ORBITAL PERTURBATIONS DUE TO RADIATION PRESSURE FOR A SPACECRAFT OF COMPLEX SHAPE*

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Abstract. We analyze the perturbations due to solar radiation pressure on the orbit of a high artificial satellite. The latter is modelled in a simplified way (axisymmetric body plus despun antenna emitting a radio beam), which seems suitable to describe the main effects for existing telecommunication satellites. We use the regularized general perturbation equations, by expressing the force in the moving Gauss' reference frame and by expanding the results in terms of some small parameters, referring both to the orbit (small eccentricity and inclination) and to the spacecraft's attitude. Some interesting results are derived, which assess the relative importance of different physical effects and of different parts of the spacecraft in determining the long-term evolution of the orbital elements.

List of the Main Symbols

S'	spacecraft's cross section
S'_A	antenna's cross section
W'	transmitted power by the antenna
m	mass of the spacecraft
$\hat{\omega}$	unit vector along the spin axis of the spacecraft
$\hat{\mathbf{n}}$	unit vector along the symmetry axis of the despun antenna
$\hat{\mathbf{e}}$	unit vector along the radiowave beam transmitted by the spacecraft
$\hat{\mathbf{s}}$	unit vector toward the Sun
ψ'	sun angle: between $\hat{\omega}$ and $\hat{\mathbf{s}}$
ψ	sun angle: between $\hat{\mathbf{n}}$ and $\hat{\mathbf{s}}$
μ	perturbation parameter; magnitude of $\mathbf{F}$ divided by Earth's gravitational acceleration on the spacecraft
$\mathbf{F}$	perturbating acceleration produced by radiation pressure force on the spacecraft
A, B, C, D, E	components of perturbing force according to the model given in Equation (1.4)
D', E'	first order terms in the expansion of D, E in D'Alembert series (see Equation (2.17))
D_k, E_k	Fourier components of D, E as function of λ
$\hat{\mathbf{e}}_S$	unit vector from the spacecraft toward the center of the Earth
$\hat{\mathbf{e}}_T$	unit vector $\hat{\mathbf{e}}_W \times \hat{\mathbf{e}}_S$
$\hat{\mathbf{e}}_W$	unit vector normal to orbital plane
S, T, W	components of the perturbing force along $\hat{\mathbf{e}}_S$, $\hat{\mathbf{e}}_T$, $\hat{\mathbf{e}}_W$
η	angle between $\hat{\mathbf{e}}_W$ and $\hat{\omega}$
ξ	angle between the projection of $-\hat{\mathbf{e}}_S$ on the equatorial plane of the spacecraft (the plane normal to $\hat{\omega}$) and $\hat{\mathbf{e}}$

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ζ	angle measured on the orbital plane from the origin of the angles to the projection of $\hat{\omega}$
σ	angle between $\hat{\mathbf{n}}$ and the equatorial plane of the spacecraft
$\mathscr{L}, \mathscr{C}$	complex representation of $\hat{\mathbf{s}}$ in the equinoctial system (see Equation (2.16))
$a, e, i, \Omega, \tilde{\omega}$	usual Keplerian elements of the spacecraft orbit
n	mean motion of the spacecraft
v, M	true and mean anomaly of the spacecraft
θ, λ	true and mean longitude of the spacecraft
ρ, ε	decomposition of λ according to Equation (2.1)
H	regularized element in complex form: $H = e \exp(j\tilde{\omega})$
P	regularized element in complex form: $P = tgi \exp(j\tilde{\omega})$
j	$\sqrt{-1}$
$\bar{H}$	complex conjugate of H
y	number of spacecraft's orbits in 1 yr
α	absorption coefficient.

1. Introduction

The main limitation to an accurate determination and prediction of satellite trajectories for high Earth's orbits is the poor modelling of the perturbations due to solar radiation pressure. This problem becomes very important when precise tracking methods are used, like laser ranging or two-frequency Doppler tracking. There are two main difficulties in constructing a good model: (a) the complex physical interaction between the sunlight and all the spacecraft surfaces and the power system; (b) the large secular effects in the orbital perturbations which may result from a small error in the force model.

In this paper we concentrate on the second aspect of the problem, but we rely upon a fairly general model of the spacecraft, suitable for most telecommunication satellites. We shall neglect the effects of the Earth's albedo (at geosynchronous distance this contributes only for about 1% of the total radiation pressure); moreover, we shall not consider eclipses. The effect of the eclipses is very complicated even for spherical satellites (Kozai, 1963) and is difficult to model accurately because of penumbra effects (Aksnes, 1976). However, a high satellite is not eclipsed for long orbital arcs (about 140 days for geosynchronous orbits) and we restrict our analysis to these arcs.

The main body of the spacecraft is assumed to be axially symmetric (or rapidly spinning) around an axis fixed in the inertial space, corresponding to the unit vector $\hat{\omega}$. In this case, the radiation acceleration is of the form:

$$\mathbf{F}' = \left(\frac{a_\odot}{r_\odot}\right)^2 [A'(\psi')\hat{\mathbf{s}} + B'(\psi')\hat{\omega}]. \tag{1.1}$$

A' and B' are accelerations depending on the angle ψ' between the spin axis and the direction of the Sun, given by the unit vector $\hat{\mathbf{s}}$. The order of magnitude of $\mathbf{F}'$ is

$$|\mathbf{F}'| \sim \frac{\Phi S'}{mc} \equiv \frac{\text{solar constant} \times \text{spacecraft's cross section}}{\text{velocity of light} \times \text{spacecraft's mass}}, \tag{1.2}$$

i.e., for $S' = 10^4\ \mathrm{cm}^2$, $m = 5 \times 10^5\ g$, about $10^{-6}\ \mathrm{cm\ s}^{-2}$. The coefficient $(a_\odot/r_\odot)^2$ in Equation (1.1) takes into account the effect due to the difference between $a_\odot = 1$ AU and the actual distance $r_\odot$ between the spacecraft and the Sun. This latter depends essentially on the orbital eccentricity of the Earth; we neglect the changes both in $r_\odot$ and in the Sun's direction due to the geocentric orbit of the spacecraft. Both $r_\odot$ and ψ' are periodic functions of the time, with the period of one year. For long missions one should take into account also the additional time dependences caused by the changes in the optical coefficients of the spacecraft's surface (due to the space environment) and by the decrease in the mass of the spacecraft due to the fuel consumption.

