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The Consistent Shapley Value for Hyperplane Games

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Abstract: A new value is defined for n-person hyperplane games, i.e., non-sidepayment cooperative games, such that for each coalition, the Pareto optimal set is linear. This is a generalization of the Shapley value for side-payment games.

It is shown that this value is consistent in the sense that the payoff in a given game is related to payoffs in reduced games (obtained by excluding some players) in such a way that corrections demanded by coalitions of a fixed size are cancelled out. Moreover, this is the only consistent value which satisfies Pareto optimality (for the grand coalition), symmetry and covariancy with respect to utility changes of scales. It can be reached by players who start from an arbitrary Pareto optimal payoff vector and make successive adjustments.

1 Introduction

Let ϕ be a 1-point solution concept for the class of cooperative games $(N;v)$ with side payments. For a non-empty subset S of N we define the *reduced games* $(S;v_{*S})$ by[3]

$$v_{*S}(T) = v(T \cup S^c) - \Sigma_{q \in S^c} \phi_q[v|T \cup S^c], \quad T \subseteq S, \tag{1.1}$$

where $v|T \cup S^c$ is the restriction of v to the set of players $T \cup S^c$ ($S^c = N \backslash S$). We regard v_{*S} as the evaluation of the members of S of their "own" game, given that they live in an environment of people who believe in the solution ϕ. Indeed, if a coalition T forms, then, without the members of $S \backslash T$, they will remain in a game $v|T \cup S^c$, the members q of S^c will demand and receive $(\phi_q[v|T \cup S^c])_{q \in S^c}$ so that the worth of T in the game on S will be v_{*S}.

Once the members of S agree that v_{*S} represents their own game, naturally they will examine whether their payoff $(\phi_q[v])_{q \in S}$ is compatible with their own game; namely, whether

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[3] We write $\phi[v]$ instead of $\phi[N;v]$, because it is clear from v what the set of players is. We adopt a similar convention for other games, such as $\phi[v|T \cup S^c]$.

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$$\phi_q[v] = \phi_q[v_{*S}] \text{ , all } q \in S. \tag{1.2}$$

A solution that has this property for every non-empty coalition S will be called *strongly consistent.* The property (1.2) will be called also *the reduced game property.*

Hart and Mas-Collel [2] proved recently that the Shapley value is strongly consistent.[4] In fact, they proved that the Shapley value can be characterized by Pareto optimality (for the coalition N), symmetry, covariance with respect to strategic equivalence and strong consistency.[5] This beautiful result is particularly interesting because it resembels a similar theorem by Sobolev [8], concerning the axiomatic characterization of the prenucleolus. The axioms are the same, except that the definition of the reduced game is different; namely,

$$\begin{cases} v_{*S}(S) = v(N) - \sum\limits_{q \in S^c} \phi_q[v] \\ v_{*S}(T) = \operatorname*{Max}\limits_{Q \subseteq S^c} \; [v(T \cup Q] - \sum\limits_{q \in Q} \phi_q[v]] , \quad \emptyset \neq T \nsubseteq S. \end{cases} \tag{1.3}$$

Thus, the Hart and Mas-Collel result, together with Sobolev's result, yield an understanding of the difference between the Shapley value and the prenucleolus in a deep sense.

It should be noted that strong consistency plays an important role in other solutions for various classes of games. We refer the reader to Peleg [5], [6], Lensberg [3] and Aumann and Maschler [1], where some applications are provided as well as references to previous occurences of this property.

Naturally, one would like to know whether a strongly consistent, Pareto optimal, symmetric and covariant solution exists for games without side payments. We tried a simple 3-person game (the one given in the example in Section 6) and found out that such a solution existed neither in the sense of (1.1), nor in the sense of (1.3). The resulting equations were inconsistent!

We then adopted a dynamic approach: Suppose someone suggests that a payoff vector x is a solution to the game of that example. Since it is not strongly consistent, some 2-person coalitions will realize that they are not receiving their Nash point in the reduced game. They will therefore ask for "corrections" which, when performed simultaneously, will yield a new payoff vector x^1. This, in turn, will lead to x^2, etc. (See Section 7 for details.) Putting these calculations in a computer, we found out to our surprise that, regardless of the starting point x, the sequences always converged to a unique outcome $\bar{x}$. If the starting payoff was $\bar{x}$, it turned out that $\bar{x}^1 = \bar{x}$ so that if all corrections were made, the players found out that all corrections cancelled out. We call this property *consistency* (see Section 4). The payoff $\bar{x}$ could also be characterized in a different way; namely, as an expected marginal contribu-

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[4] They used the word "consistent".

[5] Another proof of this result was supplied by the first author of the present paper, but was not published. It should be noted that the first three requirements are needed only for 2-person games.

tion of the players entering in a random order. These findings could easily generalize to arbitrary hyperplane games (Section 2) and this is the subject of this paper.

In Section 3 we define this new solution and prove that it is consistent in Section 4, 5. Section 6 characterizes the solution axiomatically, where one axiom will be bilateral consistency. Section 7 shows how this solution can be reached dynamically from an arbitrary Pareto optimal payoff vector. Section 8 comments on the possibility of extending this solution for arbitrary non-side-payment games.

2 Preliminary Notations and Definitions

We consider here a special type of game. Given a player set $N = \{1,2,...,n\}$, *hyperplane game* on N is a function V which assigns, to each nonempty $S \subseteq N$, a subset of $\mathbb{R}^S$ of the form

$$V(S) = \{x \in \mathbb{R}^S : \sum_{i \in S} p_i^S x_i \leqslant r_S\} . \tag{2.1}$$

It is assumed that $p_i^S > 0$ for all $i \in S$, all $S \subseteq N$. Otherwise the p_i^S, r_S are arbitrary constants, though in paractice certain values would give rise to improper (non-monotone or non-superadditive) games.

