Math. Ann. 247, 81–93 (1980)

Mathematische Annalen

Symmetric Submanifolds of Euclidean Space*

Dirk Ferus

Fachbereich Mathematik der Technischen Universität, Straße des 17. Juni 135, D-1000 Berlin 12

In the late twenties E. Cartan initiated the study of riemannian manifolds which are symmetric with respect to each of their points, i.e. for which the reflexion reversing geodesics through a given point is an isometry. In view of the rapid and rich development of this theory it seems surprising that differential geometers for fifty years overlooked the question for an extrinsic analog. If there exists a (reasonably defined and non-trivial) class of "symmetric" submanifolds in euclidean space, we expect to obtain "very detailed and extensive information about these spaces. They might therefore often serve as examples for the testing of general conjectures. On the other hand, their special nature among submanifolds should be so clear that a properly formulated extrapolation to general submanifolds should often lead to good questions and conjectures". (Adapted from the preface of Helgason [5].)

My aim in this note is twofold. At first I want to define and classify symmetric submanifolds of euclidean space. The classification is done by a trivial reduction to a problem solved earlier [4]. However I want to give a new and simpler version of the algebraic part of my proof in [4]. The second aim is to demonstrate the prominent rôle played by symmetric submanifolds in extrinsic differential geometry. I therefore show that they offer a common frame for various results, e.g. for apparently so divergent topics as tight submanifolds, and the studies of Simons, Chern, Do Carmo, Kobayashi, and Yau on minimal submanifolds of spheres.

1. Definition and Classification of Symmetric Submanifolds

I want to deal with immersed submanifolds. Let M be an n-dimensional riemannian manifold, and $f: M \to \mathbb{R}^{n+p}$ an isometric immersion into euclidean $(n+p)$-space. (Everything will be of class C^∞.) For the second fundamental form of f I shall adopt the notation of [13]. For each $x \in M$ let σ_x denote the reflexion at the normal space $\perp_x^f M$ of f at x, that is the motion of $\mathbb{R}^{n+p}$ which fixes the (affine) space through $f(x)$ normal to $df(T_xM)$, and reflects $f(x)+d_xf(T_xM)$ at $f(x)$.

* To my mother and the memory of my father

0025-5831/80/0247/0081/$02.60

Definition. $f: M \to \mathbb{R}^{n+p}$ is an *(extrinsic) symmetric submanifold (or immersion)*, if for every $x \in M$ there is an isometry i of M into itself such that $i(x)=x$ and $f \circ i = \sigma_x \circ f$.

It is *(extrinsic) locally symmetric*, if every $x \in M$ has a neighborhood U and an isometry i of U into itself, such that $i(x)=x$ and $f \circ i = \sigma_x \circ f$ on U.

Remarks. 1. If f is an imbedding, then $\sigma_x(f(M)) \subset f(M)$ for all x implies global extrinsic symmetry.

2. Symmetric submanifolds are obviously (intrinsic) riemannian symmetric. Hence we are concerned with the problem of "natural" isometric imbeddings of riemannian symmetric spaces.

Examples of Symmetric Submanifolds. Let G be a real connected semisimple Lie group of non-compact type with finite center. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of its Lie algebra, and K the corresponding maximal compact subgroup. Let $0 \neq \eta \in \mathfrak{p}$, and $K_0 := \{k \in K | \mathrm{Ad}(k)\eta = \eta\}$. Then

$$f: M := K/K_0 \to \mathfrak{p}, \qquad [k] \mapsto \mathrm{Ad}(k)\eta$$

is an embedding into the euclidean space $\mathfrak{p}$ with metric given by the Killing form of $\mathfrak{g}$. The riemannian metric induced on M turns M into a riemannian symmetric space, if $(\mathrm{ad}\,\eta)^3 = \mathrm{ad}\,\eta$. M is then called a *symmetric R-space*, and f its standard imbedding, see [11] for explicit descriptions. If f is followed by an affine conformal map into some euclidean space, this composition also will be called a standard imbedding. Symmetric R-spaces are classified in [12, 23]. They include all riemannian symmetric spaces of the compact type but some of the exceptional ones. The standard imbeddings for the hermitian symmetric spaces are those given in [18, 7], while for the classical groups they coincide with the natural imbeddings into the space of matrices of the same size. The Grassmannians are imbedded as idempotents of constant rank in the spaces of self-adjoint maps. For the projective spaces in particular one gets the *Veronese manifolds*, [6, 19, 30].