In general the functions $A'(\psi')$ and $B'(\psi')$ depend both on the geometry and on the optical properties of the spacecraft's surface; in practice their calculation is rather intricate because of the occurrence of mutual shadowing effects and multiple reflections between different parts of the spacecraft. If it is spherical or completely absorbing, $B' = 0$; $B'(\psi')/A'(\psi')$ is, in general, fairly small.

Another significant effect may be produced by the thermal re-emission of the absorbed radiation, which may occur in a highly anisotropic way due to the anisotropy of shape, surface temperature and emissivity. In particular, the surface temperature is not easy to model, depending not only on the absorbed sunlight but also on the internal heat sources and on the thermal properties of the whole spacecraft. If the spacecraft's thermal properties are axially symmetric as well (or if the body is rapidly spinning), this effect can be included in the B' term, yielding a fraction of the total acceleration of the order of the mean absorption coefficient times the relative temperature difference between the upper and lower part of the body.

To the acceleration $\mathbf{F}'$ coming from the spacecraft body we add another component due to another part of the spacecraft axially symmetric around a unit vector $\hat{\mathbf{n}}$; this vector makes a constant angle with $\hat{\omega}$ and rotates around it with the orbital period, in such a way that its projection on the orbital plane always points towards the Earth's center or, possibly, makes a small and constant offset angle with it. This component corresponds to the telecommunication system, e.g. a despun, Earth-pointing antenna with an axis $\hat{\mathbf{n}}$; one could also have a despun plate normal to $\hat{\mathbf{n}}$ reflecting the beam from an antenna fixed to the main body on its axis. If ψ is the angle between $\hat{\mathbf{s}}$ and $\hat{\mathbf{n}}$, this additional acceleration is necessarily of the type

$$\tilde{D}(\psi)\hat{\mathbf{s}} + \tilde{E}(\psi)\hat{\mathbf{n}}; \tag{1.3}$$

it depends on the Sun's anomaly and the spacecraft's true longitude θ. Its order of magnitude is $\Phi S'_A/mc$, where S'_A is the antenna's cross section. In the case of a despun plate D could be expressed as

$$R' \cos\psi + R''|\cos\psi|,$$

where the two terms take into account the possibility of different optical properties of the two surfaces and R', R'' are constant; in general, the situation can be more complicated. Moreover, we shall always neglect the mutual shadowing (and the multiple reflections and diffusions) between the body and the antenna. These phenomena cannot be treated in any simple way, and we shall assume that their orbital effect is not of outstanding importance.

If $\hat{\mathbf{e}}$ is the unit vector along the radio beam emitted by the satellite, there is an other component $C\hat{e}$ due to the transmitted power itself. Clearly, this cannot be negligible, if a relevant part of the surface is covered with solar cells, whose absorbed power is in part re-irradiated in the radio beam. $\hat{\mathbf{e}}$ is supposed to lie in the plane of $\hat{\omega}$ and $\hat{\mathbf{n}}$ (and to be orthogonal to $\hat{\omega}$) (see Figure 1); if W' is the transmitted power, the order of magnitude of C is W'/mc.

To summarize, we consider an acceleration of the general form

$$\mathbf{F} = [A + D(\theta)]\hat{\mathbf{s}} + B\hat{\omega} + E(\theta)\hat{\mathbf{n}} + C\hat{\mathbf{e}}. \tag{1.4}$$

In $A \equiv (a_\odot/r_\odot)^2 A'$, $B \equiv (a_\odot/r_\odot)^2 B'$ we have now included the slow change due to the distance from the Sun (as we have implicity done for $\tilde{D}$ and $\tilde{E}$. D and E are the same physical quantities as $\tilde{D}$ and $\tilde{E}$, but expressed as functions of θ).

The plan of our work is conceptually simple. Each of the five terms in Equation (1.4) gives a contribution to the force components in the Gauss' moving reference frame (as defined in Section 2). Using the general perturbation equations we can, in principle, evaluate the various periodic and secular terms for each orbital element (Sections 3 and 4) at the first order in the perturbation parameter μ, given by the ratio between the perturbing acceleration and the principal 'monopole' term (for a geosynchronous satellite, $\mu \approx 10^{-7}$). The separation between periodic and secular terms is useful, because orbit determination generally refers to arcs much longer than one orbital period. The arc length is anyway limited, at least for active satellites, not only by the eclipses but also by attitude and orbital manoeuvers, which are usually performed every few months. For this reason the interaction with the orbital changes caused

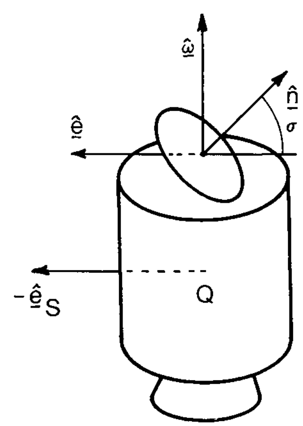

Fig. 1. Scheme of the spacecraft. The model used in this paper is showed. Q is the center of mass of the satellite, $-\hat{\mathbf{e}}_S$ is the unit vector towards the center of the Earth. The angle between $\hat{\omega}$ and $\hat{\mathbf{e}}$ is $\pi/2$; the angle between $\hat{\omega}$ and $\hat{\mathbf{n}}$ is $\frac{1}{2}\pi - \sigma$. The other symbols are explained in the text.

by other perturbations (e.g., Earth's oblateness), which have longer timescales, can be neglected (see Section 3).