As usual, we interpret V to mean that a coalition S can obtain any point $x \in V(S)$ for its members. In this case, $V(S)$ consists of all points on, or below, a certain hyperplane; hence the name. The hyperplane itself, i.e. the set of all x satisfying (2.1) as an equation, is the *Pareto-optimal surface* of $V(S)$. We point out that what distinguishes these (hyperplane) games from ordinary games with side payments is that the several hyperplanes are inclined at varying angles whereas, for side payments games, all $p_i^S = 1$, i.e. all planes are "equally inclined".

It will be of interest to study the behavior of certain concepts under *linear transformations of utility.* We say two games,[6] V and $\tilde{V}$, are *equivalent under a linear transformation of player i's utility* if there exist constants $\beta > 0$ and γ such that:

$$\left.\begin{array}{l}
\text{(a) for all } j \neq i, \text{ all } S \subseteq N, \tilde{p}_j^S = p_j^S \\
\text{(b) for all } S \text{ such that } i \in S \subseteq N, \tilde{p}_i^S = \dfrac{p_i^S}{\beta} \\
\text{(c) for all } S \text{ such that } i \notin S \subseteq N, \tilde{r}_S = r_S \\
\text{(d) for all } S \text{ such that } i \in S \subseteq N, \tilde{r}_S = r_S + \dfrac{\gamma p_i^S}{\beta} .
\end{array}\right\} \tag{2.2}$$

[6] Games are denoted by $(N;V)$, but when N is fixed we shall shorten the notation and write V.

In this case, it is easily verified that $x \in V(S)$ if and only if $\tilde{x} \in \tilde{V}(S)$, where

$$\left.\begin{aligned} \tilde{x}_i &= \beta x_i + \gamma \\ \tilde{x}_j &= x_j \ \text{ if } j \neq i. \end{aligned}\right\} \tag{2.3}$$

More generally, we will say that two games, V and W, are *equivalent under linear transformations of utility* if there is a sequence $V, V^{(1)}, V^{(2)}, \ldots, V^{(k)} = W$ such that each $V^{(\ell)}$ is equivalent to $V^{(\ell-1)}$ under a linear change of some player's utility, $\ell = 1,2,\ldots,k$, $V^0 = V$.

Let H be a solution concept which assigns either one or many payoff vectors to each game V. We will say that H is *covariant with linear changes of utility* if, whenever V and $\tilde{V}$ are related as in (2.2), we find that $x \in H(V)$ if and only if $\tilde{x} \in H(\tilde{V})$, where x and $\tilde{x}$ are related as in (2.3).

3 The Value

Let $(N;V)$ be a hyperplane game, and let π be a permutation of the set N. For $i \in N$, let $Q(i,\pi) \equiv Q(i)$ be the set of all j preceding or equal to i :

$$Q(i,\pi) = \{j \in N \mid \varrho(j) \leqslant \varrho(i)\}, \quad \varrho = \pi^{-1}. \tag{3.1}$$

Define now the vector $y(\pi)$ by

$$y_i(\pi) = [r_{Q(i)} - \sum_{\varrho(j)<\varrho(i)} p_j^{Q(i)} y_j] / p_i^{Q(i)}. \tag{3.2}$$

This equation defines $y_i(\pi)$ inductively and so $y(\pi)$ is well defined.

Heuristically, if π represents an order $(i_1, i_2, i_3, \ldots, i_n)$, the vector $y(\pi)$ gives player i_1 the most he can obtain in $V(\{i_1\})$. Then i_2 gets as much as is possible in $V(\{i_1, i_2\})$, given what player i_1 has already been given, etc. In this way, each player receives the most that he can obtain with the help of those preceding him, given what these players have already been given.

Certain properties of this vector are obvious:

Proposition 1: For each $i \in N$, the vector $y(\pi)|_{Q(i)}$ is Pareto-optimal in $V(Q(i))$. In particular, $y(\pi)$ is Pareto-optimal in $V(N)$.

Proof: From (3.2) we have:

$$\sum_{j \in Q(i)} p_j^{Q(i)} y_j = r_{Q(i)},$$

which is the condition for Pareto-optimality.

Proposition 2: The vector $y(\pi)$ is covariant with linear transformations of utility.

Proof: Let π be a fixed permutation, and $y(\pi)$ be the corresponding vector in game V. Let $\tilde{V}$ be obtained from V by a change in player i's utility, as given by (2.2), and let $\tilde{y}(\pi)$ be the corresponding vector in $\tilde{V}$.

Now, if $\varrho(j) < \varrho(i)$, then $\tilde{y}_j = y_j$, because $\tilde{V}(Q(j)) = V(Q(j))$ for all such j. Next,

$$\tilde{y}_i = [\tilde{r}_{Q(i)} - \sum_{\varrho(j)<\varrho(i)} \tilde{p}_j^{Q(i)} \tilde{y}_j]/\tilde{p}_i^{Q(i)}$$

and so, by (2.2),

$$\tilde{y}_i = [r_{Q(i)} + \frac{\gamma p_i^{Q(i)}}{\beta} - \sum_{\varrho(j)<\varrho(i)} p_j^{Q(i)} y_j]/\frac{p_i^{Q(i)}}{\beta} = \beta y_i + \gamma.$$

