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Completion Regular Measures on Product Spaces

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1. Introduction

Let X be a compact, Hausdorff space and μ a Radon probability measure supported by X, which is in some way "closely related" to the topology of X. One can ask when some other Radon probability measure ν on X is, in some sense, "isomorphic" to μ. The question can be asked, and answered, at several different levels. For instance, if μ is (normalized) Lebesgue measure on an n-cell X, what Borel probability measures ν on X are mapped to μ by a homeomorphism of X? It is a classical result of von Neumann, Oxtoby and Ulam (see e.g. [13]) that it is necessary and sufficient that ν be positive on open sets, non-atomic, and vanish on the boundary of the n-cell. A remarkable generalization has recently been obtained by Oxtoby and Prasad [12], that a similar result holds on the Hilbert cube $\mathbb{I}^{\aleph_0}$, which has no boundary: here it is necessary and sufficient only that ν be non-atomic and positive on open sets. It will follow from the results of this paper that the generalization of this to $\mathbb{I}^{\mathfrak{c}}$ is false, at least if one assumes the continuum hypothesis. The corresponding situation for the Cantor set $2^{\aleph_0}$ is much more complex.

At a more primitive level, it is a folk-lore theorem that if μ, ν are (non-atomic) Borel probability measures on Polish spaces X and Y respectively, of the same cardinal, then there is a Borel isomorphism of X to Y taking μ to ν. In this paper we investigate to what extent similar results hold for two measures on an uncountable product of compact metric spaces. In particular all our results hold for the power product space X^A, where X is a compact metric space (with at least two distinct points) and A is uncountable. The most interesting cases are those when $X = \mathbb{I} = [0, 1]$ or $X = 2 = \{0, 1\}$ – indeed we shall show that (in contrast to the homeomorphism results) it suffices to prove the results for one of these two spaces. Our results then carry over to arbitrary product spaces $\prod \{X_i : i \in A\}$, with the X_i compact metric spaces. Throughout this paper A is an *uncountable* set.

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2. Baire Isomorphism: The Conjecture

Let μ be a Radon measure on a compact, Hausdorff space X (see Fremlin [3] 73A for the definition – however since X is compact this simplifies to a finite, complete measure, which is inner regular with respect to the compact sets, this implies the measurability of all Borel sets). Let μ_0 be the completion of the restriction of μ to the Baire sets of X. We recall that μ is said to be *completion regular* if every Borel set is μ_0-measurable. Thus if $\mathscr{B}$ denotes the Borel sets, $\mathscr{B}_0$ the Baire sets, the Radon measure space is denoted by (X, Σ, μ) and (X, Σ_0, μ_0) denotes the completion of $(X, \mathscr{B}_0, \mu_0)$, then μ is completion regular if $(X, \Sigma, \mu) = (X, \Sigma_0, \mu_0)$. If $X = \prod \{X_i : i \in A\}$, X_i compact metric, and $\mu = \bigotimes \mu_i$ (the direct product measure) with $\operatorname{supp}(\mu_i) = X_i$ [where $\operatorname{supp}(\mu)$ denotes the support of μ] and $\mu_i(X_i) = 1$, then μ is completion regular (Kakutani [7]). Similarly, Haar measure is always completion regular (Kakutani and Kodaira [8]). We recall also that the Baire sets of a product space $\prod \{X_i : i \in A\}$ are of the form $E \times \prod_{i \in A \setminus C} X_i$, where C is a countable subset of A, and E is Borel in $\prod_{i \in C} X_i$. (A convenient reference for this classical result is Ross and Stone [14] Theorem 4 or Choksi [1] p. 326, Lemma.) Two compact Radon measure spaces (X, Σ^X, μ) and (Y, Σ^Y, ν), (not necessarily completion regular) are said to be *completion Baire isomorphic* if (X, Σ_0^X, μ_0) and (Y, Σ_0^Y, ν_0) are isomorphic as measure spaces; i.e. there exists a bijection T of X to Y such that $TE \in \Sigma_0^Y$ iff $E \in \Sigma_0^X$ and then $\nu(TE) = \mu(E)$. If such a T exists clearly the respective measure algebras $\mathscr{E}_\mu$ and $\mathscr{E}_\nu$ are isomorphic by a measure preserving isomorphism. If μ and ν are Radon probability measures on a product space $\prod X_i$, then the converse is also true.