I shall now show that standard imbedded symmetric R-spaces are symmetric submanifolds. Since $\mathrm{ad}\,\eta$ is a derivation of $\mathfrak{g}$, we have a vector space decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 := \ker(\mathrm{ad}\,\eta)$, and $\mathfrak{g}_1$ is the 1-eigenspace of $(\mathrm{ad}\,\eta)^2$. Moreover $\mathfrak{g}_0$ is a subalgebra, and $[\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1$, $[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0$. It follows that the reflexion $s: x_0 + x_1 \mapsto x_0 - x_1$, $x_i \in \mathfrak{g}_i$, defines an involutive automorphism of $\mathfrak{g}$, commuting with the Cartan involution corresponding to $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Tangent and normal space of f at the base point $[e] \in K/K_0$ are $\mathfrak{g}_1 \cap \mathfrak{p}$ and $\mathfrak{g}_0 \cap \mathfrak{p}$ respectively, so that $s|\mathfrak{p}$ is the reflexion at the normal space. (Note $\eta = f([e]) \in \mathfrak{g}_0 \cap \mathfrak{p}$.) $\mathfrak{g}_1 \cap \mathfrak{k}$ represents the tangent space of M at $[e]$, and for $X \in \mathfrak{g}_1 \cap \mathfrak{k}$ one has

$$\begin{aligned} \mathrm{Ad}(\exp - tX)\eta &= \mathrm{Ad}(\exp ts(X))\eta \\ &= e^{t\,\mathrm{ad}\,s(X)}\eta \\ &= e^{ts \circ \mathrm{ad}\,X \circ s}\eta \\ &= s \circ e^{t\,\mathrm{ad}\,X} \circ s(\eta) \\ &= s(\mathrm{Ad}(\exp tX)\eta). \end{aligned}$$

This proves the symmetry of the standard imbeddings.

I want to prove a converse of this fact: All (irreducible) symmetric submanifolds are standard imbedded symmetric R-spaces. To do so, I need a formulation of the symmetry condition that is more apt to calculations. For intrinsic symmetric submanifolds this is the condition $DR=0$. For symmetric submanifolds we obtain:

Lemma 1. *Locally symmetric submanifolds have covariantly constant second fundamental form.*

Proof. Let $x \in M$. Let $(\ )^{\perp}$ denote the orthogonal projection onto the normal space $\perp_x^f M$. Let $X \in T_x M$, and $c: I \to M$ be a geodesic such that $c'(0)=X$. If $\tilde{c} := f \circ c$ then, with the usual identifications,

$$\begin{aligned} \tilde{c}'(0) &= df(X) \\ \tilde{c}''(0) &= \alpha(X,X) \\ \tilde{c}'''(0) &= -df(A_{\alpha(X,X)}X) + (D_X\alpha)(X,X), \end{aligned} \tag{1}$$

where $D\alpha$ denotes the normal covariant derivative of α. Now the local symmetry of f implies $\tilde{c}^{\perp}(-t)=\tilde{c}^{\perp}(t)$. Hence $\tilde{c}'''(0)^{\perp}=0$, showing $(D_x\alpha)(X,X)=0$. Since $D\alpha$ is symmetric by Codazzi's equation, it follows $D\alpha=0$.

The converse of Lemma 1 is true. This can be seen from the classification given below. A direct proof is due to Strübing [36]. The classification of submanifolds with $D\alpha=0$ was done in [2–4]. Here I shall give an outline including details of a simplification of the central construction.

The covariant parallelity of α as well as our extrinsic symmetry are hereditary under direct products and direct factorization of immersions. I therefore restrict my attention to an irreducible symmetric f, such that $f(M)$ is not contained in an affine hyperplane of $\mathbb{R}^{n+p}$. Moreover let M be connected. According to Lemma 1 and [3], f is then a minimal immersion into some hypersphere of $\mathbb{R}^{n+p}$.

Theorem 1. *Under the above assumptions $f(M)$ is an open part of a standard imbedded symmetric R-space.*

The proof has two steps. In the first step I construct a standard imbedded symmetric R-space in $\mathbb{R}^{n+p}$ which has a point in common with $f(M)$, and such that the tangent spaces and second fundamental forms of both spaces agree at this point. In the second step, for which the reader is referred to [4], it is shown that the parallelity of both second fundamental forms implies then, that the two spaces coincide locally.

The basic algebraic fact deduced from $D\alpha=0$ is

Lemma 2. *Let $x \in M$. On $T := T_x M$ define a trilinear multiplication by*

$$\{X, Y, Z\} := R(X,Y)Z + A_{\alpha(X,Y)}Z.$$

For $U, V \in T$ put $L(U,V) := \{U, V, \cdot\} : T \to T$.

Then for all $X, Y, Z, U, V \in T$

$$\{X, Y, Z\} = \{Z, Y, X\} \tag{2}$$

$$\begin{aligned} L(U,V)\{X,Y,Z\} = \{L(U,V)X, Y, Z\} &- \{X, L(V,U)Y, Z\} \\ &+ \{X, Y, L(U,V)Z\}. \end{aligned} \tag{3}$$

This means that $\{\ldots\}$ turns T into a Jordan triple system, see [22].

Proof. (2) follows immediately from the Gauss equation

$$R(X,Y)Z = A_{\alpha(Y,Z)}X - A_{\alpha(X,Z)}Y.$$

To verify (3) we note that $D\alpha=0$ and the local symmetry $DR=0$ of M imply

$$R(U,V)\cdot R=0 \quad \text{and} \quad R(U,V)\cdot A_\alpha=0$$

for all $U, V\in T$. Here $R(U,V)\cdot$ denotes the natural extension of the curvature tensor. This implies

$$R(U,V)\cdot\{\ldots\}=0$$

which is (3) with $R(U,V)$ instead of $L(U,V)$.

