m \to -m$.

In order to express $h_m^{(\sigma)}(u)$ through special functions, we consider the the generator function $G_m(\mu, t)$ of the Jacobi polynomials $\{P^{[0,2m]}(\mu)\}$ (cf. [9] (4.4) p. 69), defined by

$$G_m(\mu, t) = \frac{2^{2m}}{R(1+t+R)^{2m}} \qquad (\mu = \mathrm{ch}\,u)$$

where $R = \sqrt{1 - 2\mu t + t^2}$. We consider $G_m(\mu, t)$ as a univalent analytic function on the complex plane cut along $[e^{-u}, e^u]$ $(u > 0)$, by defining $R = -(\sqrt{t - e^u}) \times (\sqrt{t - e^{-u}})$. Here $\sqrt{z}$ is the branch, positive for $z > 0$, and on the plane cut along $[-\infty, 0]$. Taking for $\log z$ the branch which is real for $z > 0$ and univalent in the same domain, for any complex $z \neq 0$ and a, z^a will stand for $\exp(a \log z)$. Bearing this in mind, one sees at once, that the factor of $\left(\frac{1+\mu}{2}\right)^m$ in (2.2) can be written as

$$\frac{1}{2\pi i} \int_C G_m(\mu, t) \frac{dt}{t^{z+1}}\,, \quad z = -\frac{1}{2} + i\sigma - m \tag{2.3}$$

where C is the segment $[e^{-u}, e^u]$ run over twice clockwise. Observe, incidentally, that (2.3), as function of z for a fixed $u > 0$, is integral.

With our definition of R, $1 + t + R$ maps the cut plane onto a bounded domain, the closure of which does not contain 0. Hence for $\mathrm{Re}\,z > -1$, by deforming the path of integration, (2.3) can be shown to be the same as

$$\int_{C'} G_m(\mu, t) \frac{dt}{t^{z+1}} \tag{2.4}$$

where C' is a curve, oriented counterclockwise, surrounding the interval $[-\infty, 0]$ sufficiently closely. Hence, following the standard notation for Jacobi polynomials, we denote the function, defined by (2.3) for any z, with $P_z^{[0,2m]}(\mu)$. Hence finally

$$h_m^{(\sigma)}(u) = \left(\frac{1+\mu}{2}\right)^m P^{[0,2m]}_{-\frac{1}{2}+i\sigma-m}(\mu) \qquad (\mu = \mathrm{ch}\,u)\,. \tag{2.5}$$

The factors of $\left(\frac{1+\mu}{2}\right)^m$ in (2.1) and (2.2) give the analogues of the integral representations of Laplace and Dirichlet-Mehler resp. of the Legendre polynomials for the factor of $\left(\frac{1+\mu}{2}\right)^m$ in (2.5).

Though not needed in the sequel, we remark, that starting with (2.4), a standard computation gives the following expression in terms of hypergeometric functions

$$h_m^{(\sigma)}(u) = \left(\frac{1+\mu}{2}\right)^m \left(F\left(z, 2m+z+1, 1; \frac{1-\mu}{2}\right)\right), \tag{2.6}$$

with $z = -\frac{1}{2} + i\sigma - m$. (For this cf. below 3, too.)

II. *Case of* $D_l^{\pm}$ $\left(l > \frac{1}{2}\right)$. Here the procedure is very similar. Putting

$$h_m^{(l)}(u) = (T(g_u)g_m, g_m), \qquad m = \pm(l+j)\ (j = 0, 1, 2, \ldots),$$

where we have $+$ or $-$ according to whether $T(a)$ is of type D_l^+ or D_l^-, we obtain successively ($u > 0$)

$$h_m^{(l)}(u) = \left(\frac{1+\mu}{2}\right)^m \frac{1}{2\pi} \int_0^{\pi} (\operatorname{ch} u + \cos\varphi \operatorname{sh} u)^{m-l}\, 2\operatorname{Re}\left(1 + \tanh\frac{u}{2} e^{-i\varphi}\right)^{-2m} d\varphi \tag{2.7}$$

and

$$h_m^{(l)}(u) = \left(\frac{1+\mu}{2}\right)^m \frac{2^{2m}}{2\pi} \int_{-u}^{u} \frac{e^{-\left(l-\frac{1}{2}\right)v}}{Z} \left[2\operatorname{Re}\left(2\operatorname{ch}\frac{\sigma}{2} + iZ\right)^{-2m}\right] dv. \tag{2.8}$$

We have again $h_m^{(\sigma)}(u) \equiv h_m^{(l)}(-u)$, $h_m^l(0) = 1$, and, as (2.8) shows, $h_{-m}^{(l)}(u) \equiv \equiv h_m^{(l)}(u)$. Hence in what follows we may assume $m > 0$.

To obtain representation in terms of special functions, we observe first, that the factor of $\left(\frac{1+\mu}{2}\right)^m$ in (2.8) can be written as

$$\frac{2^{2m}}{2\pi i} \int_C \frac{t^{l+m-1}}{R(1+t-R)^{2m}}\, dt$$

where C is as in (2.3). But since

$$\frac{(1+\mu)^{2m}}{R(1+t-R)^{2m}} \equiv \frac{(1+t+R)^{2m}}{2^{2m} R t^{2m}} \equiv \frac{G^{[0,-2m]}(\mu, t)}{t^{2m}}$$

this is the same as

$$\left(\frac{1+\mu}{2}\right)^{-2m} \frac{1}{2\pi i} \int_C G^{[0,-2m]}(\mu, t) \frac{dt}{t^{m-l+1}}$$

Taking into account, that $m - l$ is an integer $\geqq 0$, one immediately sees, that C can be replaced by a closed curve C' surrounding the point 0 sufficiently closely and oriented counter-clockwise. Hence finally ($|m| - l =$ integer $\geqq 0$)

$$h_m^{(l)}(u) = \left(\frac{1+\mu}{2}\right)^{-|m|} P_{|m|-l}^{[0,-2|m|]}(\mu)\ (\mu = \operatorname{ch} u). \tag{2.9}$$

Here the factor of $\left(\frac{1+\mu}{2}\right)^{-|m|}$ is a Jacobi polynomial. One can obtain expressions in terms of hypergeometric functions by computing the matrix coefficients as eigenfunctions of the Laplacian of G, in Euler coordinates formally identical with that of G_0 (cf. [1] § 10, p. 624). Since not needed in the sequel, we omit the details.

B. The main Lemma

In order to arrive at the Plancherel formula first we prove a completeness relation involving the matrix coefficients, determined in A.

Lemma. Assume, that $f(u)$ is C^∞ on $[0, +\infty]$ and vanishes outside a compact set. For a fixed $m > 0$ put

$$f(\sigma, m) = \int_0^\infty f(u)\, h_m^{(\sigma)}(u) \operatorname{sh} u\, du \qquad (\sigma > 0)$$

(2.10) *and*

$$f_j(l) = \int_0^\infty f(u)\, h_{l+j}^{(l)}(u) \operatorname{sh} u\, du$$

($j = 0, 1, 2, \ldots$). *Then we have*

$$f(0) = \int_0^\infty \sigma \operatorname{Re}[\tanh \pi(\sigma + im)]\, f(\sigma, m)\, d\sigma + \tag{2.11}$$
$$+ \sum_{0 \leqq j < m - \frac{1}{2}} \left(m - j - \frac{1}{2}\right) f_j(m - j)\,.$$

(For $0 < m \leqq \frac{1}{2}$ the second summand has to be replaced by 0.)

We shall give the proof in several steps.

1. Keeping $u > 0$ fixed, and assuming first $m - 1 < \operatorname{Re} z < m$, we denote by $I(I')$ the integral of $\frac{1}{2\pi i} G_m(t, \mu)\, t^{m-z-1}$ along the real line form $-\infty$ to $+\infty$, taking the upper (lower resp.) halves of the cuts along $[-\infty, 0]$ and $[e^{-u}, e^u]$, introduced in A.2. These integrals certainly exist, since the integrand is $O(r^{m-\operatorname{Re} z-1})$ for $|t| = r$, r large. For the same reason, since the integrand is regular in the upper (lower) half-plane, we have $I = I' = 0$. We put I_j (I'_j; $j = 1, 2, 3, 4$) for the parts of $I(I')$ corresponding to $[-\infty, 0]$, $[0, e^{-u}]$, $[e^{-u}, e^u]$ and $[e^u, +\infty]$ resp., and consider the expressions $I_j + I'_j$ separately.

a. Taking into account the definition of the function $P_z^{[0,2m]}$ given above, we have

$$I_1 + I'_1 = \frac{e^{2\pi i(z-m)} + 1}{e^{2\pi i(z-m)} - 1} P_{z-m}^{[0,2m]}(\mu)\,.$$

We observe, that this expression possesses an analytic continuation, the only singularities of which are the points $z = m \pm j$, $j = 0, 1, 2, \ldots$.

b. We have

$$I_2 = I'_2 = \frac{2^{2m}}{2\pi i} \int_0^{e^{-u}} \frac{1}{R(1 + t + R)^{2m}} \frac{dt}{t^{z-m+1}}\,.$$

Putting $t = e^{-v}$, and $V = \sqrt{2(\operatorname{ch} v - \operatorname{ch} u)}$ $(u < v < +\infty)$, we get

$$I_2 + I_2' = \frac{2^{2m}}{\pi i} \int\limits_u^\infty \frac{e^{\left(z+\frac{1}{2}\right)v}}{V\left(2\operatorname{ch}\frac{v}{2} + V\right)^{2m}}\, dv\,.$$

This is regular in the half-plane $\operatorname{Re} z < m$.

c. We put $-iR_+ (iR_+)$ for the restriction of R to the upper (lower resp.) half of the cut along $[e^{-u}, e^u]$ $(R_+ \geqq 0)$. Then

$$I_3 + I_3' = \frac{2^{2m}}{\pi i} \int\limits_{e^{-u}}^{e^u} \frac{1}{R_+} \operatorname{Im}((1 + t - iR_+)^{-2m})\, \frac{dt}{t^{z-m+1}}\,.$$

This part, as function of z is integral, and for later use we observe, that for $z = -\frac{1}{2} + i\sigma$ it is purely imaginary. Indeed, the substitution $t = e^v$ $(-u < < v < u)$ gives

$$I_3 + I_3' = \frac{2\cdot 2^{2m}}{i\pi} \int\limits_0^u \left[\operatorname{Im}\left(2\operatorname{ch}\frac{v}{2} + iZ\right)^{-2m}\right] \frac{\cos\sigma v}{Z}\, dv$$

$(Z = \sqrt{2(\operatorname{ch} u - \operatorname{ch} v)}\,)$.

d. Here $I_4 = I_4'$ and

$$I_4 + I_4' = \frac{2^{2m}}{\pi i} \int\limits_{e^u}^\infty \frac{1}{R(1 + t + R)^{2m}}\, \frac{dt}{t^{z-m+1}}$$

this being analytic for $\operatorname{Re} z > m - 1$.

Summing up, writing $F(z, \mu) = -(I_4 + I_4')$ we have the following relation

$$\frac{2^{2m}}{\pi i} \int\limits_u^{+\infty} \frac{e^{\left(z+\frac{1}{2}\right)v}}{V\left(2\operatorname{ch}\frac{u}{2} + V\right)^{2m}}\, dv = -\frac{e^{2\pi i(z-m)} + 1}{e^{2\pi i(z-m)} - 1}\, P_{z-m}^{[0,2m]}(\mu) + \tag{2.12}$$
$$+ F(z, \mu) - (I_3 + I_3') \quad (\mu = \operatorname{ch} u)\,.$$

Since in this relation, derived under the assumption $m - 1 < \operatorname{Re} z < m$, the left hand side, and the first and third summands of the right-hand side are regular analytic, except for simple poles, for $\operatorname{Re} z < m$, $F(z, \mu)$ possesses an analytic continuation; it is going to be computed explicitely below in 2.c.

2. Assume now, that $f(u) \equiv 0$ for $u > M > 0$, and put $g(u) = \left(\frac{1+\mu}{2}\right)^m f(u)$ $(g(u) \in C^\infty)$; we have $g(0) = f(0)$. Now we show, that multiplying both sides of (2.12) first by $g(u)\operatorname{sh} u$ and integrating from 0 to M, then putting $z = -\frac{1}{2} + i\sigma$, multiplying by σ, integrating from $-S$ to S $(S > 0)$ according to σ, and taking finally the real part of the lines for $S \to +\infty$, we obtain (2.11). In what follows we prove this by computing, proceeding from the left to the right, the contribution of the different terms of (2.12).

a. Taking the real part of the expression, obtained by multiplying with $g(u)\operatorname{sh} u$, integrating and putting $z = -\frac{1}{2} + i\sigma$, we get on the left hand side

$$\frac{1}{\pi}\int_0^\infty \sin\sigma v\, H(v)\, dv \tag{2.13}$$

where we put

$$H(v) = 2^{2m}\int_0^v \frac{g(u)\operatorname{sh} u\, du}{V\left(2\operatorname{ch}\frac{v}{2} + V\right)^{2m}}.$$

To justify the interchange of the order of integration according to u and v resp., we observe, that for large v, $H(v) = O\left(e^{-\left(m+\frac{1}{2}\right)v}\right)$; for small $v > 0$ however

$$H(v) < \varkappa \int_0^v \frac{\operatorname{sh} u\, du}{\sqrt{2(\operatorname{ch} v - \operatorname{ch} u)}} = \varkappa\sqrt{2(\operatorname{ch} v - 1)}$$

with $\varkappa$ not depending on v, proving $H(v) \in L^1(0, +\infty)$, and this is true even if we replace g by $|g|$.

Next we show the same for $H'(v)$. To do this, it evidently suffices to consider $v < M$. We have for $v > u > 0$

$$\frac{2^{2m}}{\left(2\operatorname{ch}\frac{v}{2} + V\right)^{2m}} = \left(\operatorname{ch}\frac{v}{2}\right)^{-2m}[1 + V\, H(u, v)]$$

with $H(u, v) \in C^\infty$. Hence

$$H(v) = \left(\operatorname{ch}\frac{v}{2}\right)^{-2m}\left[\int_0^v \frac{g(u)\operatorname{sh} u\, du}{V} + \int_0^v g(u)\, H(u, v)\operatorname{sh} u\, du\right]$$

showing, that it suffices to discuss

$$G(v) = \int_0^v \frac{g(u)}{V}\operatorname{sh} u\, du\,.$$

Partial integration gives

$$G(v) = \sqrt{2(\operatorname{ch} v - 1)}\, f(0) + \int_0^v g'(u)\, V\, du\,.$$

Writing $V = \sqrt{v^2 - u^2}\,(1 + f(u,v))$ $(f(u,v) \in C^\infty)$ and putting $u = tv$ $(0 \leq t \leq 1)$, the second summand becomes

$$v^2 \int_0^1 g'(tv)t\sqrt{1 - t^2}\,(1 + f(tv, v))\, dt\,,$$

which, along with what preceeds, clearly proves $H'(v) \in L^1(0, +\infty)$. Observe, in particular, that our considerations show, that for small $v > 0$

$$|H'(v) - f(0)| < \varkappa v \tag{2.14}$$

where $\varkappa$ does not depend on v.