The calculation becomes practical and significant for a high Earth satellite with small eccentricity e and small inclination i with respect to some reference plane; hence, we need also to express the perturbation equations in terms of a set of regularized variables (see Section 2). In addition to e and i, two more 'small parameters' will be defined in order to get linearized expressions: η, the angle between the spin axis and the normal to the orbital plane, and ξ, the misalignment angle of the antenna with respect to the Earth's center (for a high gain antenna both η and ξ must be small). Then, we have, for each orbital element and for each term in Equation (1.4), secular and periodic contributions of zero-order, first order, etc., in the four small parameters e, i, η, ξ.

As an example, for a very precise determination of the orbit of a high Earth satellite the secular change in the semi-major axis has a crucial role, resulting into a quadratically-growing longitude perturbation. It is easy to see that, if the eccentricity is neglected, the only secular effect in the semimajor axis comes from the tangential component of the radiative force; for a satellite of constant attitude this component averages to zero. In the simplest case of a spherical satellite, this result was known from the first years of the space age (Musen, 1960), but we prove that it holds for an arbitrarily shaped spacecraft of constant attitude (see Appendix). In Section 3 it is shown how this conclusion is modified by the antenna.

Finally, we discuss (Section 5) how the different terms of the resulting orbital perturbation affect the orbit determination of high Earth satellites (with a particular attention for geosynchronous satellites), when precise tracking methods are used. In the particular case of a rapidly spinning synchronous satellite carrying on board a high-gain despun antenna we derive the order of magnitude of the radiation pressure perturbations from the analytical results.

2. General Perturbation Equations

To evaluate the effects of the radiation pressure on the satellite's orbital elements $a, e, i, \Omega, \tilde{\omega}, \lambda$ (semimajor axis, eccentricity, inclination, longitude of the node, longitude of the pericenter and mean longitude of the osculating ellipse) we use the general perturbation equations in Gauss' form (see Roy, 1978, p. 184). They are obtained by expressing the perturbing acceleration $\mathbf{F}$ in its S, T, W components along the unit vectors of the righthanded orthogonal moving frame $\hat{\mathbf{e}}_S, \hat{\mathbf{e}}_T, \hat{\mathbf{e}}_W$:

$\hat{\mathbf{e}}_S$: from the center of the Earth to the spacecraft;

$\hat{\mathbf{e}}_T$: normal to $\hat{\mathbf{e}}_S$ in the orbital plane;

$\hat{\mathbf{e}}_W$: normal to the orbital plane.

The mean longitude λ is decomposed into two parts (in order to simplify the doubl integration):

$$\lambda = \int_{t_0}^{t} n \, dt' + \varepsilon = \rho + \varepsilon, \tag{2.1}$$

where n is the mean motion. The equations are:

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}}(T + Se \sin v + Te \cos v) \tag{2.2}$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na}\left[S \sin v + T\left(\frac{e \cos v + e^2}{1 + e \cos v} + \cos v\right)\right] \tag{2.3}$$

$$\frac{di}{dt} = \frac{W\sqrt{1-e^2}}{na}\cos(\tilde{\omega} - \Omega + v) \tag{2.4}$$

$$\frac{d\Omega}{dt} = \frac{W\sqrt{1-e^2}}{na \sin i}\sin(\tilde{\omega} - \Omega + v) \tag{2.5}$$

$$\frac{d\tilde{\omega}}{dt} = \frac{\sqrt{1-e^2}}{nae}[-S \cos v + T(2 + e \cos v)\sin v] + 2\frac{d\Omega}{dt}\sin^2\frac{i}{2} \tag{2.6}$$

$$\frac{d\varepsilon}{dt} = \frac{e^2}{1 + \sqrt{1-e^2}}\frac{d\tilde{\omega}}{dt} + 2\frac{d\Omega}{dt}\sqrt{1-e^2}\sin^2\frac{i}{2} - \frac{2(1-e^2)S}{na(1 + e \cos v)} \tag{2.7}$$

$$\frac{d^2\rho}{dt^2} = -\frac{3}{2}\frac{n}{a}\frac{da}{dt} \tag{2.8}$$

(v is the true anomaly of the satellite).

These equations allow some immediate conclusions about the effects of the radiation pressure force in the case in which the acceleration **F** does not depend on the spacecraft position (constant attitude satellite with no rotating antenna). These results, which do not depend on the specific adopted force model, will be discussed in the Appendix. In the following the model of Equation (1.4) is used.

Since we are interested in expanding the perturbations with respect to e, i, assumed to be small, Equations (2.2)–(2.7) are not in a suitable form because they are singular for zero e and i. The singularity is not a physical problem but only an effect of the use of a singular set of variables, hence we adopt a set of regular elements which are conveniently expressed in the complex form:

$$H = e \exp(j\tilde{\omega})$$

$$P = \operatorname{tg} i \exp(j\Omega) = i \exp(j\Omega) + 0(i^2),$$

where $j = \sqrt{-1}$ and $0(i^2)$ contains the terms of order $\geqslant 2$ in i. The regularization is performed by using H, P instead of $e, \tilde{\omega}, i, \Omega$ and expressing the remaining angular variable in terms of the true longitude $\theta = \tilde{\omega} + v$. The following step is the linearization with respect to the small parameters H, P; this requires also the expansion of θ in D'Alembert series with respect to λ, H (see Roy, 1978, p. 86):

$$\exp(j\theta) = \exp(j\lambda) - H + \bar{H}\exp(2j\lambda) + 0(H^2),$$

where $0(H^2)$ contains the terms of order $\geqslant 2$ in H, while $\bar{H}$ is the complex conjugate of H. In this way the following regular and linearized set of equations is obtained:

$$\frac{2\,\mathrm{d}a}{n\,\mathrm{d}t} = T + \mathrm{Im}[\bar{H}\exp(j\lambda)]S + \mathrm{Re}[\bar{H}\exp(j\lambda)]T \tag{2.9}$$

$$na\frac{\mathrm{d}H}{\mathrm{d}t} = (2T - jS)\exp(j\lambda) + (\tfrac{3}{2}T - jS)[\bar{H}\exp(2j\lambda) - H] \tag{2.10}$$

$$na\frac{\mathrm{d}P}{\mathrm{d}t} = [\exp(j\lambda) - \tfrac{3}{2}H + \tfrac{1}{2}\bar{H}\exp(2j\lambda)]W \tag{2.11}$$

$$na\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = -2S + \mathrm{Im}[\bar{H}\exp(j\lambda)]T - \tfrac{5}{2}\mathrm{Re}[\bar{H}\exp(j\lambda)]S + \mathrm{Im}[\bar{P}\exp(j\lambda)]W \tag{2.12}$$