[Note $R(V,U)=-R(U,V)$.]

We are therefore left to show (2) with $A_{\alpha(U,V)}$ instead of $L(U,V)$. Put $\xi:=\alpha(U,V)$. Then the left hand side minus the right hand side of (2) equals

$$\begin{aligned}
&= A_\xi R(X,Y)Z - R(A_\xi X,Y)Z + R(X,A_\xi Y)Z - R(X,Y)A_\xi Z\\
&\quad + A_\xi A_{\alpha(X,Y)}Z - A_{\alpha(A_\xi X,Y)}Z + A_{\alpha(X,A_\xi Y)}Z - A_{\alpha(X,Y)}A_\xi Z\\
&= -(R(X,Y)A_\xi Z - A_{-\alpha(A_\xi X,Y)+\alpha(X,A_\xi Y)}Z - A_\xi R(X,Y)Z)\\
&\quad + [A_\xi, A_{\alpha(X,Y)}]Z - R(A_\xi X,Y)Z + R(X,A_\xi Y)Z\\
&= -(R(X,Y)A_\xi Z - A_{R(X,Y)\xi}Z - A_\xi R(X,Y)Z)\\
&\quad + [A_\xi, A_{\alpha(X,Y)}]Z - R(A_\xi X,Y)Z + R(X,A_\xi Y)Z
\end{aligned}$$

(by the normal Gauss equation)

$$\begin{aligned}
&= -(R(X,Y)\cdot A)_\xi Z\\
&\quad + [A_\xi, A_{\alpha(X,Y)}]Z - R(A_\xi X,Y)Z + R(X,A_\xi Y)Z.
\end{aligned}$$

The first term equals 0 as we noted above. To show that the other terms cancel, we multiply by $W\in T$:

$$\begin{aligned}
&\langle [A_\xi, A_{\alpha(X,Y)}]Z - R(A_\xi X,Y)Z + R(X,A_\xi Y)Z, W\rangle\\
&\quad = \langle R(Z,W)\xi, \alpha(X,Y)\rangle - \langle R(Z,W)A_\xi X, Y\rangle + \langle R(Z,W)X, A_\xi Y\rangle\\
&\quad = \langle R(Z,W)\xi, \alpha(X,Y)\rangle - \langle A_{R(Z,W)\xi}X, Y\rangle - \langle A_\xi R(Z,W)X, Y\rangle\\
&\qquad + \langle R(Z,W)X, A_\xi Y\rangle
\end{aligned}$$

[because $R(Z,W)\cdot A=0$]

$$\begin{aligned}
&= \langle R(Z,W)\xi, \alpha(X,Y)\rangle - \langle R(Z,W)\xi, \alpha(X,Y)\rangle\\
&\quad - \langle A_\xi R(Z,W)X, Y\rangle + \langle A_\xi R(Z,W)X, Y\rangle = 0.
\end{aligned}$$

This finishes the proof of Lemma 2.

The trace form of the Jordan triple system $(T,\{\ldots\})$ is defined as

$$\lambda(X,Y) := \operatorname{trace}(L(X,Y)+L(Y,X)).$$

Above we made the assumption that f is minimal into a hypersphere. Let r denote its radius.

Lemma 3. *For all* $X, Y \in T$

$$\lambda(X, Y) = \frac{2n}{r^2}\langle X, Y\rangle.$$

Hence λ *is positive definite, which by definition says that* $(T, \{\ldots\})$ *is formal real* (= *compact*).

Proof. Let (X_i) be an orthonormal basis of T.

$$\begin{aligned}\lambda(X,X) &= 2\operatorname{trace} L(X,X)\\ &= 2\sum\langle R(X,X)X_i + A_{\alpha(X,X)}X_i, X_i\rangle\\ &= 2\sum\langle\alpha(X,X), \alpha(X_i,X_i)\rangle\\ &= 2\langle\alpha(X,X), n\eta\rangle \quad (\eta = \text{the mean curvature normal})\\ &= 2n\langle A_\eta X, X\rangle.\end{aligned}$$

By assumption on f $A_\eta = \frac{1}{r^2}\mathrm{Id}$. Hence

$$\lambda(X,X) = \frac{2n}{r^2}\langle X,X\rangle.$$

Now there is a well-known relation between formal real Jordan triple systems and symmetric R-spaces based on the construction of a semi-simple Lie algebra from such a triple system, see [21]. This construction is due to Tits [31] and Koecher [15] for Jordan algebras, and to Meyberg [22] for triple systems.

Let L be the subspace of $\operatorname{End}(T)$ spanned by $\{L(X, Y) | X, Y \in T\}$.