Theorem 1. *If X_i, $i \in A$ are compact metric spaces, $X = \prod \{X_i : i \in A\}$, μ and ν are Radon probability measures on X, and if there exists a measure preserving isomorphism ψ of the measure algebras $\mathscr{E}_\mu$, $\mathscr{E}_\nu$ of μ respectively ν, then μ, ν are completion Baire isomorphic.*

Proof. Fix $i_0 \in A$. Let $X_{i_0}^1, X_{i_0}^2$ be two copies of X_{i_0}, and let Y_{i_0} be their topological sum which is a compact metric space. Then if

$$Z_k = X_{i_0}^k \times \prod_{i \neq i_0} X_i, \; k = 1, 2; \; Y = Y_{i_0} \times \prod_{i \neq i_0} X_i \,;$$

we have that Z_1, Z_2, and X are all homeomorphic and $Y = Z_1 \oplus Z_2$. Put μ on Z_1, ν on Z_2, then $\mu \oplus \nu$ is a Radon measure on Y, whose measure algebra is $\mathscr{E}_\mu \oplus \mathscr{E}_\nu$. Put $\phi = \psi$ on $\mathscr{E}_\mu$, $\phi = \psi^{-1}$ on $\mathscr{E}_\nu$, then ϕ extends naturally to an involutary measure preserving automorphism of $\mathscr{E}_\mu \oplus \mathscr{E}_\nu$. We wish to show that ϕ is induced by a completion Baire isomorphism T of Y (onto itself) such that $T(Z_1) = Z_2$, which will necessarily be measure preserving and so will give our desired completion Baire isomorphism of (X, μ) and (X, ν). By a theorem of Choksi ([2], II, p. 101 with the extra comment for Polish spaces on the same page) such a point transformation T of Y onto itself, inducing ϕ, certainly exists, but this only satisfies $(\mu \oplus \nu)(Z_2 \triangle T(Z_1)) = 0$, and we require that $Z_2 \triangle T(Z_1) = \emptyset$. However this extra requirement is easily obtained by a slight modification of the argument in [2]. With the notation of [2], let C be a countable subset of A, containing i_0 and invariant under ϕ, which therefore induces an automorphism ϕ_C of the measure algebra of the image $(\mu\oplus\nu)_C$ of $\mu\oplus\nu$ on $Y_{i_0}\times\prod_{C\setminus\{i_0\}} X_i$, which necessarily gives a measure preserving isomorphism ψ_C of the measure algebras of the image μ_C of μ on $X^1_{i_0}\times\prod_{C\setminus\{i_0\}} X_i$ and the image ν_C of ν on $X^2_{i_0}\times\prod_{C\setminus\{i_0\}} X_i$. By von Neumann's classical theorem there exists an invertible, Borel isomorphism S_C of the Polish spaces $X^1_{i_0}\times\prod_{C\setminus\{i_0\}} X_i$ and $X^2_{i_0}\times\prod_{C\setminus\{i_0\}} X_i$, inducing this isomorphism ψ_C; and hence there exists a Borel isomorphism T_C of $Y_{i_0}\times\prod_{C\setminus\{i_0\}} X_i$ (onto itself) which induces ϕ_C and which is such that

$$T_C\Big(X^1_{i_0}\times\prod_{C\setminus\{i_0\}} X_i\Big)=X^2_{i_0}\times\prod_{C\setminus\{i_0\}} X_i.$$

In the proof of Lemma 7 of [2], I, p. 199, we restrict attention to the family of ordered pairs (B_λ, T_λ) which satisfy the additional condition (iv) $C\subseteqq B_\lambda$ and T_λ extends T_C. The point transformation T inducing ϕ obtained from the proof necessarily also extends T_C, and so satisfies our extra requirement that $T(Z_1)=Z_2$. This completes the proof of Theorem 1.

In the sequel we shall often assume that for all $i\in A$, $X_i=X_0$, a fixed compact metric space, with at least two distinct points; sometimes we shall assume further that $\mu_i=\mu_0$, some fixed probability measure on X_0 with $\operatorname{supp}(\mu_0)=X_0$, and $\mu=\bigotimes_{i\in A}\mu_i=\mu_0^A$, the power measure. Two very important special cases of this are:

(i) $X_0=[0,1]$, $\mu_0=$ Lebesgue measure on X_0. $X=\prod_{i\in A} X_i=X_0^A$ is then called a *generalized cube* and $\mu=\bigotimes_{i\in A}\mu_i=\mu_0^A$ *the power Lebesgue measure.*

(ii) $X_0=\{0,1\}$, $\mu_0(\{0\})=\mu_0(\{1\})=\frac{1}{2}$. $X=\prod_{i\in A} X_i=X_0^A$ is then called a *generalized Cantor space* and $\mu=\bigotimes_{i\in A}\mu_i=\mu_0^A$ is still called the *power Lebesgue measure.* In every case A is called the *dimension* of the product or power space; note that we denote the cardinal of the index set by the same symbol A. If A is infinite, then there is an easily constructed Baire isomorphism of the cube and the Cantor space which takes the power Lebesgue measure on one to that on the other; in particular the respective measure algebras are the same (or rather isomorphic).

