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Short Communications

Peristaltic transport of a couple-stress fluid

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Abstract: The problem of peristaltic transport of a couple-stress fluid has been investigated under a zero Reynolds number and long wavelength approximation. A comparison of the results with those for a Newtonian fluid model shows that the magnitude of the pressure rise under a given set of conditions is greater in the case of the couple-stress fluid. The pressure rise increases as the couple-stress parameter $\bar{\eta}$ decreases. The difference between the pressure rise for a Newtonian and a couple-stress fluid increases with increasing amplitude ratio at zero flow rate. However, increasing the flow rate reduces this difference.

Key words: Peristaltic transport, couple-stress fluid, long wavelength approximation

1. Introduction

The study of the mechanism of peristalsis, in both mechanical and physiological situations, has recently become the object of scientific research. Since the first investigation of Latham [1], several theoretical and experimental attempts have been made to understand peristaltic action in different situations. A review of much of the early literature is presented in an article by Jaffrin and Shapiro [2]. A summary of most of the experimental and theoretical investigations reported so far, with details of the geometry, fluid, Reynolds number, wavelength parameter, wave amplitude parameter, and wave shape, has been given in a recent paper by Srivastava and Srivastava [3].

Most theoretical investigations have been carried out for Newtonian fluids. Although it is known that most physiological fluids behave like non-Newtonian fluids, only a few recent studies [3–8] have considered this aspect of the problem since the initial investigations by Raju and Devanathan [9, 10]. The present paper considers the peristaltic transport of a couple-stress fluid, which is a special case of a non-Newtonian fluid which is intended to take into account the particle size effects.

2. Formulation and analysis

Consider the axisymmetric flow in a circular tube with a sinusoidal wave travelling down its wall. The geometry of the wall surface is described as

$$h'(x',t') = a + b \sin \frac{2\pi}{\lambda}(x' - ct'), \tag{1}$$

where a is the radius of the tube, b is the amplitude of the wave, λ is the wavelength, c is the wave propagation velocity, t' is the time and x' is the axial moving coordinate.

The circular tube is filled with a homogeneous and incompressible couple stress fluid of constant viscosity μ and density ϱ. The constitutive equations and equations of motion for couple-stress fluid flow are [11]

$$T_{ji,j} = \varrho \frac{dv_i}{dt}, \tag{2}$$

$$e_{ijk} T^A_{jk} + M_{ji,j} = 0, \tag{3}$$

$$\tau_{ij} = -p' \delta_{ij} + 2\mu_2 d_{ij}, \tag{4}$$

$$\mu_{ij} = 4\eta\, \omega_{j,i} + 4\eta'\, \omega_{i,j}, \tag{5}$$

where v_i is the velocity vector, τ_{ij} and T^A_{ij} are the symmetric and antisymmetric parts of the stress tensor T_{ij} respectively, M_{ij} is the couple-stress tensor, μ_{ij} is the deviatoric part of M_{ij}, ω_i is the vorticity vector, d_{ij} is the symmetric part of the velocity gradient, η and η' are constants associated with the couple stress, p' is the pressure, and the other terms have their usual meaning from tensor analysis. Using the long wavelength approximation and neglecting the inertia term, it follows from eqs. (2–5) that the appropriate equations in dimensionless form for describing the flow are

$$\frac{dp}{dx} = \nabla^2 \left[1 - \frac{1}{\bar{\alpha}^2} \nabla^2 \right] u \tag{6}$$

with

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right), \tag{7}$$

$$r = \frac{r'}{a}, \quad x = \frac{x'}{\lambda}, \quad u = \frac{u'}{c}, \quad t = \frac{c\,t'}{\lambda},$$

$$\bar{\alpha} = \alpha a = \sqrt{\frac{\mu}{\eta}}\, a, \quad p = \frac{p' a^2}{\lambda c \mu}, \tag{8}$$

where u' is the axial velocity, r' the radial coordinate and α the couple-stress fluid parameter.