Put

$$\mathfrak{g} := T \oplus L \oplus T,$$

and define a Lie bracket by

$$\begin{aligned}&[(X, F, Y), (\tilde{X}, \tilde{F}, \tilde{Y})]\\ &:= (F\tilde{X} - \tilde{F}X, [F, \tilde{F}] - \tfrac{1}{2}L(X, \tilde{Y}) + \tfrac{1}{2}L(\tilde{X}, Y), \tilde{F}^t Y - F^t\tilde{Y}).\end{aligned}$$

Here $[F, \tilde{F}] := F\tilde{F} - \tilde{F}F$, and F^t is the adjoint of F with respect to the metric $\langle \ldots, \ldots\rangle$.

Lemma 4. *With the above definitions we have:*

(i) $(\mathfrak{g}, [\,,])$ *is a Lie algebra.*

(ii) $\mathfrak{g}$ *is semi-simple, and a Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ *is given by*

$$\mathfrak{k} := \operatorname{span}\{(X, R(U, V), X) | X, U, V \in T\}$$

$$\mathfrak{p} := \operatorname{span}\{(X, A_{\alpha(U,V)}, -X) | X, U, V \in T\}.$$

(iii) *If* η *is the mean curvature normal of* f *at* x, *then for*

$$\bar{\eta} := (0, A_{-r^2\eta}, 0) = (0, -\mathrm{Id}, 0) \in \mathfrak{p}$$

we have

$$(\mathrm{ad}\,\bar\eta)^3 = \mathrm{ad}\,\bar\eta\,.$$

(iv) *If* $\bar f:\bar M := K/K_0 \to \mathfrak{p}$ *is the standard imbedded symmetric R-space determined by the data* (i), (ii), (iii), *then*

$$d\bar f(T_{[e]}\bar M) = \{(X,0,-X) | X \in T\}\,,$$

and the second fundamental form of $\bar f$ *is given by*

$$\bar\alpha((X,0,-X),(X,0,-X)) = (0, A_{\alpha(X,X)}, 0)\,.$$

(v) *For* $(X, A_\xi, -X)$, $(\tilde X, A_{\tilde\xi}, -\tilde X)$ *the Killing form of* $\mathfrak{g}$ *is given by*

$$\langle (X, A_\xi, -X), (\tilde X, A_{\tilde\xi}, -\tilde X)\rangle = \frac{2n}{r^2}(\langle X,\tilde X\rangle + \langle \xi, \tilde\xi\rangle)\,.$$

Proof. For (i), (ii) see [22] and [16]. In the latter paper the Killing form of $\mathfrak{g}$ is computed:

$$\begin{aligned}\langle (X,F,Y),(\tilde X,\tilde F,\tilde Y)\rangle_{\mathfrak{g}} &= \langle F,\tilde F\rangle_L + 2\,\mathrm{tr}\,(F\circ\tilde F) - \tfrac{1}{2}(\lambda(X,\tilde Y)+\lambda(Y,\tilde X))\\ &= \lambda(F,\tilde F) - \tfrac{1}{2}(\lambda(X,\tilde Y)+\lambda(Y,\tilde X))\,,\end{aligned}$$

where $\lambda(F,\tilde F)$ is the bilinear extension of

$$\lambda(L(U,V), L(X,Y)) := \lambda(L(U,V)X, Y)\,.$$

This is used to show that $\mathfrak{k}+\mathfrak{p}$ is a Cartan decomposition. (iii) is a trivial computation which also shows that $\{(X,0,-X)|X\in T\}$ is the fixed point set of $(\mathrm{ad}\,\bar\eta)^2$ on $\mathfrak{p}$. But this fixed point set is $d\!f(T_{[e]}\bar M)$ as remarked earlier. Let $(X,0,-X)\in T$, and consider $c:\mathbb{R}\to\mathfrak{p}$, $t\mapsto \mathrm{Ad}\,(\exp t(X,0,X))\bar\eta$.

Then

$$c'(0) = \mathrm{ad}\,(X,0,X)\bar\eta = (X,0,-X)\,,$$

and

$$c''(0) = (\mathrm{ad}\,(X,0,X))^2\bar\eta = (0, A_{\alpha(X,X)}, 0)$$

proving (iv). From the above formula for the Killing form (v) is obvious if $\xi = 0$ or $\tilde\xi = 0$. On the other hand, for $X, Y\in T$, $\xi = \alpha(X,Y)$ we have

$$\begin{aligned}\langle (0,A_\xi,0),(0,A_\xi,0)\rangle_{\mathfrak{g}} &= \langle [(X,0,X),(Y,0,-Y)],(0,A_\xi,0)\rangle_{\mathfrak{g}}\\ &= -\langle (Y,0,-Y),[(X,0,X),(0,A_\xi,0)]\rangle_{\mathfrak{g}}\\ &= -\langle (Y,0,-Y),(-A_\xi X,0,A_\xi X)\rangle_{\mathfrak{g}}\\ &= \frac{1}{2}(\lambda(Y,A_\xi X)+\lambda(-Y,-A_\xi X))\\ &= \frac{2n}{r^2}\langle \alpha(X,Y),\xi\rangle\\ &= \frac{2n}{r^2}\langle \xi,\xi\rangle\,.\end{aligned}$$

This proves (v).