For $\varrho(j) > \varrho(i)$, we proceed by induction on $\varrho(j)$. We assume $\tilde{y}_k = y_k$ for all $\varrho(k) < \varrho(j)$, $k \neq i$. Then,

$$\tilde{y}_j = [\tilde{r}_{Q(j)} - \sum_{\varrho(k)<\varrho(j)} \tilde{p}_k^{Q(j)} \tilde{y}_k]/\tilde{p}_j^{Q(j)} =$$

$$= [r_{Q(j)} + \frac{\gamma p_i^{Q(j)}}{\beta} - \sum_{\varrho(k)<\varrho(j),\, k\neq i} p_k^{Q(j)} y_k] - \frac{p_i^{Q(j)}}{\beta}(\beta y_i + \gamma)]/p_j^{Q(j)}$$

so

$$\tilde{y}_j = [r_{Q(j)} - \sum_{\varrho(k)<\varrho(j)} p_k^{Q(j)} y_k]/p_j^{Q(j)} = y_j\,.$$

It follows that $y(\pi)$ is covariant as desired.

Having defined the vector $y(\pi)$ for a given permutation π, we now define the expected marginal payoff vector ϕ by

$$\phi[V] = \frac{1}{n!} \sum_{\pi} y(\pi)\,, \tag{3.3}$$

where the summation is taken over all $n!$ permutations of N.

It is well known (see Shapley [7]) that if V is a game with transferable utility (side payments), then $\phi[V]$ is nothing other than the Shapley value. Thus our ϕ is a generalization of the Shapley value. We note that, as the definition implies, $\phi_i[V]$ can be interpreted as an expected payoff: *it is player i's expected marginal contribution, assuming all n! orderings of the players are given equal probabilities.*

Proposition 3: The *vector* $\phi[V]$ *is Pareto-optimal in* $V(N)$.

Proof: By Proposition 1, $y(\pi)$ is Pareto-optimal in $V(N)$ for all π. But $\phi[V]$ is merely the arithmetic mean of the $y(\pi)$. Since $V(N)$ is a half space, its Pareto-optimal surface is convex, and so $\phi[V]$ is Pareto-optimal.

Proposition 4: The vector $\phi[V]$ *is covariant with linear transformations of utility.*

Proof: Since $\phi[V]$ is the mean of the $y(\pi)$, all of which are covariant, $\phi[V]$ must also be covariant.

Given Proposition 4, there is no loss of generality in assuming that all $p_i^N = 1$, i.e. the set $V(N)$ is "equally inclined", so that $V(N)$ is simply given by

$$\sum_{i \in N} x_i \leqslant r_N .$$

Such games will be called *normalized games.*

4 The Reduced Game. Consistency

Suppose that a population N "believes" in a certain one-point solution concept Ψ. Thus, if the members participate in a game $(N;V)$, they will tend to agree to the payoff $\Psi[V]$. Now, any non-empty subset S of N may consider its "own" game, hereby denoted by V_{*S}, and its members may examine whether their payoffs $(\Psi_k[V])_{k \in S}$ agree with $\Psi[V_{*S}]$. If it does not, then, presumably, the members of S will request that adjustments be made, to bring their payoff to $\Psi[V_{*S}]$. The main question, of course, is how they evaluate their "own" game, henceforth called the *reduced game* on S, or the S-*reduced game.* There are several possibilities. In this paper we define it to be the game $(S;V_{*S})$, where, for $\emptyset \neq T \subseteq S$,

$$V_{*S}(T) = \{(x_j)_{j \in T} : ((x_j)_{j \in T}, (z_q)_{q \in S^c}) \in V(T \cup S^c)\} . \tag{4.1}$$

Here,

$$z_q = \Psi_q[V | T \cup S^c] , \quad q \in S^c , \; S^c = N \backslash S . \tag{4.2}$$

(Here, $V|T \cup S^c$ is the restriction of the game (function) V to subsets of $T \cup S^c$.)

Interpretation: The members of T evaluate their worth; namely, what they can achieve in the game $(N;V)$ without the presence of $S \backslash T$. They therefore consider themselves as participating in a game $(T \cup S^c;\ V|T \cup S^c)$ in which the members of S^c (who believe in Ψ) receive $(\Psi_q[V|T \cup S^c])_{q \in S^c}$. Every T-payoff that is feasible under this requirement belongs to their characteristic function.

Proposition 5: If V is a hyperplane game, then the games V_{*S} are all hyperplane games, regardless of Ψ.

Proof: Suppose $V(T \cup S^c)$ is given by $\sum_{j \in Q} q_j^Q x_j \leqslant r_Q$, where $Q = T \cup S^c$. Then, by (4.1), $V_{*S}(T)$ is given by

$$\sum_{j \in T} p_j^Q x_j \leqslant r_Q - \sum_{q \in S^c} p_q^Q z_q \,.$$

Thus, V_{*S} is also a hyperplane game.

Remark: We note in passing that the coefficients p_j^Q on the left side are precisely the coefficients in $V(T \cup S^c)$ for the indices in T. In particular, the coefficients in $V_{*S}(S)$ are the same as the coefficients in $V(N)$, restricted to indices in S. Thus, if $(N;V)$ is normalized, then so is $(S;V_{*S})$ and $V_*(S)$ is given by

$$\sum_{j \in S} x_j \leqslant r_N - \sum_{q \in S^c} \Psi_q[V] \,. \tag{4.3}$$

Proposition 6: If a solution Ψ is covariant with linear transformations of utility, then so are the reduced games.