The non-dimensional boundary conditions are:

$$\frac{\partial u}{\partial r} = 0 \quad \text{at } r = 0, \tag{9}$$

$$u = -1 \quad \text{at } r = h = 1 + \varphi \sin 2\pi x, \tag{10}$$

$$\frac{\partial^2 u}{\partial r^2} - \frac{\bar{\eta}}{r} \frac{\partial u}{\partial r} = 0 \quad \text{at } r = h, \tag{11}$$

$$\frac{\partial^2 u}{\partial r^2} - \frac{\bar{\eta}}{r} \frac{\partial u}{\partial r} \quad \text{is finite at } r = 0, \tag{12}$$

where $h = h'/a$, the amplitude ratio $\varphi = b/a < 1$, and $\bar{\eta} = \eta'/\eta$ is a couple-stress fluid parameter.

Boundary conditions (11) and (12) show that couple stresses vanish at the tube wall and are finite at the tube axis, as suggested by Valanis and Sun [12]. Integrating eq. (6) and using boundary conditions (9–12), we obtain the velocity profile as

$$u = -1 - \frac{1}{4} \frac{dp}{dx} \cdot \left[h^2 - r^2 + \frac{2(1-\bar{\eta}) \{ I_0(\bar{\alpha} r) - I_0(\bar{\alpha} h) \}}{\bar{\alpha} \{ I_0(\bar{\alpha} h)\, \bar{\alpha} - (1+\bar{\eta})\, I_1(\bar{\alpha} h)/h \}} \right], \tag{13}$$

where I_0 and I_1 are the modified Bessel functions of order zero and one. The dimensionless flux $q = q'/\pi a^2 c$, where q' is the flux in the moving system, is then given by

$$q = \int_0^h 2 r u \, dr = -h^2 - \frac{1}{8} \frac{dp}{dx} [h^4 + H(\bar{\alpha}, \bar{\eta}, h)], \tag{14}$$

with

$$H(\bar{\alpha}, \bar{\eta}, h) = \frac{8(1-\bar{\eta})}{\bar{\alpha}^2} \left[\frac{h I_1(\bar{\alpha} h)/\bar{\alpha} - (h^2/2)\, I_0(\bar{\alpha} h)}{I_0(\bar{\alpha} h) - (1+\bar{\eta})\, I_1(\bar{\alpha} h)/\bar{\alpha} h} \right]. \tag{15}$$

The pressure drop per wavelength $\Delta p = p(0) - p(\lambda)$ is obtained from eq. (14) as

$$\Delta p = -\int_0^1 \left(\frac{dp}{dx} \right) dx = 2q L_1 + 2 L_2, \tag{16}$$

where

$$L_1 = 4 \int_0^1 \frac{dx}{h^4 + H}, \tag{17}$$

$$L_2 = 4 \int_0^1 \frac{h^2}{h^4 + H} dx. \tag{18}$$

Following the analysis given by Shapiro et al. [13], the mean volume flow $\bar{Q}$ over a period is given as

$$\bar{Q} = q + 1 + \frac{\varphi^2}{2} \tag{19}$$

which on using eq. (16) gives

$$\bar{Q} = 1 + \frac{\varphi^2}{2} - \frac{L_2}{L_1} + \frac{\Delta p}{2 L_1}. \tag{20}$$