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}t^2} = -\frac{3n\,\mathrm{d}a}{2a\,\mathrm{d}t}. \tag{2.8}$$

In order to introduce in the Equations (2.8)–(2.12) the adopted force model we need to express the unit vectors $\hat{\mathbf{s}}$, $\hat{\omega}$, $\hat{\mathbf{n}}$, $\hat{\mathbf{e}}$ in the $\hat{\mathbf{e}}_S$, $\hat{\mathbf{e}}_T$, $\hat{\mathbf{e}}_W$ 'Gauss' frame. This can be conveniently done by expressing the force model in an inertial Earth-centered reference frame with the x and y axes in the orbital plane, the x axis being rotated by an angle $-\tilde{\omega}$ from the pericenter direction. This definition can be extended to the $e = 0$ case by a passage to the limit in which the angle between the satellite and the x axis remains θ ('equinoctial' reference frame). In this reference frame we have:

$$\hat{\mathbf{e}}_S = (\cos\theta, \sin\theta, 0)$$

$$\hat{\mathbf{e}}_T = (-\sin\theta, \cos\theta, 0)$$

$$\hat{\mathbf{e}}_w = (0, 0, 1).$$

We assume that $\hat{\omega}$ does not deviate much from the direction $\hat{\mathbf{e}}_W$ of the nominal attitude. In this hypothesis, at first order in the small parameter η (η is the angle between $\tilde{\omega}$ and $\hat{\mathbf{e}}_W$), the components of $\hat{\omega}$ in the moving frame are:

$$\hat{\omega} = (\eta\cos(\zeta - \theta), \eta\sin(\zeta - \theta), 1), \tag{2.13}$$

where ζ defines the azimuthal orientation of $\hat{\omega}$ (see Figure 2). Since we have assumed that $\hat{\mathbf{e}}\cdot\hat{\omega} = 0$ and that the misalignment angle ξ is small, again at first order we have:

$$\hat{\mathbf{e}} = (-1, -\xi, \eta\cos(\zeta - \theta)). \tag{2.14}$$

The normal to the antenna $\hat{\mathbf{n}}$ is in the plane of $\hat{\omega}$, $\hat{\mathbf{e}}$; hence

$$\hat{\mathbf{n}} = -\hat{\mathbf{e}}\cos\sigma + \hat{\omega}\sin\sigma, \tag{2.15}$$

σ being the angle between $\hat{\mathbf{n}}$ and the equatorial plane of the spacecraft (see Figure 1). The Sun direction $\hat{\mathbf{s}}$ can be expressed as:

$$\hat{\mathbf{s}} = (\mathrm{Re}[\mathscr{L}\exp(-j\theta)], \mathrm{Im}[\mathscr{L}\exp(-j\theta)], \mathscr{C}) \tag{2.16}$$

with $\mathscr{L}$ and $\mathscr{C}$ functions of the time only through the mean longitude of the Sun.

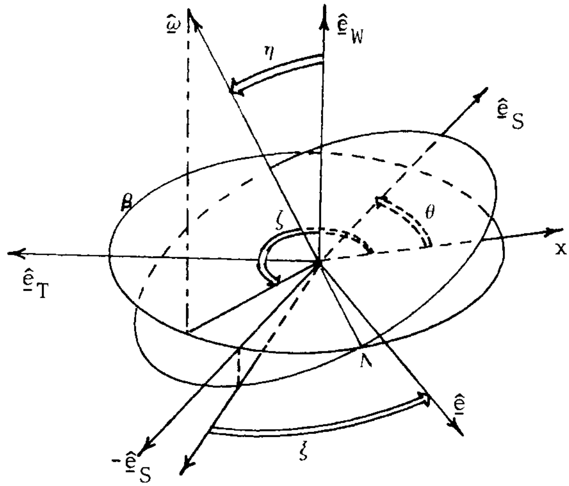

Fig. 2. Unit vectors and reference system. The figure shows the relative orientation between the Gauss' reference frame ($\hat{e}_S$, $\hat{e}_T$, $\hat{e}_W$) and the spacecraft's attitude frame ($\hat{e}$, $\hat{\omega}$). θ is the true longitude, measured from the x axis of the equinoctial orbital frame, ζ is the azimuth of the projection on the orbital plane of the $\hat{\omega}$ unit vector. N is the nodal line and β is the orbital plane.

By using Equations (2.13)–(2.16) the components of the force in the moving frame are obtained; now the trigonometric functions of θ can be expanded in power series of H with argument λ, while the periodic functions $D(\theta)$, $E(\theta)$ are expanded as follows:

$$D(\theta) = D(\lambda) + HD'(\lambda) + 0(H^2)$$
$$E(\theta) = E(\lambda) + HE'(\lambda) + 0(H^2). \tag{2.17}$$

