Math. Ann. 189, 311—316 (1970)

Rings for which Every Cyclic Module is Quasi-Projective

ANNE KOEHLER

Introduction

Let R be a ring with an identity. A ring R will be called a left (right) q^*-ring if every R-homomorphic image of R as a left (right) R-module is quasi-projective, that is, if every cyclic left (right) R-module is quasi-projective. The study of rings with the dual property (rings in which every left (right) ideal is quasi-injective) was begun by Jain, Mohamed, and Singh [5]. They call these rings with the dual property left (right) q-rings. One object of this paper is to see if some of the results for q-rings are also true for q^*-rings. S. Mohamed has a paper [10] in which the goal is to prove the following theorem: A left (right) Artinian ring is a left q-ring if and only if it is a right q-ring. The problem was suggested to him by C. Faith. By using the notion of a q^*-ring, a short proof of Mohamed's theorem is obtained in Section 3. In addition it is seen that Mohamed's theorem is true in a more general case.

1. Background

In this paper all modules are unital modules, and homomorphisms are R-homomorphisms. The Jacobson radical will be denoted by N. The results in this section will be stated for left R-modules. A module A is *large* in M if $B \cap A \neq 0$ for every nonzero submodule B of M. A module A is *small* in M if whenever $A + B = M$ for a submodule B of M, $B = M$. A projective module P is a *projective cover* of M if there is an epimorphism $\phi : P \to M$ such that $\operatorname{Ker}\phi$ is small in P. An injective module Q is the *injective hull* of M if there is a monomorphism $j : M \to Q$ such that $j(M)$ is large in Q. A ring is *semi-perfect* if every finitely generated module has a projective cover. Bass [1] has characterized these rings by

Theorem 1.1. *A ring R is semi-perfect if and only if R/N is semisimple Artinian, and idempotents modulo N can be lifted.*

It follows from the proof of this theorem that if R is semi-perfect, $R = Re_1 + \cdots + Re_n$ where $e_1, \ldots, e_n$ are orthogonal indecomposable idempotents, Re_i/Ne_i is simple, and Ne_i is a unique maximal left ideal in Re_i.

A module M is *quasi-projective* if for every epimorphism $q : M \to A$, $\operatorname{Hom}(M, A) = q \circ H(M, M)$. Miyashita [9, p. 92—93] and Wu and Jans [16, p. 440] have proved.

Theorem 1.2. *Let P be a projective module, $\phi: P \to M$ be an epimorphism, and $S = \text{End}(P)$. Then* i) *M is quasi-projective if* $\text{Ker}\,\phi$ *is invariant under S, and* ii) $\text{Ker}\,\phi$ *is invariant under S if* $\text{Ker}\,\phi$ *is small in P and M is quasi projective.*

It is clear from i) that every commutative ring is a left q^*-ring. A module is *quasi-injective* if for every monomorphism $j: A \to M$, $\text{Hom}(A, M) = \text{Hom}(M, M) \circ j$. The dual theorem of Theorem 1.2 was proved by Johnson and Wong [6, p. 261] in the form of

Theorem 1.3. *Let Q be the injective hull of M and $S = \text{End}(Q)$. Then M is quasi-injective if and only if M is invariant under S.*

A ring R is *quasi-Frobenius* if R is left Artinian and left injective. There are many equivalent forms of this definition. See [12, p. 373] for several references for studies of these rings. For this paper it is necessary to know that if R is quasi-Frobenius, it is a left injective cogenerator. A module M is a *left cogenerator* if M contains a copy of the injective hull of each simple left module [12, p. 374].

In both Sections 2 and 3, the results will be stated for left q^*-rings. It should be clear that the corresponding statements for right q^*-rings will also be true.

2. Structure of Special q^*-Rings

In this section several types of q^*-rings are investigated.

Theorem 2.1. *Let R be a semi-perfect ring. Then R is a left q^*-ring if and only if every left ideal in the radical N of R is an ideal.*

Proof. i) Assume R is a left q^*-ring. It is known that every left ideal in N is small. Hence, by Theorem 1.2 every left ideal in N is an ideal.

ii) Assume every left ideal in N is an ideal. Let L be a left ideal in R. Then R/L has a projective cover $\phi: P \to R/L$, and P can be considered to be a direct summand of R. Since $\text{Ker}\,\phi$ is small in P, it is small in R and contained in N. If $f \in \text{End}(P)$, then there is an $r \in R$ such that $f(x) = xr$ for every $x \in P$. Therefore $\text{Ker}\,\phi$ is invariant under $\text{End}(P)$. By Theorem 1.2 R/L is quasi-projective.

