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Erratum

Maximal chains in ${}^\omega\omega$ and ultrapowers of the integers

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In the above article the proof of Theorem 3.3 of [1] was flawed in that the definition of *somewhat determined* was inaccurate. This should be replaced by the following.

A condition $p \in \mathbb{P}_{\omega_2}$ will be said to be determined if there is some $\Sigma_p \in [\omega_2]^{<\aleph_0}$ such that Σ_p is the support of p and for each $\sigma \in \Sigma_p$ there is a quadruple $(a^\sigma_p, f^\sigma_p, \Delta^\sigma_p, g^\sigma_p)$ such that:
- $p \restriction \sigma \Vdash_{\mathbb{P}_\sigma}$ "$p(\sigma) = a^\sigma_p * (f^\sigma_p, \Delta^\sigma_p)$" for each $\sigma \in \Sigma_p$
- $\Delta^\sigma_p \subseteq \Sigma_p \cap \sigma$ for each $\sigma \in \Sigma_p$
- $p \restriction \sigma \Vdash_{\mathbb{P}_\sigma}$ "$g_{o(\sigma)} \restriction \mathrm{dom}(a^\sigma_p) = g^\sigma_p$" for each $\sigma \in \Sigma_p$
- for each $\{\sigma, \tau\} \in [\Sigma_p]^2$ such that $\sigma \prec \tau$ there is some $k_p(\sigma, \tau) \in 2$ such that $p \restriction \tau \Vdash_{\mathbb{P}_\tau}$ "$A^{-1}_\sigma\{k_p(\sigma,\tau)\} \in D_\tau$"
- $\mathrm{dom}(f^\sigma_p) \supseteq \mathrm{dom}(a^\sigma_p)$ for each $\sigma \in \Sigma_p$
- $\mathrm{dom}(f^\tau_p) \subseteq \mathrm{dom}(f^\sigma_p)$ for each $\{\sigma, \tau\} \in [\Sigma_p]^2$ such that $\sigma \prec \tau$.

This definition of determined differs in a substantial way from the definition of *somewhat determined* in [1]. The next lemma shows that every condition can be extended to a determined condition; this is problematic for the somewhat determined conditions.

Lemma 0.1. *The set of determined conditions is dense in* $\mathbb{P}_{\omega_2}$.

Proof. Induction on $\alpha \in \omega_2 + 1$ will be used to prove the following stronger statement: For each $m \in \omega$ and each $p \in \mathbb{P}_\alpha$ there is a determined condition $q \geq p$ such that if σ is the maximal element of Σ_q then $m \subseteq a^\sigma_q$ and σ is the maximal element of the support of p. Note that a^σ_q has the smallest domain of any function appearing in q so the requirement that $m \subseteq a^\sigma_q$ implies that m is in the domain of any function appearing in q.

To prove this, suppose the statement is true for all $\alpha \in \beta$. If β is a limit ordinal the result follows from the finite support of the iteration; therefore suppose that $\beta = \gamma + 1$. Then extend p so that $p \Vdash_{\mathbb{P}_\gamma}$ "$p(\gamma) = a*(f, \Delta)$". By extending, it may be assumed that

$m \subseteq \mathrm{dom}(a) \subseteq \mathrm{dom}(f)$. Let $\bar{m}$ be the maximal element of $\mathrm{dom}(f)$. Let $p' \geqq p \restriction \gamma$ be such that Δ is contained in the support of p'.

There are now two cases to consider: Either β is a successor in $\prec$ or it is a limit. If it is a successor then let β^* be the predecessor of β in $\prec$. Otherwise, let β^* be such that β^* is greater then the support of p' and $\beta^* \prec \beta$ and β^* is the successor of β^{**} in the ordering $\prec$. In the first case, let $p'' \geqq p'$ be such that $p'' \Vdash_{\mathbb{P}_\gamma}$ "$A_{\beta^*}^{-1} k \in D_\beta$". In the second case, choose p'' such that $p'' \Vdash_{\mathbb{P}_{\beta^*}}$ "$A_{\beta^{**}}^{-1} k \in D_{\beta^*}$" and such that β^{**} belongs to the support of p''.

Now use the induction hypothesis to find a determined condition q such that if σ is the maximal element of Σ_q then $\bar{m} \in \mathrm{dom}(a_q^\sigma)$. Moreover, in the case that β is a limit of $\prec$, then the induction hypothesis can be used to ensure that $\sigma < \beta^*$. It will be shown that the transitivity of $\prec$ guarantees that $q * p(\gamma) = r$ is a determined condition satisfying the extra induction requirements. Let $\Sigma_r = \Sigma_q \cup \{\beta\}$ and let f_r^σ, a_r^σ and Δ_r^σ have the values inherited from q and $p(\beta)$. Furthermore, $k_r(a, \tau)$ can be defined to be $k_q(\alpha, \tau)$ unless $\beta = \tau$. Here the choice of p'' helps.

In the case that β is the successor of β^*, then p'' decides that $A_{\beta^*}^{-1} k \in D_{\beta^*}$ so $k_r(\beta^*, \beta)$ can be defined to be k and, moreover $k_r(\mu, \beta)$ can be defined to be k for each $\mu \in \Sigma_q$ such that $\mu \prec \beta^*$. Since β is the successor of β^* in $\prec$ there are no new instances with which to deal. In the case that β is a limit in the partial order $\prec$, it is possible to define $k_r(\beta^{**}, \beta) = k$ because of the transitivity of $\prec$. For the same reason it is possible to define $k_r(\mu, \beta)$ to be k for each $\mu \in \Sigma_q$ such that $\mu \prec \beta^{**}$. Since the support of q is contained in β^* and β^* is the successor of β^{**} in the partial order $\prec$, it follows that there are no new instances to consider in this case as well. □

The reader who wishes to see all the details of the proof is advised to download the file *chainserrata* by anonymous ftp from the directory \pub\logic\shelahsteprans at the site *ftp.math.ufl.edu*.

References

1. Shelah, S., Steprāns, J.: Maximal Chains in ${}^\omega\omega$ and Ultrapowers of the Integers. Arch. Math. Log. **32**, 305–319 (1993)