We recall that a measure algebra (of finite magnitude) is called *homogeneous* if every non-zero principal ideal has a minimal σ-basis of the same cardinal; this cardinal is called the *Maharam type* of the homogeneous measure algebra. A famous theorem of Maharam [10] states that a homogeneous measure algebra of Maharam type A with total measure 1, is measure preserving isomorphic to the measure algebra $\mathscr{E}_A$ of the power Lebesgue measure in $[0,1]^A$ or $\{0,1\}^A$; in particular two such measure algebras are measure preserving isomorphic. It follows immediately from Maharam's theorem and Theorem 1 that

Theorem 2. (a) *If μ is a Radon probability measure on $[0,1]^A$ (or $\{0,1\}^A$) whose measure algebra is homogeneous of type A, then μ is completion Baire isomorphic to the power Lebesgue measure.*

(b) *If μ and ν are two Radon probability measures on $\prod_{i\in A} X_i$, whose measure algebras are homogeneous of the same Maharam type, then μ and ν are completion Baire isomorphic.*

Clearly, in both cases, the conditions are also necessary.

Thus, to determine which Radon probability measures on $[0,1]^A$ or $\{0,1\}^A$ are completion Baire isomorphic to the power Lebesgue measure, it is sufficient to determine which are homogeneous of Maharam type A. It might be suspected that positivity on open sets (full support) was sufficient to guarantee this. But this is easily seen to be false. Consider Wiener measure W (or the measure of the Brownian motion process) which is defined on $\mathbb{R}^{\mathfrak{c}} = \mathbb{R}^{[0,1]}$, and is clearly positive on all open sets. Since the finite dimensional marginal distributions are all non-atomic, we may replace each $\mathbb{R}$ by its two point compactification $\bar{\mathbb{R}} = [-\infty, \infty]$, and obtain Wiener measure on $\bar{\mathbb{R}}^{\mathfrak{c}}$, which is, of course, homeomorphic to the $\mathfrak{c}$ dimensional cube. However, Wiener measure is carried by the set of continuous functions on $[0,1]$, $\mathscr{C}[0,1] \subset \bar{\mathbb{R}}^{\mathfrak{c}}$, which is a Lusin space in the topology induced by $\bar{\mathbb{R}}^{\mathfrak{c}}$ (this topology being weaker than the Polish, norm topology of $\mathscr{C}[0,1]$) and hence is Borel in $\bar{\mathbb{R}}^{\mathfrak{c}}$. Thus the measure W has a separable measure algebra. The homeomorphic image of W in the cube $[0,1]^{\mathfrak{c}}$ is thus separable, non-atomic and positive on open sets; thus it is of Maharam type $\aleph_0$ and not $\mathfrak{c}$.

A much more plausible conjecture is that:

If μ is a completion regular Radon probability measure on $[0,1]^A$ or $\{0,1\}^A$, then μ has measure algebra homogeneous of type A, and so by Theorem 2 is completion Baire isomorphic to the power Lebesgue measure.

It turns out that any assumption that μ is positive on open sets (i.e. has full support) is unnecessary; it also turns out that the above conjecture is equivalent to the apparently more general conjecture:

Any two completion regular Radon probability measures on a product space $\prod\{X_i : i\in A\}$ (with each X_i compact metric, card$(X_i)\geq 2$) are completion Baire isomorphic.