Theorem 2.2. *If R is a semi-perfect left injective q^*-ring, then R is the ring direct sum of a semisimple Artinian ring and a ring $B = Re_1 + \cdots + Re_n$ where $Re_i \cong Re_j$ only if $i = j$, and $e_1, \ldots, e_n$ are orthogonal indecomposable idempotents.*

Proof. Since R is semi-perfect, $R = Re_1 + \cdots + Re_k + Re_{k+1} + \cdots + Re_n$ where $e_1, \ldots, e_n$ are orthogonal indecomposable idempotents. One can assume $Re_1, \ldots, Re_k$ are all the simple components of the decomposition. By Theorem 2.1 $Ne_i \cdot e_i Re_j = 0$ if $i \neq j$. Since $\text{Hom}(Re_i, Re_j) = e_i Re_j$, $Re_i \ncong Re_j$ for $i, j > k$ and $i \neq j$.

Let $A = Re_1 + \cdots + Re_k$, and $B = Re_{k+1} + \cdots + Re_n$. The proof will be complete if A and B are shown to be ideals. Let $i \leqq k$ and $j > k$. Then $Re_i \cdot e_i Re_j$ is 0 or simple and is contained in Re_j. Since Re_i is a simple injective module, and Re_j is not simple and is indecomposable, $Re_i \cdot e_i Re_j = 0$. Similarly $Re_j \cdot e_j Re_i = 0$ because Re_i is a simple projective module.

Corollary 2.3. *If R is a quasi-Frobenius left q^*-ring, then R is Frobenius.*

Proof. The corollary follows immediately from Theorem 2.2 and Nakayama's original definition for quasi-Frobenius and Frobenius rings [11, p. 8].

The next lemma has been proved by Golan [3] and Rangaswamy [13].

Lemma 2.4. *If a module M is the homomorphic image of a projective module P, and $P \oplus M$ is quasi-projective, then M is projective.*

Theorem 2.5. *Let R_n be the n by n matrix ring with entries from a ring R and $n>1$. Then R_n is a left q^*-ring if and only if R is semisimple Artinian.*

Proof. i) If R is semisimple Artinian, then R_n is semisimple Artinian. Every semisimple Artinian ring is, of course, a left q^*-ring.

ii) Assume R_n is a left q^*-ring. To show R is semisimple Artinian it is sufficient to show that every simple left R-module is projective [14]. Let L be a maximal left ideal in R. The set of all matrices in R_n with entries in L will be denoted by L_n, and e_{ii} has the usual meaning of the matrix with 1 in the i^{th} row and i^{th} column and 0 elsewhere. Let I be the left ideal $L_n e_{11}$ in R_n, $M = R_n e_{11}/I$, and $P = \left(\sum_{i=2}^{n} R_n e_{ii} + I \right)/I$. Observe that P is projective because $P \cong \sum_{i=2}^{n} R_n e_{ii}$. Also, $R_n/I = P \oplus M$ as R_n-modules

Define an R_n-epimorphism $\phi: P \to M$ by $\phi((a_{ij})+I) = (b_{ij})+I$ where $b_{ij}=0$ for $j>1$ and $b_{i1}=a_{i2}$ for $i=1,\ldots,n$. Since $P \oplus M$ is quasi-projective, it follows from Lemma 2.4 that there is a monomorphism $j: M \to P$ such that $\phi \circ j =$ identity on M. Let ϕ^* be the natural map from R to R/L. The simple module R/L will be projective if there is a homomorphism $j^*: R/L \to R$ such that $\phi^* \circ j^* =$ identity on R/L. Define $j^*: R/L \to R$ by $j^*(r+L) = a_{12}$ where $j(re_{11}+I) = (a_{ij})+I$. The function j^* is well defined, an R-homomorphism, and $\phi^* \circ j^* =$ identity.

The preceding theorem is true if q^*-ring is replaced by q-ring [5]. Also, a prime q-ring is a simple Artinian ring [5]. However, a prime q^*-ring does not need to be simple Artinian even if the ring is semi-perfect (i.e. a local ring which is an integral domain).

Theorem 2.6. *If R is a prime semi-perfect left q^*-ring, then R is a simple Artinian ring, or R is a local ring.*

Proof. Since R is semi-perfect, $R = Re_1 + \cdots + Re_n$ where $e_1, \ldots, e_n$ are nonzero orthogonal indecomposable idempotents. If $n=1$, then R is local because Ne_1 is a unique maximal left ideal in Re_1. From Theorem 2.1 $Ne_i \cdot Ne_j \subseteq Ne_i \cap Ne_j = 0$ for $i \neq j$. Hence $Ne_i = 0$ for all but at most one i, say $i=k$, because R is prime. Since $e_k R e_i \neq 0$ (because R is prime), there is an epimorphism from Re_k to Re_i. So $Re_k \cong Re_i$ for $i=1,\ldots,n$ because Re_i is simple and projective, and Re_k is indecomposable. Therefore R is simple Artinian if $n>1$.