Proof: Let $(N;\tilde{V})$ be related to $(N;V)$ by (2.2). Then a similar relation holds between $(Q;\tilde{V}|Q)$ and $(Q;\ V|Q)$, where $Q = T \cup S^c$. Thus, if $\tilde{z} = \Psi[\tilde{V}|Q]$ and $z = \Psi\,[V|Q]$ then $\tilde{z}_i = \beta z_i + \gamma$ (relevant, if $i \in T \cup S^c$) and $\tilde{z}_q = z_q$, otherwise. Therefore,

$$\tilde{V}_{*S}(T) = \{(\tilde{x}_j)_{j \in T} : ((\tilde{x}_j)_{j \in T}, (\tilde{z}_q)_{q \in S^c}) \in \tilde{V}(T \cup S^c)\} =$$

$$= \{(\tilde{x}_j)_{j \in T} : ((x_j)_{j \in T}, (z_q)_{q \in S^c}) \in V(T \cup S^c)\} \,,$$

where, x and $\tilde{x}$ are related as in (2.3). This shows that $\tilde{V}_{*S}(T) = [V_{*S}(T)]^{\sim}$.

We are now prepared to state our consistency requirement:

Definition: Let Ψ be a solution defined over a class of games H. The solution Ψ is called k-*consistent* if for every $(N;V) \in H$ with $n \geqslant k$, all the reduced games are in H and for every i in N,

$$\sum_{\substack{S:i\in S\\ |S|=k}} \Psi_i[V_{*S}] = \binom{n-1}{k-1} \Psi_i[V] . \tag{4.4}$$

Discussion: The players are assumed to be "believers" in Ψ. A coalition S views its own game as $(S;V_{*S})$. Thus, each player i of S would ask for an adjustment of $\Psi_i[V_{*S}] - \Psi_i[V]$. The solution is k-consistent if, whenever such adjustments are made simultaneously for all coalitions of a fixed size k, the players will find themselves back at their original payoff vector $\Psi[V]$; i.e., all credits and debits will cancel out for each player in N.

Definition: A solution is called *consistent* if such adjustments made simultaneously for *all coalitions* leave it intact.

5 Consistency of the Value

In this section we shall prove that the solution ϕ, as defined in Section 3, is k-consistent *for every k*. Note that consistency is well defined for the class of hyperplane games.

Lemma 1: If $i\in T$ and $|T| = t$, then[7]

$$\sum_{j\in T\setminus i} \phi_i[V|T\setminus j] + (r_T - \sum_{j\in T\setminus i} p_j^T \phi_j [V|T\setminus i])/p_i^T = t\phi_i[V|T] . \tag{5.1}$$

Proof: Recall that $\phi_i[V|T]$ is the expected payoff to i in the game $V|T$, given the marginal payoff scheme discussed in Section 3, assuming all permutations of T have equal probability.

In such a case, i can appear in last position (probability $1/t$) or in some other positions (probability $(t\text{-}1)/t$). If he appears in the last position, his (conditional) expected payoff will be

$$(r_T - \sum_{j\in T\setminus i} p_j^T\phi_j [V|T\setminus i])/p_i^T$$

[7] We shall write $T\setminus i$ instead of $T\setminus\{i\}$ and $T\setminus ij$ instead of $T\setminus\{i,j\}$.

since, this is what will be left for him after the other t-1 players take their expected payoff (all coming before i).

If i appears in any other position, then some player j, $j \in T\backslash i$, will be in the last position. If so, then i's conditional expected payoff is merely $\phi_i\,[v|T\backslash j]$. But each j has probability $1/t$ of being last, and so i's expected payoff in $v|T$ is

$$\frac{1}{t} \sum_{j\in T\backslash i} \phi_i\,[V|\,T\backslash j] + \frac{1}{t}\,(r_T - \sum_{j\in T\backslash i} p_j^T\ \phi_j\,[V|\,T\backslash i])/p_i^T.$$

Multiplying by t gives us, then (5.1).

In particular, if $T = N$ and all $P_i^N = 1$ then

$$n\phi_i[V] = \sum_{j\in N\backslash i} \phi_i[V|N\backslash j] + r_N - \sum_{j\in N\backslash i} \phi_j\,[V|N\backslash i]\,. \tag{5.2}$$

The following lemma plays a key role. It relates the case of removing a player j before passing to the reduced game to the case of removing him after the passage. We show that the order does not matter.

Lemma 2: If $j \in S \subseteq N$ and the reduced game is with respect to the solution ϕ then

$$(V|N\backslash j)_{*S\backslash j} = V_{*S}|S\backslash j\,. \tag{5.3}$$

Proof: Suppose $T \subseteq S\backslash j$. Then

$$V_{*S}(T) = \{(x_i)_{i\in T} : ((x_i)_{i\in T}, (z_q)_{q\in S^c}) \in V(T \cup S^c)\}\,,$$

where $z_q = \phi_q[V|\,T \cup S^c]$. Moreover,

$$(V|N\backslash j)_{*S\backslash j}\,(T) = \{(x_i)_{i\in T} : ((x_i)_{i\in T}, (z_q^1)_{q\in S^c}) \in V(T\cup S^c)\}\,,$$

where $z_q^1 = \phi_q[V|\,T \cup S^c]$, because $(N\backslash j) \setminus (S\backslash j) = S^c$. Clearly, $z_q^1 = z_q$ for all q in S^c and we see that

$$V_{*S}(T) = (V|N\backslash j)_{*S\backslash j}\,(T)\,.$$

Now V_{*S} is defined for all subsets of S, while $(V|N\backslash j)_{*S\backslash j}$ is defined only for subset of $S\backslash j$; hence, (5.3) is established.

Theorem 1: The value ϕ is k-consistent for every k, $k \leqslant n$; thus, it is a consistent solution.