Through partial integration and multiplication by σ, (2.13) gives

$$\frac{1}{\pi}\int_0^\infty \cos\sigma v\, H'(v)\, dv\,.$$

Hence finally, taking into account (2.14) the contribution of the left hand side in (2.12) will be

$$\lim_{S\to+\infty}\frac{1}{\pi}\int_{-S}^{S}\left(\int_0^\infty \cos\sigma v\, H'(v)\, dv\right) d\sigma = \lim_{S\to+\infty}\frac{2}{\pi}\int_0^S \frac{\sin S v}{v} H'(v)\, dv = f(0).$$

b. For $z = -\frac{1}{2} + i\sigma$ $(\sigma \neq 0)$ we have

$$\frac{e^{2\pi i(z-m}+1}{1-e^{2\pi i(z-u)}} = \tanh\pi(\sigma + im)\,.$$

Hence, by virtue of (2.5) and $h_m^{(\sigma)}(u) \equiv h_m^{(-\sigma)}(u)$ (cf. 2.2), ths first term on the right hand side in (2.12) yields

$$\lim_{S\to+\infty} 2\int_0^S \sigma\,\mathrm{Re}\,[\tanh\pi(\sigma+im)]\, f(\sigma, m)\, d\sigma\,.$$

We shall prove the absolute convergence in C.4, below.

c. First we assume $m \geqq \frac{1}{2}$.

Putting now R for $(1-2t\mu+t^2)^{\frac{1}{2}} > 0$ $(t > e^u$, or $t < e^{-u})$, we have for $m-1 < \mathrm{Re}(z)$ (cf. 1.d. above)

$$F(z,u) = \frac{2^{2m}}{\pi i}\int_{e^u}^\infty \frac{1}{R(1+t-R)^{2m}}\,\frac{dt}{t^{z-m+1}}\,. \tag{2.15}$$

We have

$$\frac{2^{2m}}{R(1+t-R)^{2m}} \equiv \left(\frac{1+\mu}{2}\right)^{-2m}\frac{(1+t+R)^{2m}}{2^{2m}t^{2m}R}$$
$$\equiv \frac{1+\mu}{2}\, G_{-m}(\mu,t)\, t^{-2m} \equiv \left(\frac{1+\mu}{2}\right)^{-2m}\frac{1}{t}\, G_{-m}\left(\mu,\frac{1}{t}\right).$$

From now on assume that $0 < u < M$. In order to obtain an analytic continuation of (2.15) left of the line $\mathrm{Re}(z) = m-1$, observe first, that for any integer $N \geqq 0$ we have in the domain $0 < x < e^{-u}$, $0 < u < M$:

$$G_{-m}(\mu,x) = \sum_{j=0}^{N}{}' P_j^{[0,-2m]}(\mu)\; x^j + x^{N+1}E(\mu,x)$$

where $E(\mu, x)$ is C^∞ and satisfies

$$|E(\mu,x)| \leqq \frac{C}{\sqrt{1-2\mu x+x^2}}\;; \tag{2.16}$$

C is a constant not depending on u and x. We put $N = \left[m-\frac{1}{2}\right] \gtreqless 0$; using the previous identity and (2.9), we get

$$F(z,\mu) = \frac{1}{\pi i}\left(\frac{1+\mu}{2}\right)^{-m}\left[\sum_{j=o}^{N} h_m^{(m-j)}(u)\,\frac{e^{-u(z-m+j+1)}}{z-m+j+1} + \right.$$

$$\left. + \left(\frac{1+\mu}{2}\right)\int_{e^u}^{\infty} E\left(\mu,\frac{1}{t}\right)\frac{dt}{t^{z-m+N+3}}\right]$$

whence, multiplying both sides with $g(u)\,\mathrm{sh}\,u$ and integrating according to u, we obtain

$$F(z) = \int_0^\infty F(z,\mu)\,g(u)\,\mathrm{sh}\,u\,du = F_1(z) + F_2(z)$$

where

$$F_1(z) = \frac{1}{\pi i}\left(\sum_{j=o}^{N}\frac{1}{z-m+j+1}\int_0^\infty h_m^{(m-j)}(u)\,e^{-u(z-m+j+1)}f(u)\,\mathrm{sh}\,u\,du\right)$$

and

$$F_2(z) = \frac{1}{\pi i}\int_0^\infty f_1(u)\left(\int_{e^u}^\infty E\left(\mu,\frac{1}{t}\right)\frac{dt}{t^{z-m-N+3}}\right)\mathrm{sh}\,u\,du\;\left(f_1(u) \equiv f(u)\left(\frac{1+\mu}{2}\right)^{-m}\right).$$

Up to now we have assumed $m-1<\mathrm{Re}\,z$. On the other hand, $R_1(z)$ is clearly regular, up to a finite number of simple poles, on the whole complex plane. What concerns $F_2(z)$ we show, that it is regular in a half plane $\mathrm{Re}\,z>\eta$, $\eta<-\frac{1}{2}$. To do this, observe, that by virtue of our choice of N we have $\mathrm{Re}\,z-m+N+2>\varepsilon>0$, provided $\mathrm{Re}\,z>-\frac{1}{2}-\varepsilon$; here ε depends on m only. Assuming z so chosen, interchanging the order of the two integration in the above expression of $F_2(z)$, we obtain:

$$F_2(z) = \frac{1}{\pi i}\int_0^\infty e^{-(z-m+N+2)v}H(v)\,dv$$

where

$$H(v) = \int_0^v E(\mu, e^{-v})\,f_1(u)\,\mathrm{sh}\,u\,du\,. \tag{2.17}$$

In order to justify this, we write $H_1(v)$ for the function obtained by replacing the integrand in the expression of $H(v)$ by its absolute value. It clearly suffices to show the convergence of

$$\int_0^\infty e^{-\varepsilon v}H_1(v)\,dv$$

which, however, is evident since, by virtue of (2.16), $H_1(v)$ is bounded. This implies the analyticity of $F_2(z)$ for $\mathrm{Re}\,z>-\frac{1}{2}-\varepsilon$ too, as claimed above. — Summing up all, $F(z)=F_1(z)+F_2(z)$ is analytic, except for a finite number of poles, on the same half-plane.

Now we form

$$\int\limits_{-S}^{S} \sigma F\left(-\frac{1}{2}+i\sigma\right) d\sigma = -\int\limits_{\Gamma}\left(z+\frac{1}{2}\right) F(z)\, dz \tag{2.18}$$

where Γ is the segment $z=-\frac{1}{2}+i\sigma$ $(-S \leqq \sigma \leqq S)$. Keeping $T > m-1$ fixed, we write Γ_S, Γ_{-S} and I_S for the straight segments $\left[-\frac{1}{2}+iS,\ T\pm iS\right]$ and $[T-iS,\ T+iS]$ resp. Using Cauchy's theorem we see, that (2.18) equals the sum of the integrals of $\left(z+\frac{1}{2}\right)F(z)$ along these segments (taken with the appropriate orientation) and $2\pi i$ times the sum of the residues of poles inside the rectangle defined by them. By virtue of our previous discussion, however, this is twice the second summand on the right hand side of (2.11) (cf. the expression of $F_1(z)$ above). Next we show, that the contribution of the integrals along $\Gamma_{\pm S}$ to the real part of (2.18) for $S \to +\infty$ is 0. For this it suffices to prove, that putting

$$f_{\pm S} = \int\limits_{\Gamma_{\pm S}} \left(z+\frac{1}{2}\right) F(z)\, dz$$

we have $\lim\limits_{S\to+\infty} |f_{\pm S}| = 0$, which, however, can be deduced from $\sup\limits_{z\in\Gamma_{\pm S}} |z|\,|F_j(z)| \to 0$ for $S \to +\infty$ $(j = 1, 2)$. For $j = 1$ this is a simple consequence of the Riemann-Lebesgue Lemma. To apply the same in the case $j = 2$, we observe first, that by virtue of (2.16), a reasoning used in a. above shows, that in (2.17) $\lim\limits_{v\to 0} H(v) = 0$. Hence partial integration gives

$$F_2(z) = \frac{1}{\pi i}\,\frac{1}{z-m+N+2}\int\limits_{0}^{\infty} e^{-(z-m+N+2)v}\, H'(v)\, dv$$

and it suffices to prove, that the integral tends to 0, which is certainly true, if $H'(v)$ is bounded. For large v this is evident from the definition of $E(\mu, x)$ (it is bounded along with its derivative if $0 < x \leqq C < e^{-M}$). For $0 < v \leqq M$, however, it easily follows through a discussion analogous to that of a.

Finally, we show, that the contribution of I_S for $S \to +\infty$ is $-f(0)$. Since the necessary reasonings are almost identical with those of a, we confine ourselves to a few indications. First of all, assuming $z = T + i\sigma$ $(T > m-1)$ we have

$$F(z) = \frac{1}{\pi i}\int\limits_{0}^{\infty} e^{-i\sigma v}\, H(v)\, dv$$

where now

$$H(v) = e^{-\left(T+\frac{1}{2}\right)v}\cdot 2^{2m}\int\limits_{0}^{v} \frac{f(u)\,\mathrm{sh}\, u\, du}{V\left(2\,\mathrm{ch}\frac{v}{2} - V\right)^{2m}}.$$

Observe, that for large v, $H(v) = O(e^{(m-1-T)v})$. Hence

$$\operatorname{Re}\int\limits_{I_S}\left(z+\frac{1}{2}\right)F(z)\,dz =$$

$$= \frac{T+\frac{1}{2}}{\pi}\int\limits_{-S}^{S}\int\limits_{0}^{\infty}\cos\sigma v\, H(v)\,dv\,d\sigma + \frac{1}{\pi}\int\limits_{-S}^{S}\sigma\left(\int\limits_{0}^{\infty}\sin\sigma v\; H(v)\; dv\right)d\sigma\,.$$

If $S\to+\infty$, the first expression gives zero, since $\lim\limits_{v\to+0} H(v) = 0$. Concerning the second, one shows as in a, that its limes is $f(0)$. To get the final result, one has to reverse the orientation of I_S.

If $0 < m \leqq \frac{1}{2}$, we are going to have no residues.

d. By virtue of a remark made in l.c. above, for $z = -\frac{1}{2} + i\sigma$ the last summand in (2.12) is purely imaginary, hence its contribution is 0.

Finally, putting $-f(0)$ on the left hand side and dividing by 2 we obtain 2.11.

C. Now we are ready to prove the Plancherel formula as announced in the Introduction. We fix a real C^∞ function $f(a)$ having a compact support on G_1. By I.C. we know, that if $T(a)$ is an irreducible unitary representation of G, the operator $T_f = \int\limits_G T(a)\,f(a)\,d\mu(a)$ is of trace class. Its trace will be denoted by $T_\sigma^{(\tau)}(f)$ and $T_l(f)$, if $T(a)$ is of type $C_q^{(\tau)}\left(0 \leqq \tau < 1, q > \frac{1}{4}, \sigma = \sqrt{q-\frac{1}{4}} > 0\right)$ or a direct sum of a representation of type D_l^+ with a representation of type D_l^- $\left(l > \frac{1}{2}\right)$ resp. Then, we recall, the formula to be proved is as follows

$$\text{(2.19)} \qquad f(e) = \int\limits_0^\infty\int\limits_0^1 \sigma \operatorname{Re}\tanh\pi(\sigma+i\tau)\, T_\sigma^{(\tau)}(f)\,d\sigma\,d\tau + \int\limits_{\frac{1}{2}}^{\infty}\left(l-\frac{1}{2}\right)T_l(f)\,dl$$

(e is the unit element of the group G).

1. First we observe, that it suffices to prove (2.19) for functions $\in C_c^\infty$ satisfying $f(oa) \equiv f(ao)$ $(a \in G,\ o \in O)$; in what follows we denote this family by $F(O)$. In order to see this we put $(Pf)(a) = \frac{1}{2\pi}\int\limits_0^{2\pi} f(o_\varphi a o_{-\varphi})\,d\varphi$; $f \in C_c^\infty$ implies the same for Pf, and $f(e) = (Pf)(e)$. Furthermore, if $T(a)$ is any irreducible representation, we have $\operatorname{Tr}(T_{Pf}) = \operatorname{Tr}(T_f)$. To prove this, we write $f_\varphi(a) = f(o_\varphi o o_{-\varphi})$; then $T_{f_\varphi} = T(o_{-\varphi})\,T_f\,T(o_\varphi)$, implying $\operatorname{Tr}(T_{f_\varphi}) = \operatorname{Tr}(T_f)$. From this, using the continuity of the dependence of $\operatorname{Tr}(T_f)$ on f, it follows at once, that

$$\operatorname{Tr}(T_f) = \frac{1}{2\pi}\int\limits_0^{2\pi}\operatorname{Tr}(T_{f_\varphi})\,d\varphi = \frac{1}{2\pi}\operatorname{Tr}\left(\int\limits_0^{2\pi} T_{f_\varphi}\,d\varphi\right) = \operatorname{Tr}(T_{Pf})\,.$$

Summing up all, both sides of (2.19) remain invariant if we replace f by Pf, which proves our assertion.

Suppose, that $f \in F(O)$; using the fact, that the restriction of the adjoint representation of G to $U = \{g_u\}$ is an isomorphism with its image (cf. A.1), one easily shows, that the function $f(u, \varphi) \equiv f(g_u o_\varphi)$ is C_c^∞ in (u, φ), and we have $f(o_{\varphi_1} g_u o_{\varphi_2}) \equiv f(u, \varphi_1 + \varphi_2)$.

2. Let $T(a)$ be a unitary representation and k a vector such that $(o_\varphi)k = e^{-m\varphi}k$. We are going to compute $(T_f k, k) = \int\limits_G (T(a)k, k)\, f(a)\, d\mu(a)$ in Eulerian coordinates, using $\frac{1}{(2\pi)^2} \operatorname{sh} u\, d\varphi_1\, du\, d\varphi_2$ for the element $d\mu(a)$ of the Haar measure (cf. A.1). We have, putting $h(u) = (T(g_u)k, k)$:

$$(T_f k, k) = \frac{1}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \int\limits_0^{\infty} \int\limits_0^{2\pi} h(u) e^{-im(\varphi_1 + \varphi_2)} f(u_1 \varphi_1 + \varphi_2) \operatorname{sh} u\, d\varphi_1\, du\, d\varphi_2$$

whence, if

$$F(u, m) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} f(u, \varphi) e^{-im\varphi}\, d\varphi$$

we obtain

$$(T_f k, k) = \int\limits_0^\infty h(u)\, F(u, m) \operatorname{sh} u\, du\ .$$

3. For $f, g \in C_c^\infty$, $(f \times g)(a)$ will denote the convolution of these two functions, defined by $\int\limits_g f(ab^{-1})\, g(b)\, d\mu(b)$. Observe, that $f, g \in F(O)$ implies $f \times g \in F(O)$. We write $f^\sim(a) \equiv f(a^{-1})$; again $f \in F(O)$ implies the same for $f^\sim(a)$.