The hypothesis of completion regularity rules out the counter-example given above: for Wiener measure is carried by $\mathscr{C}[0,1]$, which is Borel of cardinal $\mathfrak{c}$ in $\bar{\mathbb{R}}^{\mathfrak{c}}$, and so cannot contain a non-empty Baire set of $\bar{\mathbb{R}}^{\mathfrak{c}}$, which would have cardinal $2^{\mathfrak{c}}$ – thus Wiener measure is not completion regular. A measure *homeomorphic* to a completion regular measure (e.g. power Lebesgue measure) has to be itself completion regular. We note however that a measure Baire isomorphic to the power Lebesgue measure is not necessarily completion regular, even if it *is* positive on open sets. Let μ denote Lebesgue measure on $\mathbb{R}$, and for each measurable set $B \subseteq [0,1]$, put

$$\nu(B) = 2\mu(B \cap [0, \tfrac{1}{2}]).$$

ν is a probability measure on $[0,1]$ and the power measure ν^A is a probability measure on $[0,1]^A$ which is not completion regular, but its measure algebra is homogeneous of type A, and so, by Theorem 2, ν^A is Baire isomorphic to μ^A, the power Lebesgue measure. ν^A does not have full support; but $\eta = \frac{1}{2}(\nu^A + \mu^A)$ does have $\mathrm{supp}(\eta) = [0,1]^A$, and is also homogeneous of type A and so completion Baire isomorphic to μ^A, of course it is not completion regular.

The main purpose of this paper is to settle the above conjectures. The results are somewhat surprising. It is shown that for a large, in fact cofinal, class of cardinals A, the conjectures are true (Theorem 5). However if we assume the continuum hypothesis, the conjectures are false for $\mathbf{c}=\aleph_1$ (Theorem 7). On the other hand if we assume Martin's axiom and $\aleph_1<\mathbf{c}$ (which is consistent), the conjectures are true for $\aleph_n$, $n\in\mathbb{N}$ (Theorems 8 and 9); and if we further assume that $\mathbf{c}=\aleph_2$ (which is still consistent) the conjectures are true for $\mathbf{c}$ (Theorem 9). Thus for $\aleph_1$ and $\mathbf{c}$ the conjectures are undecidable (Theorem 10).

3. Results without Additional Set-Theoretic Hypotheses

In this section we first show the equivalence of our two conjectures, and give a further reformulation. We prove some results on completion regularity and end the section with our first main result, Theorem 5, mentioned above.

The Maharam type of an arbitrary measure algebra, $(\mathscr{E},\mu)$, of finite magnitude is defined to be the supremum of the Maharam types of its homogeneous direct summands, it is denoted by Maharam$(\mathscr{E},\mu)$, or when no ambiguity arises, simply by Maharam(μ). *Hereafter we shall tacitly assume that all compact metric spaces alluded to have at least two points, and that all topological spaces are Hausdorff.*

Lemma 1. *Let (X,μ) be a compact, completion regular, Radon measure space, Y a compact space, $f:X\to Y$ a continuous map such that for each compact Baire set $F\subseteqq X$, $f(F)$ is (compact) Baire. Then μf^{-1} is completion regular on Y, and*

$$\text{Maharam}(\mu f^{-1})\leqq\text{Maharam}(\mu).$$

Proof. Put $\nu=\mu f^{-1}$. It suffices to show that every Borel set B in Y has a Baire kernel for ν; i.e. it suffices to show that given $\varepsilon>0$, there exists B_0, Baire in Y, such that $B_0\subseteqq B$ and $\nu(B)-\nu(B_0)<\varepsilon$. Now $f^{-1}(B)$ is Borel in X, and μ is completion regular, so there exists K_0, compact Baire, $K_0\subseteqq f^{-1}(B)$ and $\mu(f^{-1}(B))-\mu(K_0)<\varepsilon$. Now $f(K_0)$ is by assumption compact, Baire and $f(K_0)\subseteqq B$; hence

$$f^{-1}(B)\supseteqq f^{-1}(f(K_0))\supseteqq K_0,$$

and

$$\mu(f^{-1}(B))-\mu(f^{-1}(f(K_0))<\varepsilon,$$

i.e.

$$\nu(B)-\nu(f(K_0))<\varepsilon.$$

The statement about the Maharam types is trivial.

Corollary. *Put $\zeta(X)$ = least Maharam type of any completion regular measure on X. If X and Y are as in the lemma, then $\zeta(Y)\leqq\zeta(X)$.*

Proposition 1. *Let $\zeta(X)$, be as in the above Corollary.*

(a) *If $B\subset A$, X is compact metric, then*

$$\zeta(X^B)\leqq\zeta(X^A).$$

(b) *If* X_i, $i\in A$, *are compact metric spaces (each with at least two points) then*

$$\zeta([0,1]^A)\leqq\zeta\left(\prod_{i\in A}X_i\right)\leqq\zeta(\{0,1\}^A).$$

(c) $\zeta(\{0,1\}^A)\leqq\zeta([0,1]^A)$;
hence in (b) *we have equality. We call this cardinal* $\tau(A)$.