Proof: By induction on k. The theorem is trivially true for $k = 1$ and arbitrary n. Assume that it is true for k-1 or less and arbitrary n. The induction hypothesis reads: if $i \in M$ and $1 \leqslant k\text{-}1 \leqslant m$, $m = |M|$, then

$$\sum_{\substack{T \subseteq M \\ i \in T, |T| = k-1}} \phi_i[(V|M)_{*T}] = \binom{m-1}{k-2} \phi_i [V|M] .$$

Here, M can be any subset of N. In particular, let $M = N\backslash j$, then

$$\sum_{\substack{T \subseteq N\backslash j \\ i \in T, |T| = k-1}} \phi_i[(V|N\backslash j)_{*T}] = \binom{n-2}{k-2} \phi_i [V|N\backslash j] . \tag{5.4}$$

We wish, now to prove that

$$k \cdot \sum_{\substack{S \subseteq N \\ i \in S, |S| = k}} \phi_i[V_{*S}] = k\binom{n-1}{k-1} \phi_i [V] , \tag{5.5}$$

which is clearly equivalent to (4.4) with $\Psi = \phi$.

To do so, we will anlyse the left side of (5.5), but first note that, in view of (4.3), Propositions 4 and 6, we can assume that all the reduced games are normalized. By Lemma 1 (see (5.2)) and (4.3),

$$k\phi_i[V_{*S}] = \sum_{j \in S\backslash i} \phi_i[V_{*S}|S\backslash j] + r_N - \sum_{q \in S^c} \phi_q[V] - \sum_{j \in S\backslash i} \phi_j[V_{*S}|S\backslash i] ,$$

and since there are $\binom{n-1}{k-1}$ S's with $i \in S$ and $|S| = k$, the left side of (5.5) is equal to

$$\sum_{\substack{S: i \in S \\ |S| = k}} \sum_{j \in S\backslash i} \phi_i[V_{*S}|S\backslash j] + \binom{n-1}{k-1} r_N - \sum_{\substack{S: i \in S \\ |S| = k}} \sum_{q \in S^c} \phi_q[V] -$$

$$- \sum_{\substack{S: i \in S \\ |S| = k}} \sum_{j \in S\backslash i} \phi_j[V_{*S}|S\backslash i] .$$

Rearranging the order of summation, we have

$$k \sum_{\substack{S:i\in S\\|S|=k}} \phi_i[V_{*S}] =$$

$$= \sum_{j\in N\setminus i} \sum_{\substack{S:i,j\in S\\|S|=k}} \phi_i[V_{*S}|S\setminus j] + \binom{n-1}{k-1} r_N - \sum_{q\in N\setminus i} \sum_{\substack{S:i\in S,q\in S\\|S|=k}} \phi_q[V] -$$

$$- \sum_{j\in N\setminus i} \sum_{\substack{S:i,j\in S\\|S|=k}} \phi_j[V_{*S}|S\setminus i] .$$

We apply Lemma 2 to the first term on the right, setting also $T = S\setminus j$. We also note that for a given q, $q \neq i$, there are $\binom{n-2}{k-1}$ possible sets S for the third term on the right. We also apply Lemma 2 to the last term on the right, setting here $T = S\setminus i$. Altogether, we obtain,

$$k \sum_{\substack{S:i\in S\\|S|=k}} \phi_i[V_{*S}] =$$

$$= \sum_{j\in N\setminus i} \sum_{\substack{T:i\in T,j\notin T\\|T|=k-1}} \phi_i[(V|N\setminus j)_{*T}] + \binom{n-1}{k-1} r_N - \binom{n-2}{k-1} \sum_{q\in N\setminus i} \phi_q[V] -$$

$$- \sum_{j\in N\setminus i} \sum_{\substack{T:i\notin T,j\in T\\|T|=k-1}} \phi_j[(V|N\setminus i)_{*T}] .$$

By Proposition 3, $r_N - \sum_{q\in N\setminus i} \phi_q[V] = \phi_i[V]$ and so

$$k \sum_{\substack{S:i\in S\\|S|=k}} \phi_i[V_{*S}] =$$

$$= \sum_{j\in N\setminus i} \sum_{\substack{T:i\in T,j\notin T\\|T|=k-1}} \phi_i[(V|N\setminus j)_{*T}] + \binom{n-2}{k-2} r_N + \binom{n-2}{k-1} \phi_i[V] -$$

$$- \sum_{j\in N\setminus i} \sum_{\substack{T:i\notin T,j\in T\\|T|=k-1}} \phi_j[(V|N\setminus i)_{*T}] .$$

We now apply the induction hypothesis (5.4) to both the first and last terms on the right side, to obtain

$$k \sum_{\substack{S: i \in S \\ |S|=k}} \phi_i[V_{*S}] =$$

$$= \sum_{j \in N\setminus i} \binom{n-2}{k-2} \phi_i[V|N\setminus j] + \binom{n-2}{k-2} r_N + \binom{n-2}{k-1} \phi_i[V] -$$

$$- \sum_{j \in N\setminus i} \binom{n-2}{k-2} \phi_j[V|N\setminus i] \, .$$

Finally, applying (5.2) to the first, second and fourth terms on the right yields

$$k \sum_{\substack{S: i \in S \\ |S|=k}} \phi_i[V_{*S}] = n\binom{n-2}{k-2} \phi_i[V] + \binom{n-2}{k-1} \phi_i[V] = k\binom{n-1}{k-1} \phi_i[V] \, .$$

This concludes the proof.