In what follows we prove (2.19) assuming f of the form $g^\sim \times g$ $(g \in F(O))$; it is known and easily verified, that the matrix coefficients of such a function are positive. The general case will be deduced from this special one.

Observe first, that, with the notations of 2 above, for any fixed m, $F(u, m)$ vanishes outside a fixed interval, independent of m, in u. We put (cf. (2.10))

$$f(\sigma, m) = \int\limits_0^\infty F(u, m)\, h_m^{(\sigma)}(u) \operatorname{sh} u\, du\ ,$$

$$f_j^{(+)}(e) = \int\limits_0^\infty F(u, l + j)\, h_{l+j}^{(l)}(u) \operatorname{sh} u\, du \tag{2.20}$$

and

$$f_j^{(-)}(e) = \int\limits_0^\infty F(u, -(l + j))\, h_{l+j}^{(l)}(u) \operatorname{sh} u\, du$$

$$(l > 0, j = 0, 1, 2, \ldots, -\infty < m < +\infty).$$

By virtue of the computations of 2, these are just integrals of products of $f(a)$ with matrix coefficients of irreducible representations of type $C_q^{(\tau)}\left(\tau = m - [m],\ q = \frac{1}{4} + \sigma^2\right)$, D_l^+ and D_l^- resp. (We recall, that $h_m^{(l)}(u) \equiv$

$\equiv h^{(l)}_{-m}(u)$, cf. (2.8).) In particular, we have

$$T^{(\tau)}_{\sigma}(f) = \sum_{j=-\infty}^{\infty} f(\sigma, \tau + j)$$

(2.21) and

$$T_l(f) = \sum_{j=0}^{\infty} (f^{(+)}_j(l) + f^{(-)}_j(l))\,.$$

For any real m, we replace in (2.11) m by $|m|$, and $f(u)$ by $F(u, m)$. Taking into account (2.20), this gives

$$\begin{aligned} F(0, m) &= \int_0^{\infty} \sigma \operatorname{Re}[\tanh \tau(\sigma + im)]\, f(\sigma, m)\, d\sigma + \\ &\quad + \sum_{0 \leq j < |m| - \frac{1}{2}} \left(|m| - j - \frac{1}{2}\right) f^{(\pm)}_j(|m| - j)\,. \end{aligned} \tag{2.22}$$

In the last expression we have to take $+$ or $-$, according to whether $m > 0$ or $m < 0$. Since

$$f(e) = f(0, 0) = \int_{-\infty}^{\infty} F(0, m)\, dm$$

using (2.21) we obtain from (2.22)

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(\int_0^{\infty} \sigma \operatorname{Re}[\tanh \pi(\sigma + im)]\, f(\sigma, m)\, d\sigma\right) dm \\ &= \int_0^1 \left[\int_0^{\infty} \sigma \operatorname{Re}[\tanh \pi(\sigma + i\tau)] \left(\sum_{j=-\infty}^{\infty} f(\sigma, \tau + j)\right) d\sigma\right] d\tau \\ &= \int_0^1 \int_0^{\infty} \sigma \operatorname{Re}[\tanh \pi(\sigma + i\tau)]\, T^{(\tau)}_{\sigma}(f)\, d\sigma\, d\tau \end{aligned}$$

and

$$\begin{aligned} &\int_{-\infty}^{0} \left(\sum_{0 \leq j < |m| - \frac{1}{2}} \left(|m| - j - \frac{1}{2}\right) f^{(-)}_j(|m| - j)\right) dm + \\ &+ \int_0^{\infty} \left(\sum_{0 \leq j < m - \frac{1}{2}} \left(m - j - \frac{1}{2}\right) f^{(+)}_j(m - j)\right) dm \\ &= \int_{\frac{1}{2}}^{\infty} \left(l - \frac{1}{2}\right) \left(\sum_{j=0}^{\infty} (f^{(+)}_j(l) + f^{(-)}_j(l))\right) dl \\ &= \int_{\frac{1}{2}}^{\infty} \left(l - \frac{1}{2}\right) T_l(f)\, dl\,. \end{aligned}$$

Observe, that, by virtue of our assumption on f, (cf. the remark at the begin of 3. above) each integrand is positive, therefore all operations (interchange of order of summation and integration etc.) used when deriving these relations, are permissible.

Hence, summing up all, integrating both sides of (2.22) according to m between $-\infty$ and $+\infty$ we obtain (2.19).

4. Now we prove (2.19) for any $f \in C_c^\infty$. By virtue of the linearity of both sides of (2.19) in f, and the identity $f \times g^\sim = \frac{1}{2}[(f+g) \times (f+g)^\sim + (f-g) \times \times (f-g)^\sim]$ it is certainly valid for any function of the form $f \times g^\sim (f, g \in F(O))$. Replace now g by an approximate identity $\{g_n\}$, such that $g_n \in F(O)$, $g_n^\sim \equiv g_n$. Then the sequence $\{f_n\} = \{f \times g_n\}$ tends uniformly, along with all of its derivatives, to f, and there exists a fixed compact set of G containing the carrier of each f_n. Then by I.C. we know, that if $\{T(a)\}$ is any irreducible representation, we have $\lim\limits_{n\to\infty} \mathrm{Tr}(T_{f_n}) = \mathrm{Tr}(T_f)$. Hence it suffices to show, that when replacing f by $\{f_n\}$ in (2.19), the limit transition under the integral signs, on the right hand side, is permissible. We shall deal with these two terms separately.

a. We observe, that, as the reasonings of 3 show, the first summand on the right hand side of (2.19) can be written as

$$\int_0^\infty \int_{-\infty}^\infty \sigma \,\mathrm{Re}[\tanh \pi(\sigma + im)]\, f(\sigma, m)\, dm\, d\sigma \tag{2.23}$$

(cf. 2.20). Now we prove the following statement. *We define* $G(\lambda, \mu, \varphi) \equiv f(so)$ $(s \in S_0,\ o \in O$; cf. I.B.) *and put*

$$H(t, \varphi) = \frac{1}{2} \int_{-\infty}^{\infty} G\left(e^{-\frac{t}{2}}, \mu, \varphi\right) d\mu$$

then

$$f(\sigma, m) = c \int_{-\infty}^{\infty} \int_{\infty} H(t, \varphi) e^{-i(\sigma t + m\varphi)}\, dt\, d\varphi \tag{2.24}$$

where c depends on the normalization of the Haar measure only.

Before giving the simple proof, observe, that $H(t, \varphi)$ is obviously C^∞ and has a compact support. Denoting by $H_n(t, \varphi)$ the functions corresponding to the members of the sequence $\{f_n\}$ considered above, their support is contained in a fixed compact set of the (t, φ) space, and they converge uniformly, along with their derivatives, to $H(t, \varphi)$. From this elementary considerations show the permissibility of the limit transition in (2.23).

Observe, too, that (2.24) proves the absolute convergence of the integral in B.2.b. above. To prove (2.24) assume, that $\{T(a)\}$ is of type $C_q^{(\tau)}$ $\left(\tau = m - [m],\ q = \frac{1}{4} + \sigma^2\right)$, and $g \equiv e^{-im\varphi} \in L_\tau^2$. Then, using the notations of A.2., we have

$$\begin{aligned}(T_f g)(o) &= \int_G f(a)\,[\lambda(oa)]^{-1+2\sigma i} g(o\bar{a})\, d\mu(a)\\ &= \int_G f(o^{-1}a)\,[\lambda(a)]^{-1+2\sigma i} g(\overline{o o^{-1} a})\, d\mu(a)\\ &= \int_{S_0 \times O} f(o^{-1} s o')\,\lambda^{-1+2\sigma i} g(o')\, d\mu_l(s)\, d\mu(o)\end{aligned}$$

(using $d_{\mu_l}(a) = d\mu_l(s)\, d\mu(a)$, cf. I.C). It is easy to see, that $d\lambda\, d\mu$ is the element of the left invariant measure on S_0 (cf. also III.A.1.δ); hence putting

$d\mu(o) = \frac{d\varphi}{2\pi}$, we have $d\mu(a) = \frac{c}{2\pi} d\lambda\, d\mu\, d\varphi$. Using the function $G(\lambda, \mu, \varphi)$ introduced above, we have

$$f(\sigma, m) = (T_f g, g) = c \int_{-\infty}^{\infty} \int_{0}^{\infty} \left[\int_{-\infty}^{\infty} G(\lambda, \mu, \varphi)\, d\mu \right] \lambda^{-1+2\sigma i} e^{-im\varphi}\, d\lambda\, d\varphi ,$$

whence, putting $\lambda = e^{-\frac{t}{2}}$, we obtain (2.24).

b. In III.A.2.b. we shall show the existence of a locally integrable function $C_l(a)$ on G such that $T_l(f) = \int_G f(a)\, C_l(a)\, d\mu(a)\ \left(l > \frac{1}{2}\right)$. From its exact form (cf. 3.16 and 3.17) we conclude, that $|T_l(f)| \leqq |l|\, M(f)$, where the constant $M(f)$ depends on the carrier and upper bound of the function $f \in C_c^\infty$ only. On the other hand, one shows through standard reasonings that denoting the left invariant infinitesimal transformation associated with l_j by L_j $(j = 0, 1, 2$; cf. (1.1) and [2] ch. IV, § II.), and putting $Q = L_0^2 - L_1^2 - L_2^2$, for any irreducible representation $\{T(a)\}$ and $f \in C_c^\infty$ we have $T_{Qf} = q\, T_f$, where q is the constant belonging to $\{T(a)\}$ (cf. (1.9)). Assuming this, we fix an integer $k > 2$; for any $l > 0$ we have $T_l(Q^k f) = [l(1-l)]^k\, T_l(f)$. Hence if $l > 1$, say, we get

$$|T_l(f)| < \frac{|l|}{[l(l-1)]^k} M(Q^k f) .$$

But for the sequence $\{f_n\}$ the constants $|M(Q^k f_n)|$ are uniformly bounded, and this clearly suffices to justify the limit transition under the integral in the second term of (2.19).

It remains to prove the relation $T_{Qf} = q\, T_f$ quoted above. We put, as in I.A., $T(\exp l_j t) = \exp(-iH_j t)$ $(j = 0, 1, 2)$, and observe, that it clearly suffices to show that, if h is any vector in the representation space, we have $iH_j T_f h = T_{L_j f} h$, since $H_1^2 + H_2^2 - H_0^2 = qI$. But, putting $g_j(t) = \exp l_j t$ again,

$$\begin{aligned} \lim_{t\to 0} \frac{T(g_j(t)) - I}{t} T_f h &= \lim_{t\to o} \int_G f(a) \frac{T(g_j(t)) - I}{t} T(a) h\, d\mu(a) \\ &= \int_G \lim_{t \to o} \frac{(f(g_j(-t)a) - f(a))}{t} T(a) h\, d\mu(a) \\ &= -\int_G (L_j f)(a)\, T(a) h\, d\mu(a) \quad \text{strongly} . \end{aligned}$$

Hence $T_f h$ is in the domain of H_j and $iH_j T_f h = T_{L_j f} h$ $(j = 0, 1, 2)$ as claimed.

Part III. The Plancherel formula as completeness relation of characters

The main idea of our second proof consists in expressing certain averages over the conjugacy classes of a fixed $f \in C_c^\infty$ through the traces of its integrals with respect to irreducible representations, making up the Plancherel formula, and then reconstructing the value assumed by our function at the unity through a certain differentiation process. The first part will be discussed below in A; in B we give a detailed description of the background of the method, and in C we deal with the second part; here we follow the ideas outlined in [4].

A. The main objective of this section is to express the traces in terms of averages over conjugacy classes; in B. it will be shown, how to invert these relations. During the course of the discussion we shall, incidentally, show, that for a fixed irreducible representation the trace, as distribution, is generated by a locally integrable function, which, in analogy with the compact groups, is called the character of the representation. By virtue of this fact, the main formula can be written in a form, closely resembling the Peter-Weyl formula, when expressed in terms of characters of irreducible representations. — We shall deal with the case of $C_q^{(\tau)}$ and $D_l^{\pm}$ separately in 1 and 2 resp. below. For some of the techniques to be employed cf. in particular [3] Ch. V and VI.

1.a. First we enumerate several subgroups, some of which had already been considered before (cf. in particular I.B.a), and fix invariant measures on them. As before, we denote the canonical homomorphism from G onto $G_1 (= SL(R,2))$ by N. In many cases the restriction of N to the subgroup to be defined turns out to be an isomorphism with its image in G_1, and we use the parameters of the latter.

α. We denote by N^+ (N^-) the connected subgroup of G lying over the subgroup of matrices of the form $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ $\left[\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix} \text{resp.}\right]$. The restriction of Φ induces and isomorphism, and we put $d\mu(n) = dx$.

β. On $O = \{o_\varphi; -\infty < \varphi < +\infty\}$ we use $d\mu(o) = \dfrac{d\varphi}{2\pi}$.

γ. We know, that the complete inverse image H under Φ of the subgroup of diagonal matrices $\begin{pmatrix}\lambda^{-1} & 0\\ 0 & \lambda\end{pmatrix}$ $(\lambda > 0)$ is the direct product of the center Z of G with the component of the identity $H_0 \subset H$, the latter being isomorphic under Φ with the diagonals. Putting again $\gamma = o_{2\pi} \in Z$, every element of H can be written uniquely as $\gamma^j h$ $(j = 0, \pm 1, \pm 2, \ldots; h \in H_0)$; sometimes we abbreviate this by writing h_j. We define $d\mu(h)$ an H by requiring $d\mu(h) = \dfrac{d\lambda}{\lambda}$ on H_0.

We recall, that on any Lie group G a left invariant measure can be defined by fixing a left invariant differential form of maximal rank. Assuming, that G is a linear group, the elements of the matrix $d_l a = a^{-1}\, da$ $(a \in G)$ form a system of linear differential forms, containing $\dim G$ linearly independent elements. Hence to obtain a left invariant form of maximal rank, it suffices to form the exterior product of members of an appropriately chosen subsystem. Similarly, one constructs right invariant forms by considering the elements of the matrix $d_r a = da \cdot a^{-1}$.