Hence, the linearized components of the perturbing acceleration are:

$$\begin{aligned} S = {} & [A + D(\lambda)]\mathrm{Re}[\mathscr{L}\exp(-j\lambda) - \mathscr{L}\bar{H} + \mathscr{L}H\exp(-2j\lambda)] + E(\lambda)\cos\sigma - \\ & - C + \eta[B + E(\lambda)\sin\sigma]\cos(\zeta - \lambda) + HD'(\lambda)\mathrm{Re}[\mathscr{L}\exp(-j\lambda)] + \\ & + HE'(\lambda)\cos\sigma; \end{aligned} \tag{2.18}$$

$$\begin{aligned} T = {} & [A + D(\lambda)]\mathrm{Im}[\mathscr{L}\exp(-j\lambda) - \mathscr{L}\bar{H} + \mathscr{L}H\exp(-2j\lambda)] + \\ & + \eta[B + E(\lambda)\sin\sigma]\sin(\zeta - \lambda) + \xi[E(\lambda)\cos\sigma - C] + \\ & + HD'(\lambda)\mathrm{Im}[\mathscr{L}\exp(-j\lambda)]; \end{aligned} \tag{2.19}$$

$$\begin{aligned} W = {} & [A + D(\lambda)]\mathscr{C} + B + E(\lambda)\sin\sigma + \eta[C - E(\lambda)\cos\sigma]\cos(\zeta - \lambda) + \\ & + HE'(\lambda)\sin\sigma. \end{aligned} \tag{2.20}$$

In the following these expressions, together with the perturbation Equations (2.8)–(2.12), will be used to study the effects of our force model on the various orbital elements.

3. Long-Periodic Effects on Semimajor Axis and Longitude

Substituting Equations (2.18) and (2.19) into Equation (2.9) we find $\mathrm{d}a/\mathrm{d}t$. The most relevant changes in the satellite position due to radiation pressure are produced by long-term changes in a; these can be approximately computed by averaging with respect to the fast angular variable λ, with an error (of the second order in the perturbation parameter μ) that – apart from short periodic effects – remains negligible for a long time (many orbital periods). This is equivalent to the classical technique of operating formally on the Fourier series expansions of the orbital elements, retaining only long-periodic or secular terms; hence difficulties can arise from resonances ('small divisors'), but in this case the resonances are not very important (see the Appendix). The results can be expressed in terms of the Fourier coefficients of the periodic functions D, E, D', E' (the latter being defined by Equation (2.17)), e.g.:

$$D_k = \frac{1}{2\pi}\int_0^{2\pi} D(\lambda) e^{-jk\lambda}\,\mathrm{d}\lambda.$$

We obtain, for the long-period and secular effects (generically indicated by the subscript $l.p.$):

$$\left.\frac{n\,\mathrm{d}a}{a\,\mathrm{d}t}\right|_{l.p.} = \mathrm{Im}[D_1\mathscr{L}] + \eta\sin\sigma\,\mathrm{Im}[E_1\exp(j\zeta)] + \xi[E_0\cos\sigma - C] +$$

$$+ \mathrm{Im}[H\mathscr{L}D_2] - \mathrm{Im}[HE_1\cos\sigma] + \mathrm{Im}[H\mathscr{L}D_1']. \tag{3.1}$$

The main term (i.e. the one containing no small parameters) is $\mathrm{Im}[D_1\mathscr{L}]$; but in the hypothesis that D is a function of $\cos\psi = \hat{\mathbf{n}}\cdot\hat{\mathbf{s}}$ only (axially symmetric antenna), this term is zero. This can be proved as follows. We have (see Equations (2.15)–(2.16)):

$$\cos\psi = \hat{\mathbf{n}}\cdot\hat{\mathbf{s}} = \mathrm{Re}[\mathscr{L}\exp(-j\theta)]\cos\sigma + \mathscr{C}\sin\sigma + 0(\xi,\eta), \tag{3.2}$$

where $0(\xi,\eta)$ indicates all the terms containing the small parameters ξ,η; hence (to order zero in ξ,η,H) D can be expressed in the following way:

$$D(\lambda) = f[\mathrm{Re}(\mathscr{L}\exp(-j\lambda))] \tag{3.3}$$

and its Fourier coefficients can be computed as follows:

$$\mathscr{L}D_k = \frac{1}{2\pi}\int_0^{2\pi} f[\mathrm{Re}(\mathscr{L}\exp(-j\lambda))]\mathscr{L}\exp(-jk\lambda)\,\mathrm{d}\lambda. \tag{3.4}$$

This expression is real for symmetry reasons. Q.E.D.

By analyzing Equation (3.1) we deduce the following relevant conclusions:

(1) The body causes no long-periodic effect on a (this is true even for a non-axisymmetric body; see the Appendix).

(2) The antenna causes no long-term effect in the case of the 'nominal' attitude and of zero eccentricity; but this is not an exact result because the mutual shadowing between the body and the antenna has been excluded from our model.

(3) The long-term effects on a, to order zero in H, are due to the E component (antenna reflection and diffusion) and to the C component (radiowaves transmission). The latter is a truly secular effect (as opposed to the essentially annual effects of the other terms).

The longitude effects are obtained by the double integration into Equation (2.8) plus the effects on ε; by substituting Equations (2.18)–(2.20) into Equation (2.12) and by averaging, we obtain the long-periodic evolution of ε:

$$
\begin{aligned}
na\frac{d\varepsilon}{dt}\bigg|_{l.p.} &= 2[C - E_0 \cos\sigma] - 2\,\mathrm{Re}[D_1\mathscr{L}] - 2\eta \sin\sigma\,\mathrm{Re}[E_1 \exp(j\zeta)] - \\
&\quad - \tfrac{1}{2}\mathrm{Im}[(D_1\mathscr{C} + E_1 \sin\sigma)P] + \tfrac{1}{4}[A + D_0]\mathrm{Re}[\bar{H}\mathscr{L}] - \\
&\quad - \tfrac{11}{4}\mathrm{Re}[HD_2\mathscr{L}] - 2\,\mathrm{Re}[HD_1'\mathscr{L}] - \\
&\quad - 2HE_0' \cos\sigma - \tfrac{5}{2}\cos\sigma\,\mathrm{Re}[HE_1].
\end{aligned} \tag{3.5}
$$

From this formula we see that the spacecraft body (with constant attitude) produces long-periodic effects on ε proportional to the eccentricity; hence for zero eccentricity there is no long-periodic effect in longitude. This also is true for more general spacecraft bodies (see Appendix).