6 Axiomatic Characterization of the Value

In this section we show how our consistency requirement can be used to define the value axiomatically. We prove:

Theorem 2: The value ϕ, for the class of hyperplane games (as defined in Section 3), *is the unique* 1*-point solution satisfying:*

(i) *Pareto optimality in N,*
(ii) *Symmetry,*
(iii) *Covariance under linear transformations of utilities,*
(iv) *Bilateral consistency* (2-consistency).

Proof: We already proved that ϕ satisfies (i), (iii) and (iv). Symmetry is straightforward. It remains to show that any 1-point solution Ψ, satisfying (i)-(iv), must coincide with ϕ. By Propositions 5 and 6, there is no loss of generality in assuming that our games, and therefore also our reduced games w.r.t. Ψ, are normalized.

The claim is trivial for $n = 1$. For $n = 2$, (i)-(iii) imply that the solution must be given by

$$\Psi[V] = \frac{1}{2}[r_{12} + r_1 - r_2 , r_{12} - r_1 + r_2] = \phi[V] . \tag{6.1}$$

Note that $\Psi[V]$ is characterized by the equations

$$\Psi_1[V] + \Psi_2[V] = r_{12} , \Psi_1[V] - \Psi_2[V] = r_1 - r_2 . \tag{6.2}$$

We now make the induction hypothesis that $\phi = \Psi$ for every hyperplane game with n-1 players. Let $(N;V)$ be an n-person hyperplane game, $n \geqslant 3$. Bilateral consistency means[8] (see (4.4)):

$$\sum_{j \in N\setminus i} \Psi_i[V_{*ij}] = (n\text{-}1)\Psi_i[V] , \text{ all } i \text{ in } N . \tag{6.3}$$

Here, V_{*ij} is the reduced game on $\{i,j\}$ with respect to the solution Ψ. Let $\tilde{V}_{*ij}$ be the reduced game on $\{i,j\}$, with respect to the solution ϕ. By the induction hypothesis, it follows from (4.1) and (4.2) that $V_{*ij}(i) = \tilde{V}_{*ij}(i)$, $V_{*ij}(j) = \tilde{V}_{*ij}(j)$, so that V_{*ij} and $\tilde{V}_{*ij}$ may differ only on the coalition $\{i,j\}$. By (*i*) and (4.3),

$$V_{*ij}\{i,j\} = \{(x_i,x_j) : x_i + x_j \leqslant \Psi_i[V] + \Psi_j[V]\} .$$

It now follows from (6.2) that

$$\Psi_i[V_{*ij}] + \Psi_j [V_{*ij}] = \Psi_i[V] + \Psi_j[V] , \tag{6.4}$$

$$\Psi_i[V_{*ij}] - \Psi_j [V_{*ij}] = \Psi_i[\tilde{V}_{*ij}] - \Psi_j[\tilde{V}_{*ij}] = \phi_i[\tilde{V}_{*ij}] - \phi_j[\tilde{V}_{*ij}] \tag{6.5}$$

For a fixed i, let us add the terms in (6.4). We obtain:

$$\sum_{j \in N\setminus i} \Psi_i[V_{*ij}] + \sum_{j \in N\setminus i} \Psi_j[V_{*ij}] = (n\text{-}1)\Psi_i[V] + \sum_{j \in N\setminus i} \Psi_j[V] .$$

8 We write V_{*ij} , $V_{*ij}(i)$, etc., instead of $V_{*\{i,j\}}$, $V_{*\{i,j\}}(\{i\})$, etc.

In view of (6.3) this reduces to

$$\sum_{j \in N\backslash i} \Psi_j[V_{*ij}] = \sum_{j \in N\backslash i} \Psi_j[V],$$

so that by Pareto optimality (i),

$$\Psi_i[V] + \sum_{j \in N\backslash i} \Psi_j[V_{*ij}] = r_N .$$

A similar analysis yields

$$\phi_i[V] + \sum_{j \in N\backslash i} \phi_j[\tilde{V}_{*ij}] = r_N .$$

Thus,

$$\Psi_i[V] + \sum_{j \in N\backslash i} \Psi_j[V_{*ij}] = \phi_i[V] + \sum_{j \in N\backslash i} \phi_j[\tilde{V}_{*ij}] . \tag{6.6}$$

For a fixed i, let us add the terms in (6.5). We obtain:

$$\sum_{j \in N\backslash i} \Psi_i[V_{*ij}] - \sum_{j \in N\backslash i} \Psi_j[V_{*ij}] = \sum_{j \in N\backslash i} \phi_i[\tilde{V}_{*ij}] - \sum_{j \in N\backslash i} \phi_j[\tilde{V}_{*ij}] ,$$

which, in view of the bilateral consistencies of ϕ and Ψ, reduces to

$$(n\text{-}1)\Psi_i[V] - \sum_{j \in N\backslash i} \Psi_j[V_{*ij}] = (n\text{-}1)\phi_i[V] - \sum_{j \in N\backslash i} \phi_i[\tilde{V}_{*ij}] . \tag{6.7}$$

Adding (6.6) and (6.7) and dividing by n yields

$$\Psi_i[V] = \phi_i[V] .$$

This holds for each i so that $\Psi[V] = \phi[V]$ and the inductive proof has been concluded.

Corollaries

(a) The symmetry axiom (ii) was employed only for 2-person games. This suggests that non-symmetric values can be obtained by fixing the solution on 2-person games and proceeding by consistency. We shall not pursue this direction here, except for noting that in the side-payment case this leads to the weighted Shapley value (see Hart and Mas-Collel [2]).

b) Covariancy (Axiom (iii)) too is only needed for 2-person games. To see this one has to repeat the proof without assuming that the games are normalized. This involves replacing ϕ_v , Ψ_v , r_v^N by $p_v^N\phi_v$, $p_v^N\Psi_v$, $p_v^N r_v$, respectively.