δ. We note the complete inverse image of the group $\left\{\begin{pmatrix}\lambda^{-1} & \mu\\ 0 & \lambda\end{pmatrix};\ \lambda > 0\right\}$ again by S. We recall, that it is the direct product of the center Z with the component of the identity S_0, the latter being isomorphic with its image under Φ. Similarly as in H we sometimes write s_j in place of $s\gamma^j$ $(s \in S_0)$. To define a left invariant measure $d\mu_l(s)$ it suffices to do it on S_0. Here λ and μ can be used as global coordinates, and, with the notations introduced above, $(d_l s)_{22} \wedge \wedge (d_l s)_{21} = d\lambda \wedge d\mu$ is a left invariant form of maximal dimension. Hence we can put $d\mu_l(s) = d\lambda\, d\mu$. The group S, however, is not unimodular. We have, indeed, $(d_r s)_{22} \wedge (d_r s)_{21} = \frac{1}{\lambda^2}(d\lambda \wedge d\mu)$; hence $d\mu_r(s) = \frac{d\lambda\, d\mu}{\lambda^2}$ defines a right-invariant measure on S_0. Putting $\delta(s) \equiv \lambda^2$, we have $d\mu_l(s) = \varrho(s)\, d\mu_r(s)$; extending δ by requiring $\delta(s) \equiv \delta(sz)$ $(s \in S,\ z \in Z)$, the same relation remains valid on S too. In particular, if $f \in L(S)$[2])

$$\int_S f(s\, s_0^{-1})\, d\mu_l(s) = \varrho(s_0) \int_S f(s)\, d\mu_l(s) \,. \tag{3.1}$$

ε. In what follows we define the invariant measure $d\mu(a)$ on G through the differential form, which is the image under $\delta\Phi$ of the form $(d_l a)_{21} \wedge (d_l a)_{22} \wedge \wedge (d_l a)_{12}$ on G_1; $d\mu(a)$ is bi-invariant.

b. We denote by G_r the open submanifold of G defined as the complete inverse image of elements, having different positive eigenvalues, of the adjoint group G. Each element of G_r can be represented in the form $g^{-1}hg$ $(g \in G,\ h \in H)$. Furthermore, the group of automorphisms of H induced by inner automorphisms of G, leaving H fixed (Weyl group), is of order 2; denoting by $\bar{h}$ the action of its nontrivial element on $h \in H$, we have $\bar{h}_j = (h^{-1})_j$ (cf. γ. above). Hence G_r can also be described as union of all conjugacy classes containing elements of $H - Z$; two elements of h fall into the same class only if they are congruent under the Weyl group. — An open subset $H_F \subset H$ is called a fundamental domain if $H_F \cap \overline{H_F} = (0)$, and $H_F \cap \overline{H_F} = H - Z$ ($\overline{H_F}$ is the image of H_F under the Weyl group).

Now we turn to the discussion of the first of our main integral relations. Denoting by $D(a)$ $(a \in G_r)$ the absolute value of the difference of the square roots of the eigenvalues $\neq 1$ of $\mathrm{Ad}(a)$, we have for $f \in L(G)$ and an arbitrary H_F

$$\int_{G_r} f(a)\, d\mu(a) = \int_{H_F} D(h)\, I_h\, d\mu(a) \,. \tag{3.2}$$

Here

$$I_h = \frac{\pi}{[\varrho(h)]^{\frac{1}{2}}} \int_{O_0 \times N^+} f(o^{-1} h n o)\, d\mu(o)\, d\mu(n)$$

where O_0 is the image of O in the adjoint group, and $\mu(O_0) = 1$ $\left(d\mu(o) = \frac{d\varphi}{2\pi},\right.$ $0 \leqq \varphi < 2\pi$ in our usual parametrization). Observe, that $I_h \in L(H)$.

Before proving (3.2) we wish to show, that

$$I_{\bar{h}} \equiv I_h \,. \tag{3.3}$$

While proving (3.2) it will turn out, that, for a fixed $h \in H - Z$, the set $\{o^{-1}hno;\ n \in N^+,\ o \in O\}$ gives the conjugacy class containing h. This together with (3.3) shows, that I_h can be interpreted as an average of f over a conjugacy class, and (3.2) gives a decomposition of the invariant integral into a continuous sum of these averages.

To prove (3.3), for $a \in G_r$ we denote by h_a the element $\in H_F$, for which $a = g h_a g^{-1}$ $(g \in G)$. Assume now, that $g(h)$ is a bounded continuous function

[2]) In general, we denote the family of all continuous functions with compact support on the group G by $L(G)$.

on H; replacing $f(a)$ by $f(a)\,g(h_a)$ in (3.2) we easily obtain

$$\int_{G_r} f(a)\,g(h_a)\,d\mu(a) = \int_{H_F} D(h)\,g(h) I_h\,d\mu(h)\,.$$

Since $\overline{H}_F$, too, is a fundamental domain we also have

$$\int_{G_r} f(a)\,g(\overline{h}_a)\,d\mu(a) = \int_{\overline{H}_F} D(h)\,g(h) I_h\,d\mu(h)\,.$$

Adding we get

$$\int_{G_r} f(a)\,(g(h_a) + g(\overline{h}_a))\,d\mu(a) = \int_{H} D(h)\,g(h) I_h\,d\mu(h)\,. \tag{3.4}$$

The left hand side of (3.4) remains invariant, if we replace $g(h)$ by $g(\overline{h})$; since, furthermore, $D(\overline{h}) \equiv D(h)$ and $d\mu(\overline{h}) = d\mu(a)$ we finally obtain

$$\int_{H} D(h)\,g(h) I_h\,d\mu(h) = \int_{H} D(h)\,g(h) I_{\overline{h}}\,d\mu(h)\,.$$

Because of the arbitrariness of $g(h)$ this implies (3.3).

We carry out the proof of (3.2) in several steps.

α. For $z \in Z$ we denote the subset of G_r, consisting of elements conjugate with elements of zH_0, by $G_r^{(z)}$. We have $G_r^{(z)} = zG_r^{(e)}$ (e = unit of G), and the sets $G_r^{(z)}$ are open submanifolds of G with disjoint closures. Hence it plainly suffices to prove, that for any $z \in Z$ we have a relation obtained from (3.2) by replacing G_r by $G_r^{(z)}$, and H_F by $H_F^{(z)} = H_F \cap zH_0$; then (3.2) follows by summation. But even here it suffices to consider $z = e$ only; the general case then follows by applying it to $f(za)$ and $z^{-1}H_F^{(z)}$, and observing, that $D(zh) \equiv D(h)$.

It is easy to see, however, that the restriction of Φ to $G_r^{(e)}$ is a diffeomorphism with its image, which is the collection of all matrices with positive eigenvalues $\neq 1$. Hence in view of $\alpha - \varepsilon$ above, it suffices to consider the corresponding problem on G_1.

β. In what follows we simply write G_r for $\Phi(G_r^{(e)})$ and H_F for $\Phi(H_F^{(e)})$. First we prove the following relation

$$\int_{G_r} f(a)\,d\mu(a) = \int_{S_F \times N^-} [\varrho(s)]^{-\frac{1}{2}} D(s)\,f(n^{-1}sn)\,d\mu_l(s)\,d\mu(n) \tag{3.4}$$

where S_F is the open submanifold of S[3]) $\subset G_1$, made up of matrices with the property, that the element of H, composed of the diagonal elements, lies in H_F. We observe, that for $a \in G_1$, $D(a)$ is just the absolute value of the difference of its eigenvalues. To prove (3.4) observe, that each $a \in G_r$ can be represented in the form $g^{-1}hg$ with a uniquely determined $h \in H_F$ ($g \in G_1$), and $a_{12} \neq 0$ implies $g_{22} \neq 0$. Assuming this case, we can write g in the form sn ($s \in S$, $n \in N^-$) to obtain for $a \in G_r$ a representation of the form $n^{-1}s_1 n$ ($s_1 \in S_F$); one easily sees, that it is unique. Summing up, we conclude, that the map $F: S_F \times N^- \to G_r$ defined by $F(s, n) = n^{-1}sn$ covers G_r up to a set of measure 0 with respect to $d\mu(a)$. This implies a relation $d\mu(a) = f(s, n)\,d\mu_l(s)\,d\mu(n)$ with some continuous function f (we denote the inverse image of $d\mu(a)$ on $S_F \times N^-$ again by $d\mu(a)$). Since $d\mu(a)$, in particular, is invariant under inner automorphisms

[3]) **Here, of course, S stands for $\Phi(S)$; it is the group of all triangular matrices in G_1 having 0 in the lower left corner.**

of G_1 implemented by elements of N^-, f does not depend on n. To determine its exact form, we have to compute the determinant of the linear transformation connecting the differentials, contributing to the differential form of maximum rank defining the resp. invariant integral, in the matrices $d_l a$, $d_l s$ and $d_l n$. Since f does not depend on n, it suffices to do this for $n = e$. Taking the differentials of both sides of $na = sn$, we have $dn \cdot a + n \cdot da = ds \cdot n + s \cdot dn$. Multiplying on the left by the corresponding sides of the relation $a^{-1}n^{-1} = n^{-1}s^{-1}$, putting $n = e$, and rearranging, we obtain

$$d_l a = d_l s + dn - s^{-1} dn s\,.$$

Assuming, as before

$$n = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}, \quad s = \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix}$$

we have

$$(d_l a)_{12} = \lambda\, d\mu - \mu\, d\lambda + \cdots$$

$$(d_l a)_{22} = \frac{d\lambda}{\lambda} + \cdots$$

$$(d_l a)_{21} = \left(1 - \frac{1}{\lambda^2}\right) dx$$

where the terms not written out are multiples of the differential dx. This gives $(d_l a)_{21} \wedge (d_l a)_{22} \wedge (d_l a)_{12} = \left(1 - \frac{1}{\lambda^2}\right)$ $(dx \wedge d\lambda \wedge d\mu)$, or $f(s) \equiv \left|1 - \frac{1}{\lambda^2}\right| \equiv$ $\equiv [\varrho(s)]^{-\frac{1}{2}} D(s)$, proving (3.4).

γ. Next we replace N^- by O_0 in (3.4) as follows. We write O for $\Phi(O)$ ($\Phi(O)$ = group of orthogonal matrices in G_1) and o_φ for $\Phi(o_\varphi)$. One easily verifies that each right coset of G_1 according to S contains two elements of O (which differ by the factor $\gamma = o_{2\pi}$), and with the exception of the coset consisting of all matrices having zero in the right lower corner, a single element of N^-. More exactly, denoting the open subset of O obtained by removing the points $o_{\pm\pi}$ by $\tilde{O}$, for each $o \in \tilde{O}$ there exists a uniquely determined $s(o) \in S$ such that $s(o)o = n \in N^-$; if o varies over $\tilde{O}$, n covers N^- twice. The connection between the parameters of n and o is given by $x = tg\,\frac{\varphi}{2}$. Let us extend the function $\varrho(s)$ (cf. δ in a. above) to G_1 by putting $\varrho(a) = a_2^2$ ($a \in G_1$); observe, that $\varrho(sa) = \varrho(s)\,\varrho(a)$. Hence finally, if $f \in L(N^-)$, the function $f_1(o) \equiv f(s(o)o)$ satisfies $f_1(\gamma o) \equiv f_1(o)$, hence it can be considered a function on O_0, and we have

$$\int\limits_{N^-} f(n)\, d\mu(n) = \frac{1}{2} \int\limits_{-\pi}^{\pi} f_1(o_\varphi) \frac{d\varphi}{\varrho(o_\varphi)} = \pi \int\limits_{O_0} f_1(o)\, d\mu(o)\,.$$

Applying this to (3.4), using (3.1) and $\varrho(s(o))\,\varrho(o) \equiv 1$, we get

$$\int\limits_{G_r} f(a)\, d\mu(a) = \int\limits_{O_0} \left[\, \int\limits_{S_F} \varrho(s)\, D(s)\, f(o^{-1}[s(o)]^{-1} s [s(o)] o)\, d\mu_l(s)\right] \frac{d\mu(o)}{\varrho(o)}$$

$$= \int\limits_{O_0} \left[\, \int\limits_{S_F} \varrho(s)\, D(s)\, f(o^{-1} s o)\, d\mu_l(s)\right] d\mu(o)\,.$$

(Observe, that trivially $D(s' s s'^{-1}) \equiv D(s)$).

Using finally the easily verified relation

$$\int_S g(s)\, d\mu_l(s) = \int_{H \times N^+} g(h, n)\, d\mu(h)\, d\mu(n)$$

we obtain a formula of the required form on G_1, and this finishes the proof of (3.2).

c. Now we proceed to obtain the expression of the trace in terms of the function I_h, and the explicit form of the character. This will be done by showing, that if $T(a)$ is of type $C_q^{(\tau)}$, the operator T_f ($f \in C_c^\infty$) is an integral operator in the representation space $L_\tau^2(O)$.

α. First we rewrite the realization of $T(a)$, given in I.B, in a form more convenient for our present purposes. We denote the character group of the abelian group H by $\hat{H}$. Using the notation of a. γ above, any $X \in \hat{H}$ can be written in the form $X(h_j) = e^{-2\pi i j \tau} \lambda^{2\sigma i}$ ($o \leqq \tau < 1$, σ arbitrary real number). We put $\chi(h) = [\varrho(h)]^{-\frac{1}{2}} X(h)$; this can be extended uniquely to a (non-unitary) character of S, by putting $\chi(hn) = \chi(h)$ ($n \in N^+$). For $g(o)$, $g'(o) \in L_\tau^2(O)$ we have $g(o\gamma) \equiv X(\gamma)\, g(o)$ and $g'(o\gamma) \equiv X(\gamma)\, g'(o)$, showing, that $g(o)\, \overline{g'(o)}$ is a function on O_0; hence the inner product in $L_\tau^2(o)$ can be written as $(g, g') = \int_{O_0} g(o)\, \overline{g'(o)}\, d\mu(o)$. Finally

$$(T(a)g)(o) \equiv \chi(s(oa))g(o\bar{a}) \tag{3.5}$$

with a χ defined through an appropriately chosen $X \in \hat{H}$. — Here the members of the family $\{C_q^{(\tau)}\}$ appear as parametrized with elements of $\hat{H}$, different from the unity. We put $X'(h) = X(\bar{h})$; the map $X \to X'$ is an automorphism of $\hat{H}$ dual to the automorphism $h \to \bar{h}$ of H, and by virtue of the discussion of I.B. we conclude, that two characters of H give rise to equivalent representations, if and only if they are congruent under this automorphism. Sometimes, in what follows, we write $T^{(X)}$ for the representation determined by $X \in \hat{H}$.