To add together the long-periodic (mainly annual) terms of Equation (3.5) and the effects on ρ due to long-periodic terms on a, we must take into account that the double integration of an annual term introduces a factor y^2, if y is the number of spacecraft's orbits in one year; on the other hand the single integration requested by Equation (3.5) introduces only a factor y. If $\xi y, \eta y \sim 1$, the terms containing ξ, η in $\Delta\rho$ are as significant as the zero order terms in $\Delta\varepsilon$. The resulting amplitudes are summarized in Table I.

The terms containing the inclination (or P) have a special status. At the first order in the perturbing parameter μ (magnitude of the radiation pressure acceleration divided by the gravitational monopole term) we can assume that P to be inserted in the formulas is the initial value; but being the reference plane completely arbitrary, the initial value can be always chosen to be zero. On the other hand, if other perturbations are present, e.g. those due to J_2 (Earth'oblateness), mixed terms containing products as $J_2\mu$ do appear. They can be estimated by putting $P = P(t)$, a solution of the long-periodic J_2 perturbation problem, into the right-hand side of Equation (3.5). The mixed terms are not at all negligible over time spans comparable to the precession period of the nodes (For the theory of the mixed effects in the spherical satellite case, see Hori, 1966). A similar argument holds for the precession of the perigee induced by J_2. However, for high satellites, whose orbit is affected by the radiation pressure as the main non-gravitational perturbation, the precession periods

TABLE I

Long-periodic perturbations in longitude λ.

All the entries must be divided by n^2 and multiplied by the corresponding parameter to give the order of magnitude of the perturbation. For the truly secular terms (as those containing C, the transmission) the effects are computed after a time of the order of 1 yr/2π.

multiplier	$a\Delta\rho$	$a\Delta\varepsilon$
1	0;	$2yE_0 \cos\sigma$; $2yC$; $2yD_1$;
H	$3y^2D_2$; $3y^2E_1$; $3y^2D'_1$;	$\frac{y}{4}(A + D_0)$; $\frac{11}{4}yD_2$; $2yD'_1$; $2yE'_0 \cos\sigma$; $\frac{5}{2}yE_1 \cos\sigma$;
P	0;	$\frac{y}{2}eD_1$; $\frac{y}{2}E_1 \sin\sigma$;
η	$3y^2E_1 \sin\sigma$;	$2yE_1 \sin\sigma$;
ξ	$3y^2E_0 \cos\sigma$; $3y^2C$;	0;

of perigee and node are very long (>1 yr), and this justifies a treatment neglecting the mixed terms over moderate intervals of time.

4. Long-Periodic Effects on Eccentricity and Inclination

The inclination and the node are affected only by the out-of-plane component W, hence substituting Equation (2.20) in Equation (2.11) and averaging we have:

$$na\frac{\mathrm{d}P}{\mathrm{d}t}\bigg|_{l.p.} = D_{-1}\mathscr{C} + E_{-1}\sin\sigma + \frac{\eta}{2}[C - E_0\cos\sigma]\exp(j\zeta) - \\ -\frac{\eta}{2}E_{-2}\cos\sigma\exp(-j\zeta) + \\ + H[D'_{-1}\mathscr{C} + E'_{-1}\sin\sigma - \tfrac{3}{2}(A + D_0)\mathscr{C} - \tfrac{3}{2}B - \tfrac{3}{2}E_0\sin\sigma] + \\ + \frac{\bar{H}}{2}[D_{-2}\mathscr{C} + E_{-2}\sin\sigma]. \qquad (4.1)$$

The qualitative informations contained in Equation (4.1) can be summarized as follows:

(1) To order zero in the eccentricity, there is no long-periodic effects in P produced by the constant attitude spacecraft body. This is also true for more general shapes (see Appendix);

(2) For a spacecraft with a despun antenna there are long-periodic effects in P containing no small parameter. This affects in a very important way the determination of some geophysical parameters (see Section 5).

The formula for the long-periodic perturbations in H, obtained by Equations

(2.10), (2.18) and (2.19), is more complex:

$$
\begin{aligned}
na\frac{\mathrm{d}H}{\mathrm{d}t}\bigg|_{l.p.} &= -\tfrac{3}{2}j[A + D_0]\mathscr{L} + \frac{j}{2}D_{-2}\mathscr{P} - JE_{-1}\cos\sigma + \\
&\quad + \eta\left[\frac{j}{2}E_{-2}\sin\sigma\exp(-j\zeta) - \tfrac{3}{2}j(B + E_0\sin\sigma)\exp(j\zeta)\right] + \\
&\quad + 2\xi E_{-1}\cos\sigma + D_{-1}[\tfrac{1}{4}\mathscr{L}\bar{H} - \tfrac{3}{4}\mathscr{P}H] - \tfrac{1}{4}D_{+1}\mathscr{L}H + \\
&\quad + \tfrac{3}{4}D_{-3}\mathscr{P}\bar{H} - \tfrac{3}{2}D'_0\mathscr{L}H + \tfrac{1}{2}D'_{-2}\mathscr{P}H + H[E_0\cos\sigma - C] - \\
&\quad - \cos\sigma[HE'_{-1} + \bar{H}E_{-2}]. \qquad (4.2)
\end{aligned}
$$

The essential feature of Equation (4.2) is the appearance of zero-order terms both from the spacecraft body and from the despun antenna. As it is known, in the spherical satellite case ($A = \text{const}, B = C = D = E = 0$) the apsidal line rotates with the Sun. In the general case the path described by H in the $(e, \tilde{\omega})$ plane is not a simple ellipse, but the main qualitative feature is the same: a forced eccentricity arises, of the order of $\frac{3}{2}A(y^2/n^2a)$ (Van der Ha and Modi, 1977).

5. Effects on the Orbit Determination Accuracy

In this section, in order to show how to use the formulas given in the paper and to asses the physical significance of the various effects, we shall compute the order of magnitude of the most important terms in the orbital perturbations for a simple example.