(c) Pareto optimality (Axiom (i)) can be replaced by Pareto optimality only for 1-person games and 1-consistency. Indeed, one obtains Pareto optimality by applying 1-consistency to the game $V_{*\{i\}}$, using the requirement of Pareto optimality for 1-person games.

(d) The Axioms (i)-(iv) can be replaced by (i)-(iii), applied only to 2-person games, together with 1-consistency and 2-consistency. Indeed, (i)-(iii) determine the solution for 2-person games and Pareto optimality for 1-person games can be deduced from applying 1-consistency to the game $(\{1,2\};V)$, where $V(1) = a - \mathbb{R}_+$, $V(2) = 0 - \mathbb{R}_+$, $V(1,2) = \{(x_1,x_2) : x_1 + x_2 \leqslant a\}$. Remarks (a)-(c) above conclude the proof[9].

(e) Theorem 2, as well as Remark (d), remain valid if one restricts the consideration to *side payment games.* The proofs are practically the same. This is, therefore, a weakening of the axioms in Hart and Mas-Collel [2] which charactgerize the Shapley value for such games (see Section 1); the 2-consistency [and 1-consistency] replace their strong consistency requirement (see Section 1). The price paid for this weakening is that 2-consistency is somewhat less intuitive than strong consistency.

In the case of general hyperplane games we cannot ask for strong consistency and must remain satisfied with 2-consistency, as the following example shows.

Example: $N = \{1,2,3\}$, $v\{i\} = 0 - \mathbb{R}_+$ for $i = 1,2,3$. $V(1,2) = \{(x_1,x_2) : 2x_1 + 3x_2 \leqslant 180\}$, $V(i,j) = (0,0) - \mathbb{R}_+^2$ for $\{i,j\} = \{1,3\}$ and $\{2,3\}$, $V(1,2,3) = \{(x_1,x_2,x_3) : x_1 + x_2 + x_3 \leqslant 120\}$. A solution satisfying (i)-(iii) and strong consistency obviously satisfies (iv), so it must be $\phi[V]$, by Theorem 2. By the definition of ϕ (Section 3), $\phi[V] = (55,50,15)$ and this vector is *not* strongly consistent. Indeed, $V_{*12}(i) = 0 - \mathbb{R}_+$ for $i = 1,2$ and $V_{*12}(12) = \{(x_1,x_2) : x_1 + x_2 \leqslant 105\}$. Thus, $\phi[V_{*12}] = (52.5,52.5)$, which is different from (55,50). For this case $\phi[V_{*13}] = (57.5,12.5)$ and $\phi[V_{*23}] = (47.5,17.5)$ and the reader can verify that it is k-consistent for $k = 1,2,3$, as it should be by Theorem 1.

We have shown that biconsistency and other axioms yield a consistent solution. A priori there is a possibility that if one replaces the axiom of biconsistency with an axiom requiring consistency one may get additional solutions. Recently Orshan [4] proved that this is not the case.

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9 A similar procedure, for side payment games was employd in Hart and Mas-Collel [2].

7 A Dynamic Approach to the Consistent Value

The axiomatic justification of our value, like many justifications of other solution concepts, is static in nature. One would like to find also dynamic processes that lead the players to ϕ, starting from an arbitrary Pareto optimal payoff vector. Such processes should make some use of the bilateral consistency. In this section we exhibit such a process. In order to do so, let us define an x-dependent reduced game:

Definition: Let x be a payoff vector on the Pareto optimal surface of $V(N)$. The *reduced game* $(S;V^{x}_{*S})$ is given by

$$\left.\begin{aligned} V^{x}_{*S}(S) &= \{(y_i)_{i \in S} : ((y_i)_{i \in S}, (x_k)_{k \in S^c}) \in V(N)\} \\ V^{x}_{*S}(T) &= V_{*S}(T), \quad \emptyset \neq T \subseteq S. \end{aligned}\right\} \tag{7.1}$$

Here, $V_{*S}(T)$ is given by (4.1) and (4.2) with $\Psi = \phi$.

Discussion: By a process of induction we assume that the players have already agreed on the solution ϕ for all m-person games, $1 \leqslant m < n$. In particular, we assume that they agreed on ϕ for 1-person games (involving only Pareto optimality) and for 2-person games (which are side-payment games after an appropriate change in the utility scale of one player). Now somebody suggests that x should be the solution for an n-person game $(N;V)$, thus suggesting a solution concept Ψ which should satisfy

$$\Psi[U] = \begin{cases} \phi[U] \text{ for all } m\text{-person games } (M;U),\ m < n \\ x \text{ for } U = V. \end{cases} \tag{7.2}$$

Then (7.1) is nothing but (4.2) with respect to this Ψ.

Of course, if $x = \phi[V]$, then V^{x}_{*S} coincides with V_{*S} for all S.

On the basis of this Ψ, the members of a coalition[10] S will examine V^{x}_{*S} for consistency. If the solution turns out to be inconsistent, they will modify x "in the direction" which is dictated by $\phi(V^{x}_{*S})$ in a manner which will be explained subsequently (see (7.3)). These modifications, done simultaneously by all 2-person coalitions, will lead to a new payoff vector x^1 and the process will repeat. The hope is that it will converge and, moreover, converge to $\phi[V]$. Since we are considering only bilateral consistency, this process makes sense only if $n \geqslant 3$. We start by extending Ψ to games $\tilde{V}$ obtained from V by changes in the utility scales of the players, by requiring it to be covariant under such changes. Then, by Proposition 6, all the games $(S;\tilde{V}^{\tilde{x}}_{*S})$ will be obtained from $(S;V^{x}_{*S})$ by the same changes in the utility scales.