β. Assume now, that $f \in C_c^\infty$. Using (3.5) we have

$$\begin{aligned}(T_f g)(o) &= \int_G f(a)\, \chi(s(oa))\, g(o\bar{a})\, d\mu(a) \\ &= \int_G f(o^{-1}a)\, \chi(s(a))\, g(\overline{oo^{-1}a})\, d\mu(a)\,.\end{aligned}$$

As one verifies easily, with our present choice of the normalization of the measures involved, we have for any $F(a) \in L(G)$ the relation

$$\int_G F(a)\, d\mu(a) = \pi \int_{O_0} \left[\int_S F(so)\, d\mu_l(s)\right] d\mu(o)\,. \tag{3.6}$$

Applying this to the previous expression, we get

$$(T_f g)(o) = \int_{O_0} K_f(o, o')\, g(o')\, d\mu(o')$$

where

$$K_f(o, o') = \pi \int_S \chi(s)\, f(o^{-1}so')\, d\mu_l(s)\,.$$

Observe, that $K_f(o, \gamma o') = \overline{X(\gamma)}\, K_f(o, o')$, and $K_f(\gamma o, o') = X(\gamma)\, K_f(o. o')$. —

Hence

$$\operatorname{Tr}(T_f) = \int_{O_0} K_f(o,o)\, d\mu(o) = \pi \int_{O_0}\int_{S} \chi(s)\, f(o^{-1} s o)\, d\mu_l(s)$$

$$= \int_H X(h) \left(\frac{\pi}{[\varrho(h)]^{\frac{1}{2}}} \int_{O_0 \times N^+} f(o^{-1} h n o)\, d\mu(n)\, d\mu(o) \right) d\mu(h)$$

by virtue of the expression of χ by X given above in α. Hence finally, recalling the definition of I_h (cf. (3.2)) and writing $T^{(X)}$ in place of T, we get

$$\operatorname{Tr}(T_f^{(X)}) = \int_H \chi(h) I_h\, d\mu(h) \tag{3.7}$$

which is an expression of the required type for the trace. It represents the left hand side, considered as function of χ, as Fourier transform of the function $I_h \in L(H)$ (and even $\in C_c^\infty$ in our case); hence, by the Fourier inversion formula

$$I_h = \int_{\hat{H}} \overline{\chi(h)}\, \mathrm{T}_r(T_f^{(X)}\, d\mu(\chi) \tag{3.8}$$

with an appropriately normalized Haar measure $d\mu(X)$ on $\hat{H}$.

γ. To obtain an expression for the character of $T^{(X)}$, we observe, that (3.4) implies

$$\int_{G_r} f(a) \frac{[g(h_a) + g(\bar{h}_a)]}{D(h_a)}\, d\mu(a) = \int_H g(h) I_h\, d\mu(h) \tag{3.9}$$

for any bounded continuous $g(h)$, say.

Putting $g(h) \equiv X(h)$, and using (3.7), we get

$$\operatorname{Tr}(T_f^{(X)}) = \int_G C_X(a)\, f(a)\, d\mu(a)\,,$$

(3.10) where

$$C_X(a) \equiv \begin{cases} \dfrac{X(h_a) + X'(h_a)}{D(h_a)} & \text{if} \quad a \in G_r \\ 0 & \text{otherwise}\,. \end{cases}$$

This is the expression for the character as function of $a \in G$. Here $h_a \in H$ is any element, conjugate to a, and $C_X(a)$ is obviously independent of its particular choice.

Finally, we wish to point out, that for a fixed X and f, the trace depends on the normalization of $d\mu(a)$, the character, however, does not. As it will turn out later, our present choice of $d\mu(a)$ differs from that of Ch. II; which will have to be taken into account when setting up the final formulas.

2. Now we turn to the discussion of the series $D_l^\pm$. Here, too, we start with an integral formula.

a. We denote the complete inverse image of elements in G_0 having non positive eigenvalues by G_c. One easily shows, that every element of G_c is conjugate to a uniquely determined element of 0. Assume, $f \in L(G)$; our next objective will be to prove the following relation

$$\int_{G_c} f(a)\, d\mu(a) = 2 \int_{-\infty}^{\infty} \sin^2 \frac{\varphi}{2}\, I_\varphi\, d\varphi \tag{3.11}$$

where

$$I_\varphi = \int_{S_0} f(s o_\varphi s^{-1})\, d\mu_l(s)\,.$$

Similarly, as in the case of G_r, during the course of the proof it will turn out, that (3.11) can be interpreted as decomposition of the invariant integral into a continuous sum of averages of f over conjugacy classes, containing elements of $0 - Z$.

α. We denote the open subset of G consisting of elements conjugate with some o_φ $(2\pi(j-1) < \varphi < 2\pi j;\, j = 1, 2, \ldots,)$ by $G_c^{(j)}$, and put $G_c^{(-j)} = [G_c^{(j)}]^{-1}$. Evidently, any two of these sets are congruent to each other mod. a translation by some element of Z, and their union is G_c. Furthermore, the restriction of Φ to $G_c^{(1)}$, say, is a diffeomorphism with its image. Therefore, since the factor of I_φ in (3.11) is periodic with 2π, a reasoning similar to that employed in 1.b.α above shows, that it suffices to prove, that if $f \in L(G_1)$, we have

$$\int_{G_c} f(a)\, d\mu(a) = 2 \int_0^{2\pi} \left(\sin\frac{\varphi}{2}\right)^2 I_\varphi\, d\varphi \tag{3.12}$$

where G_c stands for $G_c^{(1)}$ formed along with I_φ, with respect to G_1.

β. To prove (3.12) first we observe, that any $a \in G_c$ can be represented in the form gog^{-1}, where $o \in O$ is uniquely determined by a. Furthermore, we can write $g = so'$ $(s \in S_0,\ o' \in O)$, which gives $a = sos^{-1}$, and one easily checks, that in the relation s, too, is uniquely determined by a. Next we use reasonings, analogous to those of 1.b.β above. Denoting the open subset $\{o_\varphi;\, 0 < \varphi < 2\pi\}$ of O by $\tilde{O}$, the map $(o, s) \to sos^{-1}$ is a diffeomorphism between $\tilde{O} \times S_0$ and G_c and hence we can write $d\mu(a) = f(o, s)\, d\varphi\, d\mu_l(s)$. Since, however, G_c is taken into itself by any inner automorphism, and this leaves $d\mu(a)$ invariant, we conclude, that f does not depend on s. In order to determine its explicit form, we take the differentials of both sides of $as = so$, which gives $da \cdot s + a \cdot ds = ds \cdot o + s\, ds$. In addition, we have $s^{-1}a^{-1} = o^{-1}s^{-1}$; multiplying by this on the left, putting $s = e$, $o = o_\varphi$ and rearranging, we obtain $d_l a = o_{-\varphi}\, ds\, o_\varphi + d_l o - ds$. We recall, that

$$d_l o = \frac{1}{2}\begin{pmatrix} 0 & -d\varphi \\ d\varphi & 0 \end{pmatrix}$$

and

$$ds = \begin{pmatrix} -d\lambda & d\mu \\ 0 & d\lambda \end{pmatrix}.$$

To compute $o_{-\varphi}\, ds\, o_\varphi - ds$ we proceed as follows. One easily sees, that if

$$x = \begin{pmatrix} x_1 & x_2 - x_0 \\ x_2 + x_0 & -x_1 \end{pmatrix}$$

the linear transformation $\mathrm{Ad}(o_\varphi)x = o_\varphi x o_{-\varphi}$ (cf. I.A.1.) is a rotation in the (x_0, x_1, x_2) space, leaving the x_0 axis invariant, by an angle φ in the positive direction in the (x_1, x_2) plane. Hence putting $x_1 = -d\lambda$, $x_2 = -x_0 = \dfrac{d\mu}{2}$, we get

$o_{-\varphi}\, ds\, o_\varphi - ds = [\mathrm{Ad}(o_{-\varphi}) - I]x = y$, where

$$y = \begin{pmatrix} y_1 & y_2 - y_0 \\ y_2 + y_0 & -y_1 \end{pmatrix}$$

with $y_0 = 0$, $y_1 = [1 - \cos\varphi]\, d\lambda + \sin\varphi \frac{d\mu}{2}$, $y_2 = \sin\varphi\, d\lambda - \frac{(1-\cos\varphi)}{2}\, d\mu$.

Hence finally $(d_l a)_{21} \wedge (d_l a)_{22} \wedge (d_l a)_{12} = 2\left(\sin\frac{\varphi}{2}\right)^2 (d\lambda \wedge d\mu \wedge d\varphi)$, proving (3.12), and along with it (3.11) too.

Assume now, that $g(o)$ is a bounded continuous function on O, and for $a \in G_c$, denote the element of O conjugate with it, by o_a. Then (3.11) implies

$$\int_{G_c} f(a)\, g(o_a)\, d\mu(a) = 2 \int_{-\infty}^{\infty} \left(\sin\frac{\varphi}{2}\right)^2 I_\varphi\, g(o_\varphi)\, d\varphi\,. \tag{3.13}$$

b. Assume, that $\{T(a)\}$ is the direct sum of two unitary representations of type D_l^+ and D_l^- resp. $\left(l > \frac{1}{2}\right)$. In order to obtain an expression of $\mathrm{Tr}(T_f)$ $(f \in C_c^\infty)$ in terms of the averages I_h and I_φ we first compute the character, and then through an application of (3.9) and (3.13) derive the required formula. This change, when compared with the method of 1, is made necessary by the fact, that the realizations of $D_l^\pm$ constructed in I.B. where obtained by considering *subrepresentations* of representations induced by certain characters of S. The reasonings of 1.c. are applicable only when $2l$ is an integer; in this case the induced representation turns out to be a direct sum of D_l^+, D_l^- and a finite dimensional representation of $SL(R, 2)$. Hence the character of T is the difference of the character of the whole representation, computed as in 1.c. γ, and the character of the finite dimensional representation, which is easy to obtain directly. — On the other hand, the present method, too, could be used to obtain the characters of $\{C_q^{(\tau)}\}$.

α. In the following computations we are going to use Eulerian coordinates (cf. II.A.1); but, making use of the remark made at the end of 1 above, we choose as element of the invariant measure $\mathrm{sh}\, u\; d\varphi_1\, du\, d\varphi_2$. We write the function $f \in C_c^\infty$ again as $f(\varphi_1, u, \varphi_2)$, and recall, that the matrix coefficients standing in the diagonal, when using the canonical basis of I.B, are given by $h_m^{(l)}(n)\, e^{-i(\varphi_1 + \varphi_2)}$, where (cf. (2.8))

$$h_m^{(l)}(n) = \frac{1}{\pi} \int_{-u}^{u} \frac{e^{-\left(l-\frac{1}{2}\right)v}}{Z} \left[\mathrm{Re}\left(\frac{2\,\mathrm{ch}\frac{u}{2}}{\left(2\,\mathrm{ch}\frac{v}{2} + iZ\right)}\right)^{2m}\right] dv$$

$\left(u > 0, Z = (2(\mathrm{ch}\, u - \mathrm{ch}\, v))^{\frac{1}{2}};\ m = \pm(l + j), j = 0, 1, 2, \ldots, h_{-m}^{(l)}(u) \equiv h_m^{(l)}(u)\right)$.
Hence, putting for a fixed $0 < x < 1$

$$H(x;\, u,\, \varphi) = 2\left(\sum_{m \geq l} x^m \cos m\varphi\, h_m^{(l)}(u)\right)$$

we have by Abel's theorem

$$(3.14)\quad \operatorname{Tr}(T_f) = \lim_{t\to 1-0} \int_0^{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} H(x; u, \varphi)\, f(\varphi_1, u, \varphi_2)\, \operatorname{sh} u\, d\varphi_1\, du\, d\varphi_2 .$$

Putting

$$\varepsilon = \left(\frac{2\operatorname{ch}\frac{u}{2}}{2\operatorname{ch}\frac{v}{2} + iZ} \right)^2$$

we have $|\varepsilon| = 1$, and

$$H(x; u, \varphi) = 2x^l \operatorname{Re} H' ,$$

where

$$H' = \frac{e^{il\varphi}\left(2\operatorname{ch}\frac{u}{2}\right)^{2l}}{2\pi} \int_{-u}^{u} \frac{e^{-\left(l-\frac{1}{2}\right)v}}{Z} \left[\frac{\varepsilon^l}{1 - x\varepsilon e^{i\varphi}} + \frac{\varepsilon^{-l}}{1 - x\bar{\varepsilon} e^{i\varphi}} \right] dv$$

the interpretation of ε^l and $\bar{\varepsilon}^l$ being obvious.

Defining R and t^z (tz complex) as in II.A.2, and setting $F(t) = (1 + t + R)$, a simple computation, analogous to those carried out in II.A.2 shows, that

$$(3.15)\qquad H' = \frac{e^{il\varphi}\left(2\operatorname{ch}\frac{u}{2}\right)^{2l}}{2\pi i} \oint_C \frac{1}{R[F(t)]^{2(l-1)}((F(t))^2 - xbt)}\, dt$$

$\left(b = \left(2\operatorname{ch}\frac{u}{2}\right)^2 e^{i\varphi}\right)$, where C is the interval $[e^{-u}, e^u]$ run over twice *clockwise*.

β. Our next objective will be the evaluation of (3.15) for a fixed $u > 0$ and φ, if $x \to 1$. Since, as we now, $F(t)$ maps the complex plane, cut along $[e^u, e^u]$, onto a bounded domain, the closure of which lies in the open right half plane, the integrand of (3.15) is $O(|t|^{-2})$ for large $|t|$. Next we show, that $G(t) = (F(t))^2 - xbt$ possesses exactly one simple root $t(x)$, hence it will suffice to consider the limit of the residue for $t(x)$ if $t \to 1$. This will give the character $C_l(a)$ of T for $a = g_u o_\varphi$. — Observe, incidentally, that *any* character $C(a)$ necessarily satisfies $C(ab) = C(ba)$ $(a, b \in G)$.

To obtain the roots, we consider the analytic continuation of $G(t)$; it is a univalued function on the two sheeted Riemann surface of the continuation of R.

We denote the sheet corresponding to the branch of the latter used above by P_+, the other by P_-, and put $F_\pm(t)$ and $G_\pm(t)$ for the corresponding branches of the continuous $F(t)$ and $G(t)$ resp. Consider now the domain D of P_- bounded by the unit circle and the interval $[e^{-u}, 1]$. We have $|F_-(t)|^2 > x|b\,t|^2$ on the boundary; since, on the other hand, $(F_-(t))^2$ has a double zero for $t = 0$, an application of Rouche's theorem shows, that $G_-(t)$ has exactly one nontrivial zero place inside D. Observing, finally, that $G_+(t) \equiv t\, G_-\left(\frac{1}{t}\right)$, we can conclude, that $G(t)$ has exactly one zero $t(x)$, on P_+, and incidentally, $|t(x)| > 1$. If $x \to 1$, for reasons of continuity $t(x) \to t(1)$, and we have a corresponding convergence of the residues of the integrand in (3.15), provided $G'(t(1)) \neq 0$. By virtue of the previous discussion $G'(t(1)) = 0$ can occur only if $t(1) = \pm 1$.