The dimensionless friction force per wavelength $F = F'/\pi \lambda c \mu$, where F' is the friction at the wall, is given as

$$F = \int_0^1 h^2 \left(-\frac{dp}{dx} \right) dx. \tag{21}$$

Substituting eqs. (14) and (16) in eq. (21) yields

$$F = 2 L_3 - 2 \frac{L_2^2}{L_1} + \Delta P \frac{L_2}{L_1}, \tag{22}$$

where

$$L_3 = 4 \int_0^1 \frac{h^4}{h^4 + H}. \tag{23}$$

The expressions for $\bar{Q}$ and F just obtained are in terms of the integrals L_1, L_2 and L_3 (eqs. (17, 18, 23)). These integrals are not integrable in closed form, as they involve complicated functions of I_0 and I_1. For further study it is helpful to have simple analytical expressions for $\bar{Q}$, Δp and F. Therefore, we obtain expressions for $\bar{Q}$, ΔP and F, after evaluating the integrals L_1, L_2 and L_3 using expansions of the Bessel functions I_0 and I_1 (for small arguments),

$$I_0(z r) \sim \sum_{n=0}^{\infty} (n!)^{-2} \left(\frac{z r}{2} \right)^{2n}, \quad |z| r < 1, \tag{24}$$

$$I_1(z r) \sim \sum_{n=0}^{\infty} \frac{\left(\frac{2 r}{2} \right)^{2n+1}}{n! (n+1)!}, \quad |z| r < 1. \tag{25}$$

With these approximations, the expressions for $\bar{Q}$ and F are obtained as

$$\bar{Q} = \frac{2B(1-\varphi^2)^2(16\varphi^2-\varphi^4)+4A(14+89\varphi^2+74\varphi^4+12\varphi^6)+\Delta P(1-\Phi^2)^{11/2}}{8A(8+40\varphi^2+15\varphi^4)+4B(2+3\varphi^2)(1-\varphi^2)^2}, \quad (26)$$

$$F = 8\left[B+\frac{A}{(1-\varphi^2)^{3/2}}\right]+\left[\Delta P-\frac{4A(2+3\varphi^2)+8B(1-\varphi^2)^2}{(1-\varphi^2)^{7/2}}\right] \cdot \left[\frac{4(1-\varphi^2)^2\{A(2+3\varphi^2)+2B(1-\varphi^2)^2\}}{A(8+40\varphi^2+15\varphi^2+4B(2+3\varphi^2)(1-\varphi^2)^2}\right], \quad (27)$$

where

$$A = \frac{24(1-\bar{\eta})}{(7-\bar{\eta})\bar{\alpha}^2}, \quad B = \frac{3(3-\bar{\eta})}{(7-\bar{\eta})}. \quad (28)$$

In the limit $\bar{\eta} \to 1$ (i.e. $\eta' \to \eta$, no couple-stress effect) and $\bar{\alpha} \neq 0$, eqs. (26) and (27) give the corresponding expressions for a Newtonian fluid as

$$\bar{Q} = \frac{16\varphi^2-\varphi^4}{2(2+3\varphi^2)}+\frac{\Delta P}{4}\,\frac{(1-\varphi^2)^{7/2}}{2+3\varphi^2}, \quad (29)$$

$$F = 8\left[1-\frac{2(1-\varphi^2)^{1/2}}{2+3\varphi^2}\right]+\Delta P\frac{2(1-\varphi^2)^2}{2+3\varphi^2}. \quad (30)$$

Eq. (29) is the same as that obtained by Barton and Raynor [14] and Jaffrin and Shapiro [2].

To observe the effect of the amplitude ratio φ on the flow behaviour analytically, we approximate eqs. (26) and (27) for small squeeze ($\varphi < 1$) as

$$\bar{Q} \cong \frac{\varphi^2}{32A+4B}\left[16B+A\left(178-\frac{14(80A-B)}{8A+B}\right) - \frac{\Delta P}{4}\left(11+\frac{(80A-B)}{8A+B}\right)\right]+\frac{\Delta P+56A}{64A+8B}, \quad (31)$$

$$F \cong \varphi^2\left[20B^2-36A^2-16AB+52A-4B - \frac{(11A+7B)}{2(A+B)}\Delta P\right]+\Delta P. \quad (32)$$