As an idealized, but representative case we choose a geosynchronous satellite with $e = 0.001$; the spinning spacecraft body is assumed to be spherical* with an area-to-mass ratio of 0.05 cm^2 g^{-1}; the despun antenna is assumed to be flat with an area equal to 1/3 of the cross section of the sphere and with $\sigma = 45°$. In this case the order of magnitude of the total accelaration due to the radiation pressure is (see Equation (1.2)):

$$\frac{\phi S'}{mc} = 2.3 \times 10^{-6}\ \text{cm s}^{-2}.$$

To compute the various terms of Equation (1.4) we make the following assumptions:

(1) The optical properties are specified by a reflectivity of 0.2 and an absorptivity α of 0.8 (no diffusivity); hence, in CGS units:

$$A = 2.3 \times 10^{-6},$$

$$D = 6.2 \times 10^{-7}|\cos\psi|,$$

$$E = 3.1 \times 10^{-7}|\cos\psi|\cos\psi\,;$$

* The cylindrical case does not differ in any essential way.

(2) The power system dissipates the energy in the lower part of the spacecraft, yielding (with the anisotropic solar illumination) an average temperature difference ΔT of the lower half-sphere with respect to the upper one, such that $|\Delta T|/T = 1/30$. From radiation balance considerations we find

$$B = \frac{4}{3}\alpha\frac{\Delta T}{T}A \simeq 0.035A;$$

(3) The emitted power in the radio beam is supposed to be 3% of the radiation incident on the sphere; this means

$$C = 0.03A;$$

(4) We assume for the spacecraft's orientation $\xi = \eta = 0.01$ rad. The long-term effects in the semimajor axis can be evaluated with Equation (3.1), noting that the first term is zero (see Equation (3.4)): the main terms have the amplitudes

$$\frac{\sqrt{2}}{2}\eta E_1, \qquad \frac{\sqrt{2}}{2}\xi E_0, \qquad \xi C,$$

while the terms containing H are smaller by one order of magnitude. The exact value of E_0, E_1 is a function of the Sun's mean anomaly $M_{\odot}$ and of the initial inclination with respect to the ecliptic, but their order of magnitude is $\frac{1}{2} \times 3.1 \times 10^{-7}$ CGS units. Then the effect in 1/4 of a year is, for each of the E terms:

$$\Delta a \sim \frac{y\sqrt{2}}{n^2}(1.1 \times 10^{-9}) \simeq 1 \text{ m}.$$

For the C term we have a truly secular effect with an amplitude over 1/4 of a year of:

$$\Delta a \sim \frac{2y}{n^2}(7 \times 10^{-10}) \simeq 0.5 \text{ m}.$$

The corresponding longitude effects are given by

$$a|\Delta\rho| = \tfrac{3}{2}y|\Delta a| \sim 550|\Delta a|.$$

Hence, the combined effect is of the order of 1 km. By comparison, the short-periodic (daily) effects due to the spacecraft body only are

$$\Delta a \sim \frac{2A}{n^2} \simeq 8.6 \text{ m}$$

$$a|\Delta\rho| \sim \frac{3A}{n^2} \simeq 13 \text{ m}.$$

For ε the inspection of Equation (3.5) shows that the main terms produce, over 1/4 of the year, longitude displacements $a\Delta\varepsilon$ of the order of

$$\frac{2y}{n^2}C, \qquad \frac{y\sqrt{2}}{n^2}E_0, \qquad \frac{2y}{n^2}D_1, \qquad \frac{A}{4}\frac{y}{n^2}e,$$

whose respective amplitude is approximately

$$95\text{ m}, \qquad 151\text{ m}, \qquad 426\text{ m}, \qquad 40\text{ m}.$$

Note that the term coming from the body of the spacecraft would become dominant if e were larger than η and ξ.

As regards the inclination, the main terms (see Equation (4.1)) contributing to $a|\Delta i|$ are of the type

$$\frac{y}{n^2}D_1, \qquad \frac{y\sqrt{2}}{2n^2}E_1,$$

respectively of amplitude

$$213\text{ m}, \qquad 75\text{ m},$$

while the most important term due to the body of the satellite is about 2 m (containing a factor e).

The forced eccentricity evaluated in Section 4 gives a term

$$a|\Delta e| \sim \tfrac{3}{2}a\frac{y}{n^2} \sim 237\text{ m}.$$

From this example we can derive useful considerations about the achievable orbit determination accuracy and, in particular, the possibility of using the tracking data of high satellites to recover geophysical informations. Since it is very difficult to model the perturbing force due to the radiation pressure to a high accuracy (in practice, uncertainties of some percents seem unavoidable unless ad hoc satellites are used), the following limitations must be considered:

(1) Even for very short ($\sim 1d$) orbital arcs, it is extremely difficult to predict the satellite position to an accuracy below 1 m.

(2) The recovery of geophysical parameters using a resonant effect in longitude (like the geopotential coefficients with $(l-m)$ even in the case of a synchronous orbit) is limited to a longitude accuracy of the order of 100 m (for an orbital arc of ~ 3 months).

(3) The recovery of geophysical parameters using secular effects in the node (like the zonal geopotential coefficients) or in the inclination (like the polar motion) have an accuracy limitation of tens of meters. For the polar motion, the effect to be determined has, for a geosynchronous orbit, an amplitude of ~ 40 m, with a nearly annual periodicity. It follows that the effects of the radiation pressure, having a similar signature, cannot be easily separated (at least if there is a despun antenna). We note that for a lower satellite the situation is even worse, because the ratio between the effects of radiation pressure and of polar motion is proportional to $1/a^2$.

(4) The long-term effects in eccentricity produce obviously only short-periodic changes in the satellite coordinates. However, if the tracking data have a non-uniform distribution in mean anomaly, some 'systematic' error in the orbit determination can arise.