[10] Only 2-person coalitions will be studied here, because we only employ the concept of Bilateral consistency.

Here, $\tilde{x}$ is related to x in accordance with the changes in the scales. This procedure enables us to discuss only normalized hyperplane games.

Let $(N;V)$ be an n-person normalized hyperplane n-person game, $n \geqslant 3$. Let $\bar{V}(N)$ be the Pareto surface of $V(N)$. We define now the *correction function* $f: \bar{V}(N) \rightarrow \bar{V}(N)$ by

$$f_i(x) = x_i + \sum_{j \in N \backslash i} \alpha(\phi_i[V^x_{*ij}] - x_i) \text{ , all } i \in N, \tag{7.3}$$

where α is a fixed positive number, which reflects the assumption that player i does not ask for full correction (when $\alpha_i = 1$) but only (usually) a fraction of it. Note that $\phi_i[V^x_{*ij}] - x_i + \phi_j[V^x_{*ij}] - x_j = 0$ (see, e.g., (6.2) and use the fact that $V^x_{*ij}(ij) = \{(y_i,y_j) : y_i + y_j \leqslant x_i + x_j\}$, because x is Pareto optimal); therefore, $f(x)$ is also Pareto optimal.[11]

We can now consider the dynamic sequence $x = x^0, x^1, x^2, \ldots$, where $x^{q+1} = f(x^q)$, $q = 1,2,\ldots$

Theorem 3: If $0 < \alpha < 4/n$, *then for each* x *in* $\bar{V}(N)$, *the above dynamic sequence converges geometrically to* $\phi[V]$.

Proof: It follows from (6.2), and (7.1) that

$$\phi_i[V^x_{*ij}] + \phi_j[V^x_{*ij}] = x_i + x_j$$

$$\phi_i[V^x_{*ij}] - \phi_j[V^x_{*ij}] = \phi_i[V_{*ij}] - \phi_i[V_{*ij}] \text{ ;}$$

therefore,

$$2(\phi_i[V^x_{*ij}] - x_i) = \phi_i[V_{*ij}] - \phi_j[V_{*ij}] - x_i + x_j \text{ .}$$

Summing up for j in $N \backslash i$, we obtain from (7.3) that for each i in N,

$$f_i(x) = x_i + \frac{\alpha}{2}\{ \sum_{j \in N \backslash i} \phi_i [V_{*\text{ij}}] - \sum_{j \in N \backslash i} \phi_j [V_{*ij}] - (n\text{-}1)x_i + (r_N\text{-}x_i)\} \text{ .}$$

11 Note also that changes in utility scales affect $f(x)$ the same way they affect x.

Now, ϕ is 2-consistent and Pareto optimal; therefore,

$$f_i(x) = x_i + \frac{\alpha}{2}\{(n\text{-}1)\phi_i[V] + r_N - nx_i - \sum_{j \in N\setminus i} \phi_j[V_{*ij}]\} =$$

$$= x_i + \frac{\alpha}{2}\{n\phi_i[V] - nx_i\} .$$

We have proved that

$$x_i^{q+1} = x_i^q + \frac{\alpha n}{2}[\phi_i[V] - x_i^q]$$

or

$$x_i^{q+1} - \phi_i[V] = (1 - \frac{\alpha n}{2})(x_i^q - \phi_i[V]) .$$

If $0 < \alpha < 4/n$, then $-1 < 1 - \alpha n/2 < 1$ and the sequence $(x_i^q - \phi_i[V])_{q=1,2,\ldots}$ is a geometric sequence converging to zero.

8 Concluding Remarks

At present, the extension of the consistent value to arbitrary non-side-payment games is an open problem. Apparently, for unanimity games such a value should be the Nash point, which is even strongly consistent (Section 1). See Lensberg [3], who uses strong consistency to characterize the Nash point for such games. It seems that such extension should be a fixed point of a differential relation, at least in the dynamic approach, because one cannot usually perform discrete simultaneous corrections without leaving the Pareto optimal surface of $V(N)$, unless it is a hyperplane. We consider such an extension as a challenging and interesting enterprise, and we plan to propose one extension in a subsequent paper.

References

[1] Aumann RJ and Maschler M (1985) "Game Theoretic Analysis of a Bankruptcy Problem from the Talmud". Journal of Economic Theory 36, pp. 195-213

[2] Hart S and Mas-Collel A (1989) "Potential, Value and Consistency". To appear in Econometrica.

[3] Lensberg T (1988) "Stability and the Nash Solution." Journal of Economic Theory 45, pp. 330-341.

[4] Orshan G "The Consistent Shapley Value in Hyperplane Games from a Global Standpoint". M. Sc. Thesis, Dept. of Mathematics, The Hebrew University of Jerusalem, November 1987.

[5] Peleg B (1986) "On the Reduced Game Property and Its Converse." International Journal of Game Theory 15, pp. 187-200.

[6] Peleg B (1985) "An Axiomatization of the Core of Coperative Games Without Side Payments." Journal of Mathematical Economics 14, pp. 203-214.

[7] Shapley LS (1953) "A Value for n-Person Games." Contributions to the Theory of Games II, Annals of Mathematics Study 28, ed. Tucker AW and Kuhn HW, pp. 307-317.

[8] Sobolev AI (1975) "The Characterization of Optimality Principles in Cooperative Games by Functional Equations." Mathematical Methods in the Social Sciences (Russian language), ed. N.N. Vorobjev, Vilnius, pp. 94-151.

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