Eqs. (31) and (32) show that $\bar{Q}$ and F increase with φ. The pressure rise for zero time-mean flow and the time-mean flow for zero pressure rise can be obtained from eq. (26) as (33)

$$(-\Delta P)_{\bar{Q}=0} = \frac{2B(16\varphi^2-\varphi^4)}{(1-\varphi^2)^{7/2}}+\frac{A(96\varphi^2+102\varphi^4-9\varphi^6)}{2(1-\varphi^2)^{11/2}},$$

$$(\bar{Q})_{\Delta P=0} = \frac{4B(1-\varphi^2)^2(16\varphi^2-\varphi^4)+A(96\varphi^2+102\varphi^2-9\varphi^6)}{2A(8+40\varphi^2+15\varphi^4)+8B(2+3\varphi^2)(1-\varphi^2)^2}. \quad (34)$$

For $\bar{\eta} \to 1$, eqs. (33) and (34) reduce to Jaffrin and Shapiro's [2] results.

3. Numerical results and discussions

In order to be able to discuss the results obtained above quantitatively, the expressions for $\bar{Q}$, $(\bar{Q})_{\Delta P=0}$, $(-\Delta P)_{\bar{Q}=0}$, and F from eqs. (26, 27, 33, 34) are plotted in figures 1–4 for various values of φ and the couple-stress parameter $\bar{\eta}$. The average pressure rise $(-\Delta P)$ versus the time-mean flow rate is plotted for various values of the parameter $\bar{\eta}$ for $\varphi = 0.4$ in figure 1, which shows a linear relation between them as for a Newtonian fluid. As expected, an increase in flow rate reduces the pressure rise and thus maximum flow rate is achieved at zero pressure rise and maximum pres-