Proof. (a) is immediate from Lemma 1, Corollary.

(b) will follow from Lemma 1, Corollary if we can show that $\prod_{i\in A}X_i$ is a continuous image of $\{0,1\}^A$, and $[0,1]^A$ is a continuous image of $\prod_{i\in A}X_i$, both with the additional property of preserving compact $\mathscr{G}_\delta$s. Now every compact metric space X_i is a continuous image, under a map f_i, of the Cantor set $\{0,1\}^\omega$; $f=\prod f_i$ gives a continuous surjection of $\{0,1\}^A$ onto $\prod\{X_i:i\in A\}$. Further, the image under f of a set based in countably many coordinates, is itself a set based in countably many coordinates: hence f maps compact Baire sets (i.e. compact $\mathscr{G}_\delta$ sets), to compact Baire sets.

Since each X_i contains at least two distinct points, $\{0,1\}^\omega$ is embedded in $\prod\{X_i:i\in C\}$ for each countable subset C of A. There exists a continuous map of $\{0,1\}^\omega$ onto $[0,1]$, which, by Tietze's extension theorem, extends to a continuous map of $\prod\{X_i:i\in C\}$ onto $[0,1]$. If we divide A into countable disjoint sets C_i, the method of the previous paragraph yields a continuous surjection of $\prod\{X_i:i\in A\}$ onto $[0,1]^A$ taking compact $\mathscr{G}_\delta$s to compact $\mathscr{G}_\delta$s.

(c) Let μ be a completion regular Radon probability measure on $[0,1]^A$. Let K denote the Cantor set $\{0,1\}^\omega$. For each $i\in A$, let $Z_i\subset[0,1]$ be a countable subset, dense in $[0,1]$, such that $\mu(\pi_i^{-1}(Z_i))=0$. (Here π_i is the canonical projection.) Then $Y_i=[0,1]\setminus Z_i$ is homeomorphic to the irrationals (see Kuratowski [9], Chapter III, §36.II, p. 442, where the result is proved for $\mathbb{R}\setminus Z_i$, but the same proof works for $[0,1]\setminus Z_i$). Hence, there exists a continuous surjection $f_i:Y_i\to K$ ([9], p. 440–441). If $Y=\prod\{Y_i:i\in A\}$, then $Y\subset[0,1]^A$ and $\mu^*(Y)=1$. So μ induces a measure μ_1 on Y given by $\mu_1(Y\cap E)=\mu(E)$ for all Borel $E\subseteqq[0,1]^A$. Let $f:Y\to X=K^A$ be given by

$$f(\langle x_i\rangle_{i\in A})=\langle f_i(x)\rangle_{i\in A},$$

and let $\nu=\mu f^{-1}$ be the Borel measure induced on K^A. We assert that ν extends to a completion regular Radon measure on K^A (which is homeomorphic to $\{0,1\}^A$). For let $F\subseteqq K^A$ be a Borel set. Then $f^{-1}(F)=Y\cap E$ for some Borel set $E\subseteqq[0,1]^A$. Let $\varepsilon>0$. Then since μ is completion regular, there exists a compact $\mathscr{G}_\delta$ set (i.e. a compact Baire set) $H\subseteqq E$ such that

$$\mu(H)\geqq\mu(E)-\varepsilon=\nu(F)-\varepsilon.$$

Express H as $H_B\times[0,1]^{A\setminus B}$, where B is countable, $\subseteqq A$, and $H_B\subseteqq[0,1]^B$ is compact. Then since Y_i is Borel in $[0,1]$, $\prod\{Y_i:i\in B\}$ is Borel in $[0,1]^B$, and so $\pi_B^{-1}\left(H_B\cap\prod_{i\in B}Y_i\right)$ is Borel in $[0,1]^A$ (where π_B is the canonical projection). Further since $\mu(\pi_i^{-1}(Y_i))=1$, for all $i\in A$, it follows that $\mu\left(\pi_B^{-1}\left(\prod_{i\in B}Y_i\right)\right)=1$, and so