$t(1) = -1$ implies $b = -\left(2\operatorname{ch}\frac{u}{2}\right)^2$ or $e^{i\varphi} = -1$; and $G'(-1) \neq 0$. Hence what remains is $t(1) = 1$, in which case $\left(\operatorname{ch}\frac{u}{2}\cos\frac{\varphi}{2}\right)^2 = 1$. It is easy to find the group theoretical interpretation of this condition by observing, that $\det(\Phi(g_u o_\varphi) - yI) \equiv y^2 - 2\operatorname{ch}\frac{u}{2}\cos\frac{\varphi}{2}y + 1$, and the roots of $\operatorname{Ad}(g_u o_\varphi)$ are 1 and the squares of the roots of this equation. Hence we conclude, that $\left(\operatorname{ch}\frac{u}{2}\cos\frac{\varphi}{2}\right)^2 = 1$, if and only if all roots of $\operatorname{Ad}(g_u o_\varphi)$ are 1, or if $g_u o_\varphi$ lies on the common boundary of G_r and G_c (cf. 1.b and 2.a); we call these elements of G singular. Obviously these form a set of measure 0 with respect to the Haar measure. — Summing up, in order to obtain the character $C_l(a)$ of T, which is a direct sum of two representations of type D_l^+ and D_l^- resp., it suffices to compute twice the real part of the residue for $t(1)$ of the integrand in (3.15)[4]), assuming $\left(\cos\frac{\varphi}{2}\operatorname{ch}\frac{u}{2}\right)^2 \neq 1$. What concerns the possibility of the interchange of the limit transition and integration in (3.17), we observe, that it is enough to justify it *locally*. In the neighborhood of a nonsingular element it is clear from the previous discussion; otherwise it is easy to obtain a dominating function (cf. the subsequent computations).

γ. Now we proceed to determine $t(1)$ and the corresponding residue as indicated above; we distinguish two cases.

γ_1. $\left(\operatorname{ch}\frac{u}{2}\cos\frac{\varphi}{2}\right)^2 > 1$, (or $g_u o_\varphi \in G_r$). It is clear from the expression for $H(x; u, \varphi)$ that $C_l(g_u o_\varphi) \equiv C_l(g_u o_{-\varphi})$ hence it suffices to consider $\varphi > 0$. We put $\varphi' = \varphi - \left[\frac{\varphi+\pi}{2\pi}\right] 2\pi$; observe, that $-\pi < \varphi' < \pi$. In order that $[F(e_v)]^2 = \left(2\operatorname{ch}\frac{u}{2}\right)^2 e^{i\varphi'} e^v$, $v > 0$, we must have $v < u$, and

$$\frac{\operatorname{ch}\frac{v}{2}}{\operatorname{ch}\frac{u}{2}} \pm i\,\frac{Z}{2\operatorname{ch}\frac{u}{2}} = e^{i\frac{\varphi'}{2}} \quad \left(Z = (2(\operatorname{ch}u - \operatorname{ch}v))^{\frac{1}{2}}\right)$$

where we have $+$ or $-$ according to whether $\varphi' > 0$ or $\varphi' < 0$. Hence $\operatorname{ch}\frac{v}{2} = \operatorname{ch}\frac{u}{2}\cos\frac{\varphi'}{2} = \left|\operatorname{ch}\frac{u}{2}\cos\frac{\varphi}{2}\right| > 1$, which determines $v > 0$ and $t(1) = e^v$; it lies on the upper (lower) part of the cut along $[e^{-k}, e^u]$ if $\varphi' > 0$ ($\varphi' < 0$ resp.). At the same time, one sees at once, that $e^{\frac{v}{2}}$ is the root > 1 of $\Phi(g_u o_\varphi)$, hence $e^{\pm v}$ are the roots of $\operatorname{Ad}(g_u o_\varphi)$. — To obtain the residue, we observe, that according to our present interpretation of powers, $[F(t(1))]^{2(l-1)} = \left(2\operatorname{ch}\frac{u}{2}\right)^{2(l-1)} \times e^{(l-1)v} e^{i(l-1)\varphi'}$. Furthermore, since $\operatorname{Re}[F_+(t)] > 0$, we have $F(t(1)) = 1 + t(1) + R(t(1)) = 2\operatorname{ch}\frac{u}{2}\, e^{\frac{v}{2}} e^{i\frac{\varphi'}{2}}$; which gives $R\,\frac{d}{dt}G(t)\Big|_{t=t(1)} = 2\left(2\operatorname{ch}\frac{u}{2}\right)^2 e^{i\varphi'} e^{\frac{v}{2}}\operatorname{sh}\frac{v}{2}$.

[4]) Of course $e^{il\varphi}\left(2\operatorname{ch}\frac{u}{2}\right)^2$ included.

Hence, putting $\left[\frac{\varphi+\pi}{2\pi}\right] = j$ we get finally

$$C_l(g_u o_\varphi) = \frac{2\cos(2\pi l j)\, e^{-\left(l-\frac{1}{2}\right)v}}{e^{\frac{v}{2}} - e^{-\frac{v}{2}}}.$$

To obtain a group theoretic formulation of the right hand side, observe, that using the notations of 1.b.α, we have $g_u o_\varphi \in G_r^{(e)}$ for small φ, hence $g_u o_\varphi \in G_r^{(z)}$, $z = \gamma^j \in Z$ (j defined as above) for any $\varphi > 0$. Similarly, if $\varphi < 0$, $g_u o_\varphi \in G_r^{(z)}$, $z = \gamma^{-j}$, $j = \left[\frac{|\varphi| + \pi}{2\pi}\right]$. Denoting one of the eigenvalues ∓ 1 of $\mathrm{Ad}(a)$ by λ_a, we can write

$$(3.16) \qquad C_l(a) = \frac{2\cos(2\pi j l)\, \lambda_a^{-\left(l-\frac{1}{2}\right) sg(\lambda_a - 1)}}{D(a)} \quad (a \in G_r^{(z)}, z = \gamma^j).$$

γ_2. $\left(\mathrm{ch}\frac{u}{2}\cos\frac{\varphi}{2}\right)^2 < 1$ (or $g_u o_\varphi \in G_c$). We define φ_0, $(o < \varphi_0 < \pi)$ through $\cos\frac{\varphi_0}{2} = \left(\mathrm{ch}\frac{u}{2}\right)^{-1}$; hence, for $0 < \varphi < 2\pi$ our condition is satisfied if $\varphi_0 < \varphi < 2\pi - \varphi_0$. Next we observe, that for any such φ there exists a uniquely determined α, $(o < \alpha < 2\pi)$, such that $[F_+(e^{-i\alpha})]^2 = \left(2\,\mathrm{ch}\frac{u}{2}\right)^2 e^{i(\varphi-\alpha)}$. Indeed, this condition is equivalent to

$$\frac{\cos\frac{\alpha}{2}}{\mathrm{ch}\frac{u}{2}} + i\frac{Z'}{2\,\mathrm{ch}\frac{u}{2}} = e^{i\frac{\varphi}{2}}, \quad \left(Z' = (2\,\mathrm{ch}\,u - \cos\alpha)^{\frac{1}{2}}\right)$$

or $\cos\frac{\alpha}{2} = \mathrm{ch}\frac{u}{2}\cos\frac{\varphi}{2}$, which determines α. Keeping $u > 0$ fixed, let us vary φ from φ_0 to $2\pi - \varphi_0$; we denote the corresponding α by $\alpha(\varphi)$. Then $\alpha(\varphi)$ will vary from 0 to 2π, and moreover we have 1. $-\pi < \varphi - \alpha(\varphi) < \pi$, since $\mathrm{Re}[F_+] > 0$, 2 $\cdot\, g_u o_\varphi$ is conjugate with $o_{\alpha(\varphi)}$. The last statement can be proved e.g. by checking the analogous situation for $\Phi(g_u o_\varphi)$ in G_1. We extend now the definition of $\alpha(\varphi)$ first for any $\varphi > o$ $\left(\left(\cos\frac{\varphi}{2}\,\mathrm{ch}\frac{u}{2}\right)^2 < 1\right)$ by setting $\alpha(\varphi) = \alpha(\varphi')$ if $\varphi = 2\pi j + \varphi'$ $(j = 0, 1, 2, \ldots, o < \varphi' < 2\pi)$ and then for a negative φ by $\alpha(\varphi) = \alpha(-\varphi)$. It is clear, that we always have $t(1) = \exp(-i\alpha(\varphi))$, along with properties 1 and 2 described above. Turning now to the computation of the residue, we have by 1.: $[F_+(t(1))]^{2(l-1)} = \left(2\,\mathrm{ch}\frac{u}{2}\right)^{2(l-1)} e^{i(l-1)\varphi} e^{-i(l-1)\alpha}$ $(\alpha = \alpha(\varphi))$, and $F_+(t(1)) = 1 + t(1) + R(t(1)) = 2\,\mathrm{ch}\frac{u}{2} e^{i\frac{\varphi}{2}} e^{-i\frac{\alpha}{2}}$. Hence, similarly, as in γ_1 above this implies $R\frac{d}{dt}G(t)\Big|_{t=t(1)} = 2\left(2\,\mathrm{ch}\frac{u}{2}\right)^2 e^{i\frac{\varphi}{2}} e^{-i\frac{\alpha}{2}} \cdot i\sin\frac{\alpha}{2}$. This finally gives

$$(3.17) \qquad C_l(a) = -\frac{\sin\left(l - \frac{1}{2}\right)\alpha}{\sin\frac{\alpha}{2}} \quad (a \in G_c \text{ and conjugate to } o_\alpha).$$

(3.16) and (3.17) determine $C_l(a)$ completely, with the exception of the set of all singular elements; this, however, as already observed, is of a measure zero.

c. As in II, we put $T_l(f) = \mathrm{Tr}(T_f)$; assuming, that T is a direct sum of two representations of type D_l^+ and D_l^- resp. ($f \in C_c^\infty$). In what follows, we express it in terms of I_h and I_φ.

The discussion of b. shows, that

$$T_l(f) = \int_G f(a)\, C_l(a)\, d\mu(a) = \int_{G_r} f(a)\, C_l(a)\, d\mu(a) + \int_{G_c} f(a)\, C_l(a)\, d\mu(a);$$

we shall deal with the last two summands separately.

α. By virtue of (3.9), we have for any bounded continuous function $g(h)$ on H, satisfying $g(\bar{h}) \equiv g(h)$:

$$\int_{G_r} f(a)\, \frac{2g(h_a)}{D(a)}\, d\mu(a) = \int_H g(h)\, I_h\, d\mu(h)\,.$$

Using the notations of 1.a.γ, we put $g(h_j) = (\cos 2\pi l j)\, \lambda^{-(2l-1)\, sg\,(\lambda-1)}$; since $\bar{h}_j = (h^{-1})_j$, we have $g(h) \equiv g(\bar{h})$. Hence, using (3.16), we may write

$$\int_{G_r} f(a)\, C_l(a)\, d\mu(a) = \sum_{j=-\infty}^{\infty} (\cos 2\pi l j) \int_{H_0} \lambda^{-(2l-1)\, sg\,(\lambda-1)}\, I_{h_j}\, d\mu(h)\,.$$

In the subsequent considerations it will be more convenient to use certain parameters on H_0. To do this, we put

$$h_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \quad (-\infty < t < +\infty)$$

and $f_j(t) \equiv I_{(h_t)_j}$ ($j = 0, \pm 1, \pm 2, \ldots$). We observe, that since $I_h \in C_c^\infty$ on H, we have $f_j(t) \in C_c^\infty$ and $f_j(t) \equiv 0$ but for a finite set of indices. Furthermore, $I_h \equiv I_{\bar{h}}$ (cf. (3.3)) gives $f_j(t) \equiv f_j(-t)$. Hence we can write finally

$$\int_{G_r} f(a)\, C_l(a)\, d\mu(a) = \sum_{j=-\infty}^{\infty} \cos(2\pi l j) \int_0^\infty e^{-\left(l-\frac{1}{2}\right)t} f_j(t)\, dt \tag{3.18}$$

$\left(\text{Observe, that } d\mu(h) = \dfrac{d\lambda}{\lambda} = \dfrac{dt}{2}\right)$.

β. Using (3.11) and (3.17) we obtain

$$\int_{G_c} f(a)\, C_l(a)\, d\mu(a) = -2 \int_{-\infty}^{\infty} \sin\left(l - \frac{1}{2}\right) \varphi\, G(\varphi)\, d\varphi \tag{3.19}$$

where $G(\varphi) \equiv \sin\frac{\varphi}{2}\, I_\varphi \cdot \equiv \sin\frac{\varphi}{2} \int_{S_0} f(s o_\varphi s^{-1})\, d\mu_l(s)$.

Observe, that I_φ vanishes outside a finite interval, but it becomes infinite for $\varphi = 2\pi j$ ($j = 0, \pm 1, \pm 2, \ldots$), since the measure of S_0 is infinite. The properties of $G(\varphi)$, as function of φ, will be discussed later in C.

B. To describe the main idea of the second proof of the Plancherel formula, it will be useful to make a comparison with the case of the group $SL(C, 2)$. Denoting the group of all diagonal matrices by H, here, too, one can associate an irreducible representation with each element of the character group $\hat{H}$ of H, characters, equivalent under the Weyl symmetry $h \to h^{-1}$ $(h \in H)$, giving rise to equivalent representations. Furthermore, defining N^+ (with complex coefficients) as in A.1.a.α and replacing O by the maximal compact group of all 2×2 unimodular unitary matrices, we can introduce I_h as in (3.2), and with it one has the analogue of (3.7). To obtain the Plancherel formula, one shows the existence of a differential operator L on H, such that for any $f \in C_c^\infty$, $L I_h\Big|_{h=e} = c f(e)$ (c is a constant $\neq 0$, not depending on f); finally, applying L on both sides of (3.8) for $h = e$, and defining a measure on $\hat{H}$ by $d\nu(X) = L\overline{X(h)}\Big|_{h=e} d\mu(X)$ one obtains a formula of the required type (cf. e.g. [3] Anhang III). The indicated procedure can be extended to any complex semi-simple Lie group.