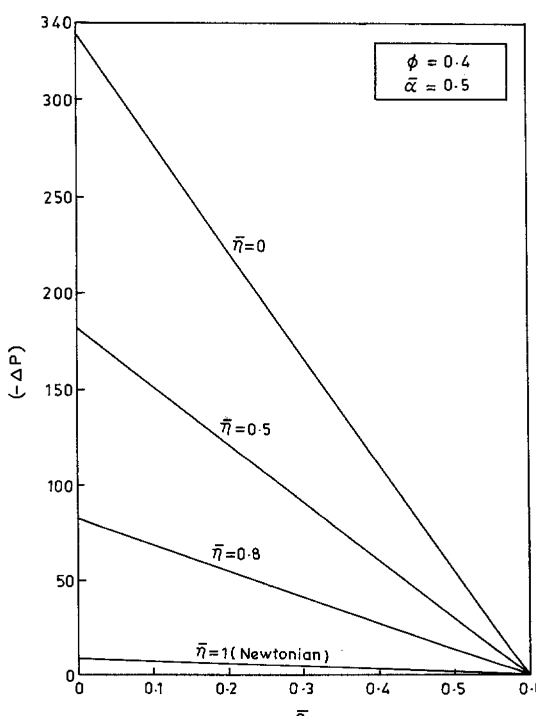

Fig. 1. Pressure-flow rate relationship

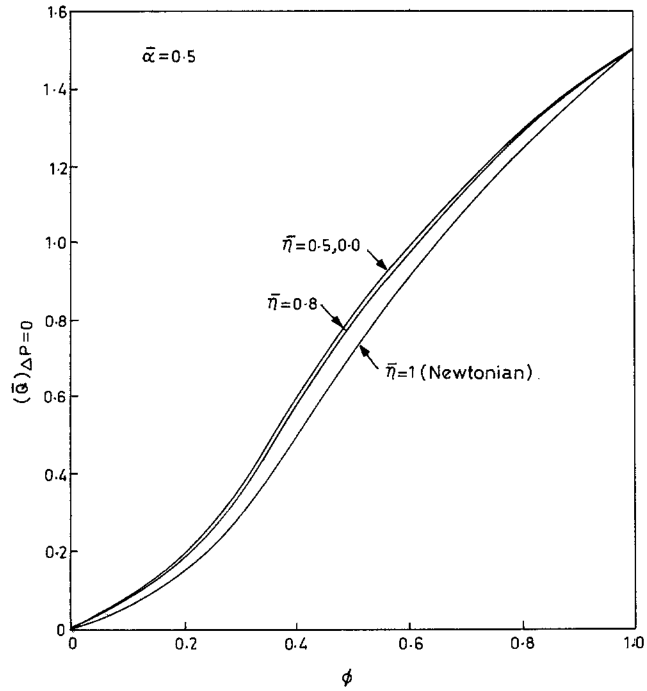

Fig. 2. Variation of flow rate with amplitude ratio

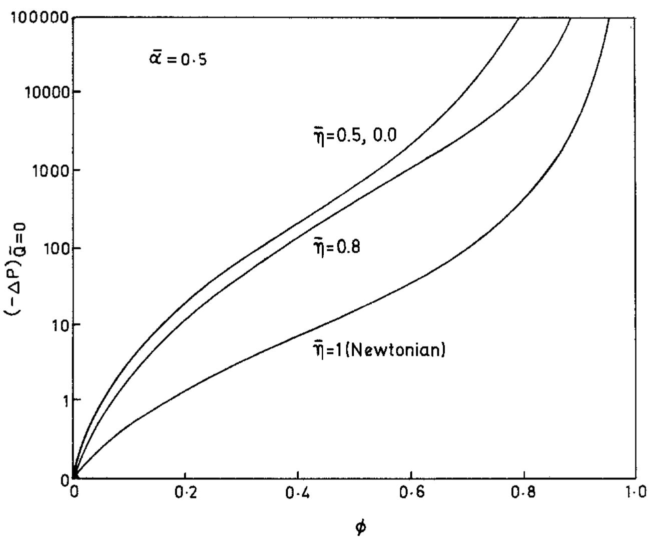

Fig. 3. Variation of pressure rise with amplitude ratio

sure occurs at zero flow rate. A significant result is that the magnitude of the pressure rise under a given set of conditions is greater for the couple-stress fluid than for a Newtonian fluid.

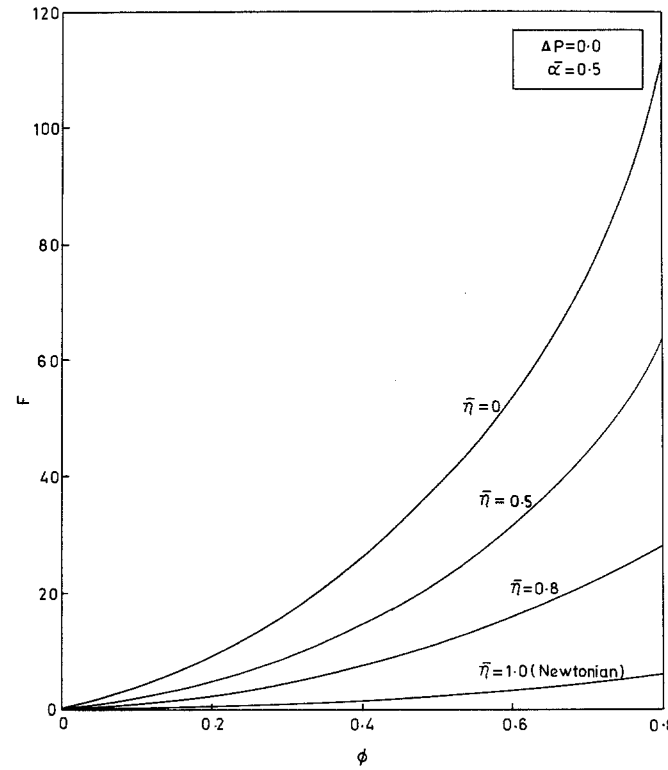

Fig. 4. Variation of friction force with amplitude ratio

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