3. q^*-Rings and q-Rings

A few relationships between left q^*-rings, right q^*-rings, left q-rings, and right q-rings will now be studied. The first theorem appears in [5], and it can be proved in a manner dual to the proof of Theorem 2.1.

Theorem 3.1. *Let R be a left injective ring. Then R is a left q-ring if and only if every large left ideal is an ideal.*

Theorem 3.2. *If R is a semi-perfect right q-ring, then R is a left q^*-ring.*

Proof. Mohamed has shown that left ideals in N are ideals if R is a right q-ring in the following way [10]. Let L be a left ideal in N and $a \in L$. Since R is right injective, $N = \{a : r(a)$ is large right ideal in $R\}$ and $l(r(Ra)) = Ra$ [2, p. 26]. In addition $r(Ra) = r(a)$ which is a large ideal in R. Therefore, Ra is an ideal, that is, L is an ideal. By Theorem 2.1 R is a left q^*-ring.

Theorem 3.3. *Let R be a left injective and semi-perfect ring. If R is a left q^*-ring, then R is a left q-ring.*

Proof. Let I be a large left ideal in R. The module R/I has a projective cover $P \xrightarrow{\phi} R/I \to 0$. Let $f : R \to R/I$ be the canonical homomorphism. Then there is an epimorphism $f' : R \to P$ such that $\phi \circ f' = f$ because P is projective and $\operatorname{Ker}\phi$ is small. Since P is projective and f' is onto, $R = Re_1 \oplus Re_2$ with $Re_1 \cong P$. Also, it can be seen that $I = K \oplus Re_2$ where $K \cong \operatorname{Ker}\phi$ and $K \subseteq Re_1$. The left ideal K is small in R, and R is a left q^*-ring. Thus $K \subseteq N$, and K must be an ideal by Theorem 2.1. The left socle S of R is contained in I because I is large. The left ideal Ne_2 is an ideal in R. So $Re_2 \cdot e_2 Re_1 \subseteq S$ because $Ne_2 \cdot e_2 Re_1 = 0$ and R is semi-perfect. Therefore I is an ideal, and R is a left q-ring by Theorem 3.1.

Theorem 3.4. *Let R be a left injective and semi-perfect ring. If R is a left q^*-ring, then R is a right q^*-ring.*

Proof. The result follows immediately from Theorem 3.3 and Theorem 3.2.

Theorem 3.5. *Let R be semi-perfect and both right and left injective. Then the following statements are equivalent.*

1. *R is a left q^*-ring.*
2. *R is a left q-ring.*
3. *R is a right q^*-ring.*
4. *R is a right q-ring.*

Proof. Use Theorems 3.2 and 3.3.

The equivalence of 2 and 4 in Theorem 3.5 is a generalization of S. Mohamed's theorem (see the introduction),

It has not been shown that a semi-perfect left q-ring is a q^*-ring. However, the next theorem and Theorem 3.5 indicate that this statement may be true.

Theorem 3.6. *If R is a left cogenerator and a left q-ring, then R is a left q^*-ring.*

Proof. A left quasi-injective ring is left injective. So R is semi-perfect [12, p. 377]. Let I be a left ideal in N. It follows that I is quasi-injective. By Theorem 2.1 it is sufficient to show I is an ideal. First it will be shown that one can assume I is indecomposable. The injective hull of I, $E(I)$, can be chosen in R. Then $E(I)=Re$ where e is an idempotent. Since R is semi-perfect Re is a direct sum of indecomposable modules. Hence I is a direct sum of indecomposable quasi-injective left ideals [4, p. 354].

Suppose I is an indecomposable left ideal in N. Then $R=Re_1+Re_2+\cdots+Re_n$ where $E(I)\cong Re_1$, and $e_1, e_2, \ldots, e_n$ are orthogonal indecomposable idempotents. In addition it is known that each Re_i, $i=1,\ldots,n$, is the injective hull of a minimal left ideal. This last statement can be proved by using the facts that R is a cogenerator, each simple module is isomorphic to Re_i/Ne_i for some i, $Re_i\cong Re_j$ iff $Re_i/Ne_i\cong Re_j/Ne_j$, and each Re_i can contain at most one minimal ideal.

The left ideal I is contained in Ne_1. Obviously, $Ne_1\cdot e_iRe_j=0$ if $i\neq 1$. Also, $Ie_1Re_1\subseteq I$ by Theorem 1.3. All that remains to be shown is that $Ie_1Re_j=0$ when $j\neq 1$. Let A_j be the minimal left ideal in Re_j, and consider the large left ideal $A=Re_1+\cdots+A_j+\cdots+Re_n$. This left ideal must be an ideal by Theorem 3.1. Thus $Ne_1\cdot e_1Re_j=0$ because Ne_1 is the unique maximal left ideal in Re_1. Therefore, $Ie_1Re_j=0$, and I is an ideal.