We finally identify $\mathbb{R}^{n+p} = T_x M \oplus \perp_x^f M$ and $\mathfrak{p}$ by

$$(X, \xi) \mapsto (X, A_\xi, -X).$$

According to (v) of the preceeding lemma this is a homothetic isomorphism, which maps $f(x) = -r^2\eta$ into $\bar{\eta} = \bar{f}([e])$, the tangent space $df(T_x M)$ onto $d\bar{f}(T_{[e]}\bar{M})$, and preserves the second fundamental form at x.

This completes the construction of the model symmetric R-space, which in the second step of the proof (see [4]) is shown to coincide locally with the given submanifold. This fact can also immediately be seen from [36].

From Theorem 1 and Lemma 1 we obtain

Theorem 2. *If $f: M \to \mathbb{R}^{n+p}$ is a connected, locally symmetric submanifold, then $f(M)$ is an open part of a standard imbedded symmetric R-space, or of an affine subspace, or of a product of such spaces.*

For later use I conclude this section with the determination of all flat symmetric submanifolds.

Theorem 3. *A connected, flat, locally symmetric submanifold $f: M \to \mathbb{R}^{n+p}$ maps M onto an open part of a Clifford torus or of an orthogonal cylinder over a Clifford torus. (A Clifford torus is the product of plane circles, not necessarily of the same radii.)*

Proof. Let f be irreducible and not totally geodesic, $f(M)$ not contained in any hyperplane. Then, with the notations of the proof of Theorem 1, we have

$$L = \{A_\xi | \xi \text{ normal vector at } x\}.$$

But L is a Lie algebra, see [22], and therefore $[A_\xi, A_{\tilde{\xi}}] = 0$ for all normal $\xi, \tilde{\xi}$. But then the normal bundle is flat. This together with the irreducibility of f implies that $\dim M = 1$, and $f(M)$ is a plane circle, see [3].

2. The Geometry of Symmetric Submanifolds

In this section I want to give a catalogue of results, most of them known, in which symmetric submanifolds play a central rôle. I shall outline proofs which emphasize the common core of these theorems.

Kähler Submanifolds. The first result seems to be new. Studying real isometric immersions of Kähler manifolds it is natural to ask for ones that in some reasonable sense respect the complex structure. One possible condition to look for would be

$$\alpha(JX, JY) = -\alpha(X, Y)$$

for all tangent vectors X, Y. This condition however is satisfied by every complex analytic submanifold of $\mathbb{C}^m$ considered as a Kähler manifold in $\mathbb{R}^{2m}$. Hence there is not enough rigidity for strong results. The more surprising is

Theorem 4. *Let $f: M \to \mathbb{R}^{n+p}$ be an isometric immersion of a connected Kähler manifold with complex structure J and second fundamental form α. Suppose*

$$\alpha(JX, JY) = \alpha(X, Y) \tag{4}$$

for all X, Y. Then $f(M)$ is an open part of a standard imbedded hermitian symmetric R-space, of an affine subspace, or of a product of such spaces.

Proof. Equation (4) is equivalent with

$$\alpha(JX, Y) = -\alpha(X, JY). \tag{5}$$

Differentiation using $DJ = 0$ yields

$$D_Z\alpha(JX, Y) = -D_Z\alpha(X, JY). \tag{6}$$

Therefore

$$\begin{aligned} D_Z\alpha(X, Y) &= D_Y\alpha(X, Z) = -D_Y\alpha(X, J^2Z) \\ &\underset{(6)}{=} D_Y\alpha(JX, JZ) = D_{JZ}\alpha(JX, Y) \\ &\underset{(6)}{=} -D_{JZ}\alpha(X, JY) = -D_X\alpha(JZ, JY) \\ &\underset{(6)}{=} D_X\alpha(J^2Z, Y) = -D_X\alpha(Z, Y) = -D_Z\alpha(X, Y). \end{aligned}$$

Hence $D\alpha = 0$.

Remarks. 1. The J-invariance (4) of α was first studied by Rettberg [26], see Theorem 8 below.

2. According to [12], p. 877, the complex structure on a standard imbedded R-space $M = K/K_0$ with canonically decomposed Lie algebra $\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{m}$ is given by $JX = [I\eta, X]$ for $X \in \mathfrak{m}$ where I is a complex structure on $\mathfrak{g}$. But an easy computation shows that $\alpha(X, X) = [X, [X, \eta]]$, see [2]. From this we conclude that the standard imbedded hermitian symmetric R-spaces really satisfy (4).