All these limitations are important because they show that the accuracy of the tracking data is presently better that the accuracy in the modeling of the orbital perturbations. This is not true if special efforts are made in the spacecraft and mission design, as in the case of perfectly spherical satellites without antennas. On the other hand, if the spherical model is used to determine the orbit of spacecraft whose real shape is complex, the recovered parameters (interstation baselines, geopotential coefficients, polar motion) will be biased; some cases of discrepancy between different determinations of these parameters can be attributed to this problem.

Appendix

In this Appendix we prove some results about the long-term perturbations due to radiation pressure for a spacecraft of completely arbitrary shape and thermo-optical properties, in the following hypotheses: (a) the attitude is constant, i.e., the spacecraft's orientation in an inertial reference frame is fixed (implying no despun antenna; the spacecraft can be spun, provided that the spin axis is fixed and that the spin period is much shorter than the orbital period); (b) the surface optical properties do not change with time; (c) the temperature can change, but only as a function of the Sun's position. We summarize these assumptions by stating that the perturbing acceleration **F** induced by the radiation pressure can be expressed as function of the Sun's position vector **s** only:

$$\mathbf{F} = \mathbf{F}(\mathbf{s}).$$

Then the long-term perturbations of the orbital elements have the following properties:

THEOREM 1. *The semimajor axis undergoes no long-periodic or secular perturbation.*

Proof. From Equation (2.2)

$$\frac{n\sqrt{1-e^2}}{2}\frac{\mathrm{d}a}{\mathrm{d}t} = \mathbf{F}(\mathbf{s})\cdot(\hat{\mathbf{e}}_T + e\sin v\,\hat{\mathbf{e}}_S + e\cos v\,\hat{\mathbf{e}}_T).$$

Now we suppose $e \neq 0$ (the result holds even for $e = 0$ by a continuity argument), and use a right-handed reference system with the x axis in the pericenter direction and the z axis in the angular momentum direction. We have

$$\hat{\mathbf{e}}_S = (\cos v, \quad \sin v, \quad 0)$$

$$\hat{\mathbf{e}}_T = (-\sin v, \quad \cos v, \quad 0).$$

If $F_x(\mathbf{s})$, $F_y(\mathbf{s})$ are the components of the perturbing acceleration in this frame, we have:

$$\frac{n\sqrt{1-e^2}}{2}\frac{\mathrm{d}a}{\mathrm{d}t} = -F_x(\mathbf{s})\sin v + F_y(\mathbf{s})(e+\cos v).$$

By averaging over the osculating mean anomaly M of the satellite, since

$$\int_0^{2\pi} \sin v \, \mathrm{d}M = 0$$

and

$$\int_0^{2\pi} \cos v \, \mathrm{d}M = -2\pi e,$$

we find

$$\int_0^{2\pi} \frac{\mathrm{d}a}{\mathrm{d}t} \mathrm{d}M = 0. \tag{A.1}$$

To illustrate the relationship between this average and the long term evolution of a, we have to consider $\mathrm{d}a/\mathrm{d}t$ as a function of both M and $M_\odot$; then it can be expanded in Fourier series:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \sum_{h,k} \left[\frac{\mathrm{d}a}{\mathrm{d}t}\right]_{hk} \exp\left[j(hM_\odot + kM)\right].$$

Hence, (A.1) means that for every integer h

$$\left[\frac{\mathrm{d}a}{\mathrm{d}t}\right]_{h0} = 0.$$

Since

$$\frac{\mathrm{d}M_\odot}{\mathrm{d}t} = n_\odot \ll n = n_\odot y,$$

the long-periodic terms in the perturbation of a are of two kinds: those with $k = 0$ and those with the small divisor $hn_\odot + kn \sim 0$ (i.e. $-h/k \simeq y$). But the latter are of very high order, because $|h| \geqslant y$ (e.g., $\geqslant 366$ for a geosynchronous orbit). Hence, the harmonics of order h in the Fourier expansion of $\mathbf{F(s)}$ will be negligible for every reasonable shape. We can then conclude that Equation (A.1) is equivalent, for every practical case, to

$$\left.\frac{\mathrm{d}a}{\mathrm{d}t}\right|_{l.p.} = 0.$$

THEOREM 2. *The mean longitude $\lambda = \varepsilon + \rho$ undergoes no long-periodic or secular perturbation at zero order in the eccentricity.*

Proof. From Equation (2.7)

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{2}{na}\mathbf{F(s)} \cdot \hat{\mathbf{e}}_S + 0(e) + 0(i),$$

where the term $0(i)$ containing the inclination can be assumed to be zero because the reference plane is arbitrary; $0(e)$ contains the terms of order $\geqslant 1$ in e. Using the same frame as before, we have

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{2}{na}[F_x(\mathbf{s}) \cos v + F_y(\mathbf{s}) \sin v] + 0(e).$$

By averaging, since

$$\int_0^{2\pi} \cos v \, \mathrm{d}M = 0(e),$$

we have

$$\left.\frac{\mathrm{d}\varepsilon}{\mathrm{d}t}\right|_{l.p.} = 0(e).$$

On the other hand, the long-periodic or secular perturbations on $\rho = \int n \, \mathrm{d}t$ are zero as a consequence of Theorem 1.

THEOREM 3. *The inclination and the node undergo no long-periodic or secular perturbation to zero order in the eccentricity.*

Proof. From Equation (2.4) and (2.5)

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{\cos(\tilde{\omega} - \Omega + M)}{na} \mathbf{F}(\mathbf{s}) \cdot \hat{\mathbf{e}}_W + 0(e)$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{\sin(\tilde{\omega} - \Omega + M)}{na \sin i} \mathbf{F}(\mathbf{s}) \cdot \hat{\mathbf{e}}_W + 0(e).$$

Since $\mathbf{F}(\mathbf{s}) \cdot \hat{\mathbf{e}}_W$ does not depend on M, by averaging we get

$$\left.\frac{\mathrm{d}i}{\mathrm{d}t}\right|_{l.p.} = 0(e)$$

$$\left.\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right|_{l.p} = 0(e).$$

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