On the other hand, it is clear that in our case any attempt, based on (3.7), to imitate the previous procedure is doomed to failure. Indeed, unlike the complex case, here the complement of the collection G_r of conjugacy classes containing elements of H contains an open set G_c. Also, the support of the characters C_X (cf. (3.10)) is contained in the complement of G_c. Since, however, for the Plancherel formula we evidently have to take into account the whole carrier of f, and since it is only the characters of the series $D_l^\pm$ which do not vanish on G_c, these, too, must be taken into consideration when setting up our formula. In order to gain a better picture of the method to be followed, we transform the relations of A.2.c. as follows. First, we get through partial integration from (3.18)

$$\left(l-\frac{1}{2}\right) \int_{G_r} f_j(t) e^{-\left(l-\frac{1}{2}\right)t} dt = f_j(0) + \int_0^{+\infty} f'_j(t) e^{-\left(l-\frac{1}{2}\right)t} dt$$

and

$$\left(l-\frac{1}{2}\right) \int_{G_r} f(a)\, C_l(a)\, d\mu(a) = \sum_{j=-\infty}^{\infty} \cos(2\pi j l)\, f_j(0) +$$

$$+ \sum_{j=-\infty}^{\infty} \cos(2\pi j l) \int_0^{\infty} f_j(t) e^{-\left(l-\frac{1}{2}\right)t} dt\,.$$

Next, assuming for a moment, that $G(\varphi)$ possesses the required properties, we have

$$-\left(l-\frac{1}{2}\right) \int_{2\pi(j-1)}^{2\pi j} \sin\left(l-\frac{1}{2}\right)\varphi\, G(\varphi)\, d\varphi = (-1)^j \cos(2\pi l j)\, G_j^{(-)} -$$

$$-(-1)^{j-1} \cos(2\pi l(j-1))\, G_{j-1}^{(+)} - \int_{2\pi(j-1)}^{2\pi j} \cos\left(l-\frac{1}{2}\right)\varphi\, G'(\varphi)\, d\varphi$$

where we put $G_j^{(\pm)} = \lim\limits_{\varphi \to 2\pi j \pm 0} G(\varphi)$.

This gives (cf. (3.19))

$$\left(l-\frac{1}{2}\right)\int_{G_c} f(a)\, C_l(a)\, d\mu(a) = -2\int_{-\infty}^{\infty} G'(\varphi)\cos\left(l-\frac{1}{2}\right)\varphi\cdot d\varphi -$$
$$-2\left(\sum_{j=-\infty}^{\infty}(-1)^j\cos(2\pi l j)\,(G_j^{(+)}-G_j^{(-)})\right).$$

Hence finally

$$\left(l-\frac{1}{2}\right)T_l(f) = \sum_{j=-\infty}^{\infty}\cos(2\pi j l)\int_0^{\infty} f_j'(t)\, e^{-\left(l-\frac{1}{2}\right)t}\,dt -$$
$$-2\int_{-\infty}^{\infty}\cos\left(l-\frac{1}{2}\right)\varphi\, G'(\varphi)\,d\varphi + R \tag{3.20}$$

where

$$R = \sum_{j=-\infty}^{\infty}\cos(2\pi l j)\,[f_j(0) - 2(-1)^j[G_j^{(+)}-G_j^{(-)}]].$$

Below in C, we show, that $G(\varphi)$ possesses the following properties

(P)

P.1. *$G(\varphi)$ is C^∞ in the neighborhood of any point, which is not multiple of 2π.*

$G_j^{(\pm)} = \lim_{\varphi\to 2\pi j\pm 0} G(\varphi)$ *exists, and we have* $f_j(0) = 2(-1)^j[G_j^{(+)}-G_j^{(-)}]$.

P.2. *For each fixed j, and φ sufficiently small*

$$|G'(2\pi j+\varphi) + \frac{\pi}{2}(-1)^j f(\gamma^j)| < A\,|\varphi|\,|\log\varphi|$$

where A does not depend on φ. $(\varphi \gtrless 0)$.

Taking all this for granted, we immediately have $R = 0$ in (3.20). Moreover, by virtue of P.2. the Fourier transform of $G'(\varphi)$ is summable[5]); hence eliminating the functions $\{f_j'(t)\}$ from (3.20) by aid of (3.8) and integrating both sides according to l from $\frac{1}{2}$ to $+\infty$, we obtain the final formula. We shall carry out the necessary computations in D. — We observe, that the identity in 1.P. possesses a simple group theoretic meaning (cf. C3.a).

C. Now we turn to prove the statements (P). First we remark, that it suffices to consider the case $j = 0$ only. Indeed, denoting by $\{\bar f_k(t)\}$ and $\bar G(\varphi)$ the functions corresponding to $f(\gamma^j a)$ ($j \neq 0$, fixed), we have $\bar f_0(0) = \bar f_j(0)$ and $(-1)^j\bar G(\varphi) \equiv G(\varphi + 2j\pi)$. — Moreover, since in I_φ, for a fixed $\varphi \neq 0$, only values of $f\in C_c^\infty$, assumed on elements $\{s o_\varphi s^{-1};\ s\in S_0\}$ occur, and since the restriction of Φ to a sufficiently small neighborhood of the closure of the set $\{s o_\varphi s^{-1};\ s\in S_0,\ |\varphi|<\delta,\ 0<\delta<\pi\}$ is a diffeomorphism with its image, it is again enough to discuss the analogous problems for G_1. This we shall do in several steps.

[5]) This can be proved through an easy adaptation of the reasonings leading to the classical theorem of S. Bernstein an Fourier series; cf. e. g. A. Zygmund, Trigonometrical series (New York 1952) 6.3, p. 135.

1. In what follows we put the function $I_\varphi = \int\limits_{S_0} f(s o_\varphi s^{-1})\, d\mu_l(s)$ $(\varphi > 0)$ in a more convenient form. We recall (cf. A.2.a. β), that for $0 < \varphi < 2\pi$ $s o_\varphi s^{-1} = o_{\varphi'}$ implies $\varphi = \varphi'$ and $s = e$. Since $o_\varphi = \exp(\varphi l_0)$ (I.A.1.), we can write $s o_\varphi s^{-1} = \exp(\varphi \operatorname{Ad}(s) l_0)$. Putting $\operatorname{Ad}(s) l_0 = x$, where

$$x = \frac{1}{2}\begin{pmatrix} x_1 & x_2 - x_0 \\ x_2 + x_0 & -x_1 \end{pmatrix},$$

$H_\varphi = \{x;\ x_0^2 - x_1^2 - x_2^2 = \varphi,\ x_0 > 0\}$ and representing x as a point in R^3, one easily checks, that the map F: $S_0 \to R^3$ defined by $F(s) = \operatorname{Ad}(s) l_0 = x$ is a diffeomorphism between S_0 and H_1. From this we conclude, that assuming, as usual

$$s = \begin{pmatrix} \lambda^{-1} & \mu \\ o & \lambda \end{pmatrix} \quad (\lambda > 0)$$

in the domain D of R^3 bounded by H_0 and $H_{2\pi}$, through the relation $\exp x = s o_\varphi s^{-1}$ we have a one-to-one differentiable correspondence between (x_0, x_1, x_2) and the parameters (λ, μ, φ). — The map $x \to \exp x$ is a diffeomorphism between D and its image D_1 in G_c, and incidentally, we have $\varphi = \sqrt{x_0^2 - x_1^2 - x_2^2}$. We put $x_0 = \varphi \operatorname{ch} u$, $x_1 = \varphi \operatorname{sh} u \cos\psi$, $x_2 = \varphi \operatorname{sh} u \sin\psi$ $(0 < \varphi < 2\pi,\ 0 \leqq \psi < 2\pi,\ u \geqq 0)$ and using the previous observations, we express $d\mu_l(s) = d\lambda\, d\mu$ (cf. A.1.a. δ) in terms of u and ψ as follows. If the carrier of the function $f \in L(G_1)$ lies in D_1, we can write

$$\int\limits_G f(a)\, d\mu(a) = \int\limits_D f(\exp x)\, \alpha(x)\, dx$$

$(dx = dx_0\, dx_1\, dx_2)$, where $\alpha(x)$ is some smooth function in x_0, x_1, x_2. Since the Haar measure on G_1 is bi-invariant, and $\det(\operatorname{Ad} g) \equiv 1$ $(g \in G_1)$, we conclude, that $\alpha([\operatorname{Ad} g] x) \equiv \alpha(x)$, implying, that $\alpha(x)$ depends on φ only; in what follows we write $\alpha(\varphi)$. Observe, incidentally, that by virtue of $d_l a = dx$ for $a = e$, we have $\alpha(o) = \frac{1}{4}$. We have furthermore $dx = \varphi^2 \operatorname{sh} u\, d\varphi\, du\, d\psi$, hence on D: $d\mu(a) = \alpha(\varphi)\, \varphi^2 \operatorname{sh} u\, d\varphi\, du\, d\psi$. On the other hand (3.11) gives $d\mu(a) = 2\left(\sin\frac{\varphi}{2}\right)^2 f(u, \psi)\, d\varphi\, du\, d\psi$, where f is the Jacobian of (λ, μ) according to (u, ψ). Comparing the two expressions for $d\mu(a)$, we finally get $f \equiv \frac{1}{2} \operatorname{sh} u$.

We now put $g(x) = f(\exp x)$; it is defined and C^∞ on R^3 if $f \in C_c^\infty$ on G_1. Defining

$$\text{(3.21)}\quad F(y_1, y_2) = \frac{1}{8}\int\limits_0^{2\pi} g\left(\frac{1}{2}(y_1 + y_2), \frac{1}{2}(y_1 - y_2)\cos\psi, \frac{1}{2}(y_1 - y_2)\sin\psi\right) d\psi$$

$F(y_1, y_2)$ is again C^∞, and by virtue of the previous discussion we can write

$$\text{(3.22)}\qquad I_\varphi = 2\int\limits_0^\infty F(\varphi e^u, \varphi e^{-u})\,(e^u - e^{-u})\, du \quad (\varphi > 0)\,.$$

2. Our main tool when proving statements (P) will be the following lemma (cf. [4] Lemma):

Suppose $H(x_1, x_2)$ is a C^∞ function with a compact support. Put

$$g(\varphi) = \int_0^\infty H(\varphi e^u, \varphi e^{-u})(e^u - e^{-u})\, du \quad (0 < \varphi < 1)\,.$$

Then we have

$$\lim_{\varphi \to 0} \varphi\, g(\varphi) = \int_0^\infty H(s, 0)\, ds \tag{3.23}$$

and

$$|(\varphi\, g(\varphi))' + 2H(0, 0)| < A\, \varphi |\log \varphi| \tag{3.24}$$

for sufficiently small φ; A is independent of φ.

Hence, in particular $\lim_{\varphi \to 0} (\varphi\, g(\varphi))' = -2H(0, 0)$. To prove all this, we put $g_1(\varphi) = \varphi \int_0^\infty H(\varphi e^u, \varphi e^{-u}) e^u\, du$, and $g_2(\varphi) = -\varphi \int_0^\infty H(\varphi e^u, \varphi e^{-u}) e^{-u}\, du$, such that $\varphi g(\varphi) = g_1(\varphi) + g_2(\varphi)$. Making the substitution $\varphi e^u = s$ in the first expression, we get

$$f_1(\varphi) = \int_\varphi^\infty H\left(s, \frac{\varphi^2}{s}\right) ds = \int_\varphi^M H\left(s, \frac{\varphi^2}{s}\right) ds$$

with a sufficiently large M, independent of φ, since the support of H is compact. Similarly, the substitution $\varphi e^{-u} = s$ gives

$$g_2(\varphi) = -\int_0^\varphi H\left(\frac{\varphi^2}{s}, s\right) ds = -\int_{\varepsilon\varphi^2}^\varphi H\left(\frac{\varphi^2}{s}, s\right) ds$$

where $0 < \varepsilon < 1$ does not depend on φ. But these expressions immediately imply $\lim_{\varphi \to 0} G_1(\varphi) = \int_0^\infty H(s, 0)\, ds$ and $\lim_{\varphi \to 0} g_2(\varphi) = 0$, proving (3.23).

On the other hand

$$g_1'(\varphi) = -H(\varphi, \varphi) + 2\varphi \int_\varphi^M H_2\left(s, \frac{\varphi^2}{s}\right) \frac{ds}{s}$$

with $H_2(x_1, x_2) \equiv \dfrac{\partial H}{\partial x_2}(x_1, x_2)$. Hence for $0 < \varphi < \dfrac{1}{2}$, say,

$$\left|\frac{1}{\varphi}(g_1'(\varphi) + H(0, 0))\right| \leqq \left|\frac{1}{\varphi}(H(\varphi, \varphi) - H(0, 0))\right| +$$

$$+ 2\int_\varphi^M \left|H_2\left(s, \frac{\varphi^2}{s}\right)\right| \frac{ds}{s} < A_1 + A_2 \int_\varphi^M \frac{ds}{s} < A_3 |\log \varphi|$$

where A_1, A_2 and A_3 do not depend on φ.

Furthermore

$$g_2'(\varphi) = -H(\varphi, \varphi) + 2\varepsilon\varphi \cdot H\left(\frac{1}{\varepsilon}, \varepsilon\varphi^2\right) -$$

$$-2\varphi \int_{\varepsilon\varphi^2}^{\varphi} H_1\left(\frac{\varphi^2}{s}, s\right)\frac{ds}{s}$$

$\left(H_1(x_1, x_2) \equiv \frac{\partial H}{\partial s_1}(x_1, x_2)\right)$. This gives

$$\left|\frac{1}{\varphi}(g_2(\varphi) + H(0,0))\right| \leqq \left|\frac{1}{\varphi}(H(\varphi, \varphi) - H(0,0))\right| +$$

$$+ 2\varepsilon\left|H\left(\frac{1}{\varepsilon}, \varepsilon\varphi^2\right)\right| + B\int_{\varepsilon\varphi^2}^{\varphi}\frac{ds}{s} <$$

$$< B_1 + B_2|\log\varphi| < B_3|\log\varphi|\,.$$

These two estimates together prove (3.24).

Finally, we observe, that in the lemma it evidently suffices to assume H to be of a compact support, when restricted to $0 \leqq x_1x_2 \leqq \delta$, $x_1, x_2 \geqq 0$, with some fixed $\delta > 0$.

3. Now we are ready to prove (P).

a. First we remark, that since $g(x_0, x_1, x_2)$ is of a compact support, when restricted to the closure of the domain $0 < \sqrt{x_0^2 - x_k^2 - x_2^2} < \delta < 2\pi$, $x_0 > 0$, so is $F(y_1, y_2)$ (cf. 3.21), when considered for $0 \leqq y_1y_2 \leqq \delta^2$, $y_1, y_2 \geqq 0$. Hence we can apply the lemma above to F in place of H.

a. Using (3.22) and (3.23) we get

$$G_0^{(+)} = \lim_{\varphi\to 0} G(\varphi) = \lim_{\varphi\to 0} \sin\frac{\varphi}{2} I_\varphi$$

$$= \lim_{\varphi\to 0} 2\sin\frac{\varphi}{2}\int_0^\infty F(\varphi e^u, \varphi e^{-u})(e^u - e^{-u})\,du = \int_0^\infty F(s, 0)\,ds\,.$$

By virtue of (3.21) the last integral is $\frac{1}{8}\int_0^\infty\int_0^{2\pi} g\left(\frac{s}{2}, \frac{s}{2}\cos\psi, \frac{s}{2}\sin\psi\right)d\psi\cdot ds$.