Minimal Submanifolds of the Sphere. Simons' paper [28] initiated an extensive study of the Laplacian of $\|\alpha\|^2$, and of submanifolds of constant mean curvature in the sphere. If $f: M \to \mathbb{R}^{n+p}$ has image contained in the unit sphere in $\mathbb{R}^{n+p}$, let h denote the second fundamental form with respect to the sphere. Note that $D\alpha = 0$, if and only if $Dh = 0$. One has

$$\Delta\langle h, h\rangle = 2\langle h, \Delta h\rangle + 2\langle Dh, Dh\rangle,$$

and integration over a compact M yields

$$0 \leqq \int \langle Dh, Dh\rangle = -\int \langle h, \Delta h\rangle. \tag{7}$$

Let M be minimal in the sphere, and put $S := \langle h, h\rangle$. Simons showed

$$-\langle h, \Delta h\rangle \leqq S\left(\left(2 - \frac{1}{p-1}\right)S - n\right).$$

Hence, if $S \leqq n \Big/ \left(2 - \frac{1}{p-1}\right)$ everywhere on M, then $Dh=0$ (and therefore $D\alpha=0$), and either $S=0$ (i.e. f is totally geodesic into the sphere), or $S = n \Big/ \left(2 - \frac{1}{p-1}\right)$. In the latter case one can show $p \leqq 3$, see [1]. A view on the list of standard imbedded symmetric R-spaces in [11] shows that spheres and the 2-dimensional Veronese in euclidean 5-space are the only possible irreducible examples. This is the main result of Chern et al. [1], who give a direct proof of this fact.

Opposite to Simon's inequality Yau [35] proved

$$-\langle h, \Delta h\rangle \leqq S\left(n - 2nK - \frac{1}{p-1} S\right),$$

where K denotes the minimal sectional curvature at the point considered. If $S \geqq (p-1)n(1-2K)$ everywhere on M, then we see from (5), that again $Dh=0$, and $S=0$ or $S=(p-1)n(1-2K)$, where K is now a constant on M. Yau shows, that $K=0$ only for a product of spheres. Since symmetric R-spaces are of compact type, we have $K \geqq 0$, and the only possible case left is that of a standard imbedded symmetric space of rank 1, a Veronese. Computation of S for the Veronese manifolds shows, that $S=(p-1)n(1-2K)$ only for the real case. This proves a conjecture of [35]:

Theorem 5 (Yau). *A compact minimal submanifold of dimension n in S^{n+q} with*

$$S \geqq qn(1-2K)$$

is a product of spheres (of appropriate radii) or a real Veronese manifold.

Submanifolds with Plane Geodesics. Hong [8] was the first to ask for all submanifolds of euclidean space, whose geodesics are plane curves. He showed the following

Lemma 5. *If M is connected, and $f: M \to \mathbb{R}^{n+p}$ an isometric immersion which is not totally geodesic, and such that for every geodesic c in M $f \circ c$ is a plane curve in $\mathbb{R}^{n+p}$, then $f \circ c$ is a plane circle.*

Proof. From (1) in the proof of Lemma 1 we see that $A_{\alpha(X,X)}X \in \mathbb{R}X$, and $D_X\alpha(X,X)$ and $\alpha(X,X)$ are linearly dependent for each tangent vector X. Hence $\langle X, Y\rangle = 0$ implies

$$\langle \alpha(X,X), \alpha(X,Y)\rangle = \langle A_{\alpha(X,X)}X, Y\rangle = 0.$$

So the map $X \mapsto \langle \alpha(X,X), \alpha(X,X)\rangle$ is constant on each unit tangent sphere, that is all geodesic emanating from a point have the same curvature in $\mathbb{R}^{n+p}$. Now, if $\langle X, Y\rangle = 0$, and X, Y are parallel unit fields along geodesics emanating from $x \in M$, then at x

$$\begin{aligned} X\cdot\langle\alpha(X,X),\alpha(X,X)\rangle &= X\cdot\langle\alpha(Y,Y),\alpha(Y,Y)\rangle \\ &= 2\langle(D_X\alpha)(Y,Y),\alpha(Y,Y)\rangle \\ &= 2\langle(D_Y\alpha)(X,Y),\alpha(Y,Y)\rangle \\ &= 2Y\cdot\langle\alpha(X,Y),\alpha(Y,Y)\rangle \\ &\quad -2\langle\alpha(X,Y),(D_Y\alpha)(Y,Y)\rangle \\ &= 0. \end{aligned}$$

Hence $t\mapsto f\circ\exp tX$ is of constant curvature. This proves the lemma.

But f is obviously locally symmetric, if all geodesics are mapped into circles. The latter property is not preserved under direct products. In particular, in view of Theorem 3, such a manifold cannot contain a flat totally geodesic submanifold of dimension greater than 1. This yields

Theorem 6. *A connected immersed submanifold $f: M\to\mathbb{R}^{n+p}$ which is not totally geodesic, and maps geodesics into plane circles, maps M onto an open part of a standard imbedded symmetric R-space of rank one, i.e. onto a sphere or a Veronese manifold. Conversely all these submanifolds have plane geodesics.*

The last statement is proved in Little [19], who also gives an elementary direct proof of Theorem 6. That theorem was independently found by Nomizu and this author. A weaker result was obtained by Hong [8].

Submanifolds with Isometric α. Steiner [29] considers submanifolds of spheres for which $h: T_xM\otimes T_xM\to\perp_x^f M$ yields an isometry of the symmetric tensor product i.e.

$$\langle h(X,Y),h(U,V)\rangle=\langle X,U\rangle\langle Y,V\rangle+\langle X,V\rangle\langle Y,U\rangle.$$

It follows immediately that

$$\langle D_Z h(X,Y),h(U,V)\rangle=0$$

for all tangent vectors. He then assumes the spherical normal space to be of dimension $=\frac{1}{2}n(n+1)$, which implies it to be spanned by $\{h(U,V)|U,V \text{ tangent}\}$.