Corollary 3.7. *If R is a quasi-Frobenius left q-ring, then R is a Frobenius ring.*

Proof. Use Theorem 3.6 and Corollary 2.3.

The paper will now be ended with an example of a ring which is both left and right Artinian, a right q^*-ring, but not a left q^*-ring.

Example 3.8. Let $R=\left\{\begin{bmatrix}\lfloor a\rfloor & \bar{b}\\ 0 & \bar{c}\end{bmatrix}: \lfloor a\rfloor\in Z_4 \text{ and } \bar{b},\bar{c}\in Z_2\right\}$ where Z_2 is the ring of integers modulo 2, and Z_4 is the ring of integers modulo 4. Define

$$\begin{bmatrix}\lfloor a\rfloor & \bar{b}\\ 0 & \bar{c}\end{bmatrix}+\begin{bmatrix}\lfloor d\rfloor & \bar{e}\\ 0 & \bar{f}\end{bmatrix}=\begin{bmatrix}\lfloor a+d\rfloor & \overline{b+e}\\ 0 & \overline{c+f}\end{bmatrix}$$

and

$$\begin{bmatrix}\lfloor a\rfloor & \bar{b}\\ 0 & \bar{c}\end{bmatrix}\begin{bmatrix}\lfloor d\rfloor & \bar{e}\\ 0 & \bar{f}\end{bmatrix}=\begin{bmatrix}\lfloor ad\rfloor & \overline{ae+bf}\\ 0 & \overline{cf}\end{bmatrix}.$$

The radical of R is $N=\left\{\begin{bmatrix}\lfloor a\rfloor & \bar{b}\\ 0 & \bar{0}\end{bmatrix}: \lfloor a\rfloor=\lfloor 0\rfloor \text{ or } \lfloor 2\rfloor, \text{and } \bar{b}\in Z_2\right\}$. Every right ideal in N is an ideal. However, the left ideal $L=\left\{\begin{bmatrix}\lfloor 0\rfloor & \bar{0}\\ 0 & \bar{0}\end{bmatrix},\begin{bmatrix}\lfloor 2\rfloor & \bar{1}\\ 0 & \bar{0}\end{bmatrix}\right\}$ is not an ideal. Hence, by Theorem 2.1 R is a right q^*-ring but not a left q^*-ring.

References

1. Bass, H.: Finitistic dimension and a homological generalization of semi-primary rings. Trans. Amer. Math. Soc. **95**, 466—488 (1960).
2. Faith, C.: Lectures on injective modules and quotient rings. Berlin-Heidelberg-New York: Springer 1967.
3. Golan, J. S.: Characterization of semisimple and perfect rings using quasi-projective modules. Israel J. Math. (to appear).
4. Harada, M.: Note on quasi-injective modules. Osaka J. Math. **2**, 351—356 (1965).
5. Jain, S. K., Mohamed, S. H., Singh, Surjeet: Rings in which every right ideal is quasi-injective. Pacific J. Math. **30**, 73—79 (1969).
6. Johnson, R. E., Wong, E. T.: Quasi-injective modules and irreducible rings. J. London Math. Soc. **36**, 260—268 (1961).
7. Lambek, J.: Lectures on rings and modules. Toronto: Blaisdell Publishing Co., 1966.
8. Miyashita, Y.: On quasi-injective modules. J. Fac. Sci. Hokkaido Univ. **18**, 158—187 (1965).
9. — Quasi-projective modules, perfect modules, and a theorem for modular lattices. J. Fac. Hokkaido Univ. (I) **29**, 86—110 (1966).
10. Mohamed, S. H.: q-rings with chain conditions. J. London Math. Soc. (to appear).
11. Nakayama, T.: On Frobeniusean algebra II. Annals of Math. **42**, 1—21 (1941).
12. Osofsky, B. L.: A generalization of quasi-Frobenius rings. J. Algebra **4**, 373—387 (1966).
13. Rangaswamy, K. M.: Quasi-projective modules (to appear).
14. Satyanarayana, M.: Semisimple rings. Amer. Math. Monthly **74**, 1086 (1967).
15. Utumi, Y.: Self-injective rings. J. Algebra **6**, 56—64 (1967).
16. Wu, L. E. T., Jans, J. P.: On quasi-projectives. Ill. J. Math. **11**, 439—447 (1967).

Dr. Anne B. Koehler
Department of Mathematics
Miami University
Oxford, Ohio 45056, USA

(Received December 8, 1969)