To find its group theoretic meaning, we put $n_s = \begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix} \in N^+$ (A.1.a.α); then if

$$z = \frac{1}{2}\begin{pmatrix} z_1 & z_2 - z_0\\ z_2 + z_0 & -z_1\end{pmatrix}$$

with $z_0 = \frac{s}{2}$, $z_1 = 0$, $z_2 = -\frac{s}{2}$, we have $z = n_{-\frac{s}{2}}$. Hence (cf. the similar reasoning in A.2.a.β) Ad $\left(o_{\psi+\frac{\pi}{2}}\right) z = x$, with $x_0 = \frac{s}{2}$, $x_1 = \frac{s}{2}\cos\psi$, $x_2 = \frac{s}{2}\sin\psi$.

Returning to the function $f(a)$, this gives $G_0^{(+)} = \frac{1}{4} \int\limits_0^\infty \int\limits_0^{2\pi} f(o_\psi n_{-s} o_{-\psi})\, d\psi\, ds$ $= \frac{1}{4} \int\limits_{-\infty}^0 \int\limits_0^{2\pi} f(o_\psi n_s o_{-\psi})\, d\psi\, ds.$

Replacing $f(a)$ by $f(a^{-1})$ and denoting the corresponding function by $\overline{G}(\varphi)$ we have $\overline{G}(\varphi) = -G(-\varphi)$. Hence

$$G_0^{(-)} = \lim_{\varphi \to -o} G(\varphi) = -\, G_0^{(+)} = -\frac{1}{4} \int\limits_0^\infty \int\limits_0^{2\pi} f(o_\psi n_s o_{-\psi})\, d\psi\, ds\,.$$

Summing up, we have $2(G_0^{(+)} - G_0^{(-)}) = \frac{1}{2} \int\limits_{-\infty}^\infty \int\limits_0^{2\pi} f(o_\psi n_s o_{-\psi})\, d\psi\, ds$. But by (3.2) this is $I_e = f_0(0)$.

Evidently $G_0^{(+)}$ and $G_0^{(-)}$ are averages of f taken on the common boundary of $G_r^{(e)}$ and $G_c^{(1)}$, $G_c^{(-1)}$ resp. (cf. A.1.b.α and A.2.a.α). The above discussion shows, that they can be obtained through an appropriate limit transition from the averages over conjugacy classes determined by elements of O.

The C^∞ character of $G(\varphi)$ follows immediately from (3.21) and (3.22).

b. We have $F(0, 0) = \frac{\pi}{4} g(0) = \frac{\pi}{4} f(e)$. Writing $\sin\frac{\varphi}{2} = \frac{\varphi}{2} + \varphi^3 g_1(\varphi)$, and applying (3.24), we get for $0 < \varphi < \frac{1}{2}$, say,

$$\left|G'(\varphi) + \frac{\pi}{2} f(e)\right| < \left|\frac{1}{2}(\varphi I_\varphi)' + \frac{\pi}{2} f(e)\right| + \left|\left((\varphi^2 g_1(\varphi))\,(\varphi I_\varphi)\right)'\right| < A\,\varphi\,|\log\varphi|$$

where the appropriately chosen A does not depend on φ.

Substituting again $f(a^{-1})$ in place of $f(a)$, and observing $\overline{G}'(\varphi) \equiv G'(-\varphi)$, we get a similar estimate for $-\frac{1}{2} < \varphi < 0$.

By virtue of the remarks made at the beginning of C, a and b. together prove (P) for any $j = 0, \pm 1, \pm 2, \ldots$, and $f \in C_c^\infty$ on G.

D. Now we are ready to derive our final formula.

1. We know, (cf. B), that by virtue of the results of C. we have $R = 0$ in (3.20); hence

$$\begin{aligned}(3.25)\qquad \left(l - \frac{1}{2}\right) T_l(f) &= \sum_{j=-\infty}^{\infty} \cos(2\pi j l) \int\limits_0^\infty f_j'(t)\, e^{-\left(l-\frac{1}{2}\right)t}\, dt\, - \\ &\quad - 2\int\limits_{-\infty}^{\infty} G'(\varphi) \cos\left(l - \frac{1}{2}\right) \varphi\, d\varphi = \Sigma_1 + \Sigma_2\,.\end{aligned}$$

Next we eliminate the functions $\{f_j'(t)\}$ from Σ_1, by aid of the traces corresponding to the family $\{C_q^{(\tau)}\}$ in the following fashion. By virtue of (3.8)

$$(3.26)\qquad \mathrm{Tr}(T_f^{(X)}) = \int\limits_H X(h) I_h\, d\mu(h)\,,$$

where $X(h_j) = e^{-2\pi i j \tau} \lambda^{2\sigma i}$ with an appropriately chosen real σ, and $0 \leqq \tau < 1$ (cf. A.1.c.α and β). The left hand side of (3.26), incidentally, up to a constant factor, is $T_\sigma^{(\tau)}(f)$ (cf. II.C and the remark at the end of A.1.c). Using the notations of A.2.c, (3.26) gives readily

$$\mathrm{Tr}(T_f^{(X)}) = \frac{1}{2} \sum_{j=-\infty}^{\infty} e^{-2\pi j \tau i} \int_{-\infty}^{\infty} e^{i\sigma t} f_j(t)\, dt .$$

We denote the right hand side by $F(\sigma, \tau)$ for a while. Observe, that $F(\sigma,\tau) \equiv F(-\sigma,\tau)$ (since $f_j(t) \equiv f_j(-t)$), $F(\sigma,\tau) \equiv F(\sigma, \tau+1)$, and $f_j(t) \neq 0$ but for a finite number of j's. An easy computation gives

$$\int_{-\infty}^{\infty} f_j'(t) e^{i\sigma t}\, dt = \frac{2\sigma}{i} \int_0^1 F(\sigma,\tau) e^{2\pi i \tau}\, dt$$

and, using the Plancherel formula for the real line $\left(l > \frac{1}{2}\right)$

$$\int_0^\infty f_j'(t)\, e^{-\left(l-\frac{1}{2}\right)t}\, dt = \int_0^1 e^{2\pi i j \tau} \left(-\frac{1}{\pi} \int_{-\infty}^{\infty} F(\sigma,\tau) \frac{\sigma^2}{\sigma^2 + \left(l - \frac{1}{2}\right)^2}\, d\sigma \right) d\tau .$$

Substituting this in the expression of Σ_1 (3.25), we get

$$\Sigma_1 = -\frac{1}{2\pi} \int_{-\infty}^{\infty} [F(\sigma, l) + F(\sigma, -l)] \frac{\sigma^2}{\sigma^2 + \left(l - \frac{1}{2}\right)^2}\, d\sigma . \tag{3.27}$$

2. Next we prove the following simple lemma:

Suppose, that the continuous function $g(u)$, defined on the real line satisfies α. $g(u) \equiv g(-u)$, β. $g(u+1) \equiv g(u)$. Then we have for any fixed real σ

$$\int_0^\infty \frac{\sigma}{\sigma^2 + u^2}\, g\left(u + \frac{1}{2}\right) du = \frac{\pi}{2} \int_0^1 g(\tau)\, [\mathrm{Re} \tanh \pi(\sigma + i\tau)]\, d\tau . \tag{3.28}$$

For the proof we observe, that

$$\pi \tanh \pi z = \sum_{j=0}^{\infty} \frac{2z}{z^2 + \left(j + \frac{1}{2}\right)^2} = \sum_{j=0}^{\infty} \left[\frac{1}{z + i\left(j + \frac{1}{2}\right)} + \frac{1}{z - i\left(j + \frac{1}{2}\right)} \right]$$

gives

$$= \pi\, \mathrm{Re}[\tanh \pi(\sigma + i\tau)] \sum_{j=0}^{\infty} \left[\frac{\sigma}{\sigma^2 + \left(\tau + j + \frac{1}{2}\right)^2} + \frac{\sigma}{\sigma^2 + \left(\tau - j - \frac{1}{2}\right)^2} \right] .$$

On the other hand

$$\int_0^\infty \frac{\sigma}{\sigma^2 + u^2}\, g\left(u + \frac{1}{2}\right) du = \int_0^1 g\left(u + \frac{1}{2}\right) \left(\sum_{j=0}^{\infty} \frac{\sigma}{\sigma^2 + (u+j)^2} \right) du$$

$$= \int_{\frac{1}{2}}^{\frac{3}{2}} g(u) \left(\sum_{j=0}^{\infty} \frac{\sigma}{\sigma^2 + \left(u + j - \frac{1}{2}\right)^2} \right) du .$$

Taking into account, that by virtue of properties α and β

$$\int_{\frac{1}{2}}^{1} g(u) \frac{\sigma}{\sigma^2 + \left(u + j - \frac{1}{2}\right)^2} du = \int_{0}^{\frac{1}{2}} g(u) \frac{\sigma}{\sigma^2 + \left(u - j - \frac{1}{2}\right)^2} du$$

and

$$\int_{1}^{\frac{3}{2}} g(u) \frac{\sigma}{\sigma^2 + \left(u + j - \frac{1}{2}\right)^2} du = \int_{0}^{\frac{1}{2}} g(u) \frac{\sigma}{\sigma^2 + \left(u + j + \frac{1}{2}\right)^2} du$$

we obtain finally

$$\int_0^\infty \frac{\sigma}{\sigma^2 + u^2} g\left(u + \frac{1}{2}\right) du = \pi \int_0^{\frac{1}{2}} g(\tau) \left[\operatorname{Re} \tanh \pi(\sigma + i\tau)\right] d\tau$$
$$= \frac{\pi}{2} \int_0^1 g(\tau) \left[\operatorname{Re} \tanh \pi(\sigma + i\tau)\right] d\tau .$$

3. Before proceeding, we wish to compare the normalization of the Haar measures as introduced in II.A.1. and III.A.1.ε; in what follows we denote them by $d\mu(a)$ and $d\nu(a)$ resp. Again, it suffices to check the relation between the corresponding measures on G_1. The subsequent computations will be very similar e.g. to those of A.2.a.

In any case we have $d\nu(a) = f(\varphi, u, \varphi')\, d\varphi\, du\, d\varphi'$ where (φ, u, φ') are Eulerian coordinates (II.A.1) (for reasons of convenience, we now write φ, φ' in place of φ_1, φ_2 resp.). Using the invariance of $d\nu(a)$ under left and right translations by elements of O, we conclude, that f does not depend on φ and φ'. Taking the differentials of both sides of $a = o g_u o'$ $(o, o' \in O, g_u \in V)$, multiplying on the left by $a^{-1} = o'^{-1} g_{-u} o^{-1}$ and putting $o = o' = e$ we get

$$d_l a = g_{-u}\, do\, g_u + d_l g_u + do' .$$

Here

$$g_u = \begin{pmatrix} \operatorname{ch} \frac{u}{2} & \operatorname{sh} \frac{u}{2} \\ \operatorname{sh} \frac{u}{2} & \operatorname{ch} \frac{u}{2} \end{pmatrix}$$

and

$$d_l g_u = \frac{1}{2} \begin{pmatrix} 0 & du \\ du & 0 \end{pmatrix}, \quad do = \frac{1}{2} \begin{pmatrix} 0 & -d\varphi \\ d\varphi & 0 \end{pmatrix},$$

and similarly for do'.

This gives

$$2(d_l a)_{21} = \operatorname{ch} u\, d\varphi + du + d\varphi'$$
$$2(d_l a)_{22} = -\operatorname{sh} u\, d\varphi$$
$$2(d_l a)_{12} = -\operatorname{ch} u\, d\varphi + du - d\varphi'$$

whence

$$(d_l a)_{21} \wedge (d_l a)_{22} \wedge (d_l a)_{12} = -\frac{\operatorname{sh} u}{4}(d\varphi \wedge du \wedge d\varphi')$$

or $|f| \equiv \frac{\operatorname{sh} u}{4}$, and $d\nu(a) = \frac{\operatorname{sh} u}{4} d\varphi\, du\, d\varphi'$. Since, on the other hand $d\mu(a) = \frac{1}{(2\pi)^2} \operatorname{sh} u\, d\varphi\, du\, d\varphi'$ we finally conclude that $d\mu(a) = c \cdot d\nu(a)$ with $c = \pi^{-2}$.

4. (3.25) and (3.27) give

$$-2\int_{-\infty}^{\infty} g'(\varphi)\cos\left(l-\frac{1}{2}\right)\varphi\, d\varphi = \frac{1}{2\pi}\int_{-\infty}^{\infty}[F(\sigma,l)+F(\sigma,-l)]\frac{\sigma^2}{\sigma^2+\left(l-\frac{1}{2}\right)^2}d\sigma +$$

$$+\left(l-\frac{1}{2}\right)T_l(f)\,. \tag{3.29}$$

By virtue of P.1 and P.2 in B above, and since $G(\varphi)$ vanishes outside a finite interval, the Fourier transform of $G'(\varphi)$ is summable. Hence, since the first summand on the right hand side, too, is summable in l over $\left[\frac{1}{2}, +\infty\right]$, so is the second. Integrating both sides over this interval, the left hand side gives $\pi^2 f(0)$ (cf. P.2). Observing again, that in the definition of $F(\sigma,\tau)$ (cf. D.1), $f_j(t) \in C_c^\infty$ and $f_j \equiv 0$ but for a finite number of j's, in the first summand on the right hand side we can interchange the order of the two integrations. Keeping σ fixed, and applying the lemma of D.2 with $g(u) \equiv \frac{1}{2}(F(\sigma,u)+F(\sigma,-u))$, integration according to l gives

$$\frac{1}{2}\sigma\int_0^1 F(\sigma,\tau)\,[\operatorname{Re}\tanh\pi(\sigma+i\tau)]\,d\tau$$

whence, taking into account $F(\sigma,\tau) \equiv F(-\sigma,\tau)$, and integrating according to σ we get

$$\int_0^\infty\int_0^1 \sigma F(\sigma,\tau)\,[\operatorname{Re}\tanh\pi(\sigma+i\tau)]\,d\tau\, d\sigma\,.$$

But by virtue of D.3 $\frac{1}{\pi^2}F(\sigma,\tau)$ and $\frac{1}{\pi^2}T_l(f)$ in (3.29) coincide with $T_\sigma^{(\tau)}(f)$ and $T_l(f)$ resp., as these were defined in II.C. — Summing up, integrating (3.29) over $\left[\frac{1}{2}, +\infty\right]$, and dividing both sides with π^2, we finally get

$$f(e) = \int_0^\infty\int_0^1 \sigma[\operatorname{Re}\tanh\pi(\sigma+i\tau)]\,T_\sigma^{(\tau)}(f)\,d\sigma\,d\tau + \int_{\frac{1}{2}}^\infty\left(l-\frac{1}{2}\right)T_l(f)\,dl\,.$$

But this is the same as (2.19), finishing our second proof for the Plancherel theorem.

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