Therefore $Dh=0$. Again from the isometry condition we conclude

$$\langle h(X,X),h(X,Y)\rangle=\langle\alpha(X,X),\alpha(X,Y)\rangle=0$$

for orthonormal X,Y. From (1) we see that the geodesics are plane curves, and Theorem 6 together with the information on the codimension restrict the possibilities to real Veronese manifolds, which in fact have the assumed isometry property.

0-Tight Submanifolds. A compact, connected immersion $f: M\to\mathbb{R}^{n+p}$ is called *0-tight*, if for almost all $\xi\in\mathbb{R}^{n+p}$ the height function $\xi f: M\to\mathbb{R}$, $x\mapsto\langle\xi,f(x)\rangle$ has only one local minimum. I shall use the notation ξf also for normal vectors ξ of f, using the canonical identification. Then x is a critical point of ξf, if and only if

$\xi \in \perp_x^f M$. In this case $\text{Hess}_x \xi f = A_\xi$. A point $x \in M$ is called an *extreme point*, if there exists $\xi \in \perp_x^f M$ such that A_ξ is positive definite.

The following lemma is well-known, see for example [20].

Lemma 6. *Let $f: M \to \mathbb{R}^{n+p}$ be 0-tight, and $x \in M$ an extreme point. Let ξ be normal at x. Then $\xi f \geqq \xi f(x)$ if and only if $A_\xi \geqq 0$. If f is substantial in the sense that $f(M)$ is not contained in an affine hyperplane, then $\zeta \mapsto A_\zeta$ is injective on the normal space, and the image of α is the full normal space.*

It follows that the codimension of a 0-tight substantial immersion of an n-manifold is $\leqq \frac{1}{2}n(n+1)$, which is the dimension of the space of symmetric endomorphisms on a tangent space of M.

Theorem 7 (Kelly [10]). *Let G/K be a compact riemannian symmetric space, and $\pi: G \to SO(n+p)$ a real class one representation such that π induces a 0-tight isometric immersion $f: G/K \to \mathbb{R}^{n+p}$, $[g] \mapsto \pi(g)\eta$ for some $\eta \in \mathbb{R}^{n+p}$. Then f is extrinsic symmetric.*

Proof. α and $D\alpha$ are computed easily in terms of $d\pi$, and one obtains

$$\langle D_Z \alpha(X, Y), \alpha(U, V) \rangle = 0$$

for all X, Y, Z, U, V, see Lemma 6.1 of [10]. We now restrict our attention to the affine space spanned by $f(M)$, that is, we assume f to be substantial. Since $f(M)$ is contained in a hypersphere, every point is extreme, whence $D\alpha = 0$ by Lemma 6. By Theorem 1 f covers a globally symmetric submanifold, and is therefore itself globally symmetric.

Conversely compact connected symmetric submanifolds are 0-tight, as an easy geometric argument shows. Kobayashi and Takeuchi proved that they are in fact not only 0-tight but tight [14].

The Kuiper-Little-Pohl theorem ([17, 20]) states that a 0-tight submanifold of maximal codimension, i.e. of codimension $\frac{1}{2}n(n+1)$, n the dimension of the submanifold, is projectively equivalent with a real Veronese manifold. Looking for an analogous characterization of the complex Veronese, one can try to make the appropriate codimension (namely n^2 for a manifold of real dimension $2n$) the maximum possible. This is achieved by the additional assumption that M is hermitian (not necessarily kählerian), and α is J-invariant in the sense of (4). Then Rettberg [26] could show by a nice induction argument on the dimension, that all geodesics are plane curves, which led him to

Theorem 8 (Rettberg [26]). *Let $f: M \to \mathbb{R}^{2n+p}$ be an isometric immersion of a compact connected hermitian manifold of real dimension $2n$, which satisfies (4). If f is substantial, 0-tight, and of codimension $p = n^2$, then $f(M)$ is a complex Veronese.*

It might be possible to give a similar proof for the real case under the additional assumption that $f(M)$ lies in a sphere (or is 0-taut).

Remark. One may be tempted to study symmetric submanifolds in more general spaces, which of course must admit sufficiently many symmetries themselves. The spherical case coincides with the euclidean, but I have not thought about the hyperbolic case. Kähler submanifolds with $D\alpha = 0$ in complex projective space have been considered in [25, 24].

Acknowledgements. I want to thank N. Kuiper for pointing out an error in a previous version of this paper, and H. Reckziegel for several helpful comments.

References

1. Chern, S.S., Do Carmo, M., Kobayashi, S.: Minimal submanifolds of a sphere with second fundamental form of constant length. Functional analysis and related fields, ed. F. E. Browder. Berlin, Heidelberg, New York: Springer 1970
2. Ferus, D.: Immersionen mit paralleler zweiter Fundamentalform: Beispiele und Nicht-Beispiele. manuscripta math. **12**, 153–162 (1974)
3. Ferus, D.: Produkt-Zerlegungen von Immersionen mit paralleler zweiter Fundamentalform. Math. Ann. **211**, 1–5 (1974)
4. Ferus, D.: Immersions with parallel second fundamental form. Math. Z. **140**, 87–93 (1974)
5. Helgason, S.: Differential geometry and symmetric spaces. New York, London: Academic Press 1962
6. Hirzebruch, U.: Über Jordan-Algebren und kompakte Riemannsche symmetrische Räume vom Rang 1. Math. Z. **90**, 339–354 (1965)
7. Hirzebruch, U.: Über eine Realisierung der hermiteschen, symmetrischen Räume. Math. Z. **115**, 371–382 (1970)
8. Hong, S.L.: Isometric immersions of manifolds with plane geodesics into euclidean space. J. Differential Geometry **8**, 259–278 (1973)
9. Houh, C.S.: Pseudo-umbilical surfaces with parallel second fundamental form. Tensor **26**, 262–266 (1972)
10. Kelly, E.F.: Tight equivariant imbeddings of symmetric spaces. J. Differential Geometry **7**, 535–548 (1972)
11. Kobayashi, S.: Isometric imbeddings of compact symmetric spaces. Tôhoku Math. J. **20**, 21–25 (1968)
12. Kobayashi, S., Nagano, T.: On filtered Lie algebras and geometric structures. I. J. Math. Mech. **13**, 875–907 (1964)
13. Kobayashi, S., Nomizu, K.: Foundations of differential geometry. II. New York, London, Sydney: Interscience 1969
14. Kobayashi, S., Takeuchi, M.: Minimal imbeddings of *R*-spaces. J. Differential Geometry **2**, 203–215 (1968)
15. Koecher, M.: Imbedding of Jordan algebras into Lie algebras. II. Amer. J. Math. **89**, 787–816 (1967)
16. Koecher, M.: Gruppen und Lie-Algebren von rationalen Funktionen. Math. Z. **109**, 349–392 (1969)
17. Kuiper, N.: On convex maps. Nieuw Arch. Wisk. **10**, 147–164 (1962)
18. Lichnerowicz, A.: Géométrie des groupes de transformations. Paris: Dunod 1958
19. Little, J.A.: Manifolds with planar geodesics. J. Differential Geometry **11**, 265–285 (1976)
20. Little, J.A., Pohl, W.: On tight immersions of maximal codimension. Inventiones math. **13**, 179–204 (1971)
21. Loos, O.: Jordan triple systems, *R*-spaces, and bounded symmetric domains. Bull. Amer. Math. Soc. **77**, 558–561 (1971)
22. Meyberg, K.: Jordan-Tripelsysteme und die Koecher-Konstruktion von Lie-Algebren. Math. Z. **115**, 58–78 (1970)
23. Nagano, T.: Transformation groups on compact symmetric spaces. Trans. Amer. Math. Soc. **118**, 428–453 (1965)
24. Nakagawa, H., Takagi, R.: On locally symmetric Kaehler submanifolds in a complex projective space. J. Math. Soc. Japan **28**, 638–667 (1976)
25. Nomizu, K.: Eine Kennzeichnung der projektiven Unterräume in dem komplexen projektiven Raum nach ihren Geodätischen. Math. Z. **137**, 147–150 (1974)

26. Rettberg, M.: Immersionen komplexer Mannigfaltigkeiten in einen euklidischen Raum. Diplomarbeit, Bielefeld 1978
27. Simon, U., Weinstein, A.: Anwendungen der de Rham Zerlegung auf Probleme der lokalen Flächentheorie. manuscripta math. **1**, 139–146 (1969)
28. Simons, J.: Minimal varieties in riemannian manifolds. Ann. of Math. **88**, 62–105 (1968)
29. Steiner, S.: Eine lokale Charakterisierung der Veronesemannigfaltigkeiten. manuscripta math. **23**, 155–160 (1978)
30. Tai, S.S.: Minimum imbeddings of compact symmetric spaces of rank one. J. Differential Geometry **2**, 55–66 (1968)
31. Tits, J.: Une classe d'algebras de Lie en relation avec les algebras de Jordan. Koninkl. Nederlandsche Akademie van Wetenscappen, Amsterdam **65**, 530–535 (1962)
32. Vilms, J.: Submanifolds of euclidean space with parallel second fundamental form. Proc. Amer. Math. Soc. **32**, 263–267 (1972)
33. Walden, R.: Untermannigfaltigkeiten mit paralleler zweiter Fundamentalform in euklidischen Räumen und Sphären. manuscripta math. **10**, 91–102 (1973)
34. Yano, K., Ishihara, S.: Submanifolds of codimension 2 or 3 with parallel second fundamental tensor. J. Korean Math. Soc. **9**, 1–11 (1972)
35. Yau, S.T.: Submanifolds with constant mean curvature. II. Amer. J. Math. **97**, 76–100 (1975)
36. Strübing, W.: Symmetric submanifolds of riemannian manifolds. Math. Ann. **245**, 37–44 (1979)

Received January 15, 1979; in revised form October 2, 1979