$\mu\left(\pi_B^{-1}\left(H_B \cap \prod_{i\in B} Y_i\right)\right) = \mu(H)$. Hence there exists a compact $D \subseteqq H_B \cap \prod_{i\in B} Y_i$, such that $\mu(\pi_B^{-1}(D)) \geqq \mu(H) - \varepsilon$. Now if $f_B = \prod_{i\in B} f_i$, then $f_B(D)$ is a compact $\mathscr{G}_\delta$ in K^B and $f\left[D \times \prod_{i\in A\setminus B} Y_i\right] = f_B(D) \times K^{A\setminus B}$ is a compact $\mathscr{G}_\delta$ in K^A. Now

$$\begin{aligned} \nu\left(f\left[D \times \prod_{i\in A\setminus B} Y_i\right]\right) &\geqq \mu^*\left(D \times \prod_{i\in A\setminus B} Y_i\right) \\ &= \mu^*(\pi_B^{-1}(D) \cap Y) = \mu(\pi_B^{-1}(D)) \\ &\geqq \mu(H) - \varepsilon \geqq \nu(F) - 2\varepsilon . \end{aligned}$$

Since $D \subseteqq H_B$, $D \times \prod_{i\in A\setminus B} Y_i = \pi_B^{-1}(D) \cap Y \subseteqq H \cap Y$ and $f\left(D \times \prod_{i\in A\setminus B} Y_i\right) \subseteqq F$. Since $f\left[D \times \prod_{i\in A\setminus B} Y_i\right]$ is Baire (being based in countably many coordinates in K^A), and ε is arbitrary, this proves our assertion that ν is completion regular. Clearly the Maharam type of ν is no greater than that of μ, which proves (c).

Since the power Lebesgue measure (on $[0,1]^A$ or $\{0,1\}^A$) is completion regular of Maharam type A, we clearly have $\tau(A) \leqq A$. Further τ is monotone increasing, i.e. $B \leqq A$ implies $\tau(B) \leqq \tau(A)$. Also note that if μ is a completion regular Radon measure on a compact space X, and $\mu = \bigoplus \mu_i$ is its Maharam decomposition into measures with homogeneous measure algebras (see [10]), then each μ_i is (extends to) a completion regular Radon measure. Theorem 2 and Proposition 1 now show at once that our conjecture is equivalent to:

$$\tau(A) = A .$$

It is thus purely a property of A, and independent of what compact metric spaces the X_i are; in the sequel we often assume that they are all $\{0,1\}$.

We now prove some results which we shall need in the sequel, but which are of independent interest.

Theorem 3. *Let (X, μ) be a compact, completion regular Radon measure space and $\langle (X_i, \mu_i) \rangle$, $i \in A$. A countable or uncountable, a family of compact, metric, Radon probability measure spaces, such that* $\operatorname{supp}(\mu_i) = X_i$, *for all $i \in A$. Then the product Radon measure $\nu = \mu \otimes \left(\bigotimes_{i\in A} \mu_i\right)$ is completion regular on $X \times \prod_{i\in A} X_i$.*

(*Note.* This theorem generalizes that of Kakutani [7], mentioned at the beginning of §2.)

Proof. (a) Suppose first $A = \{i\}$. Let $\{V_n\}_{n\in\mathbb{N}}$ be a base for the topology of X_i. If $G \subseteqq X \times X_i$ is open, it is expressible as $\bigcup_{n\in\mathbb{N}} G_n \times V_n$, where each $G_n \subseteqq X$ is open. Choose Baire sets $H_n \supseteqq G_n$ such that $\mu(H_n) = \mu(G_n)$ and put $H = \bigcup_{n\in\mathbb{N}} H_n \times V_n$; then H is Baire in $X \times X_i$, $H \supseteqq G$ and $\nu(H) = \nu(G)$.

(b) Now we turn to the general case. For each $B \subseteqq A$, let $\mu_B = \bigotimes_{i\in B} \mu_i$, and let $\nu_B = \mu_B \otimes \mu$, π_B the canonical projection $X \times \prod_{i\in A} X_i \to X \times \prod_{i\in B} X_i$. Let G be open in

$X \times \prod_{i \in A} X_i$. It suffices to show that there exists a Baire cover of G, i.e. a Baire set H containing G of the same measure. Let G_0 be any open Baire kernel of G, i.e. $G_0 \subseteqq G$, $\nu(G \setminus G_0) = 0$. There exists a countable set $J \subseteqq A$ such that $G_0 = G_1 \times \prod_{i \in A \setminus J} X_i$, where G_1 is Baire (in fact also open) in $X \times \prod_{i \in J} X_i$. Put $H_1 = \pi_J^{-1}(\pi_J(G))$. Then H_1 is open, $H_1 \supseteqq G$; we assert that $\nu(H_1) = \nu(G)$. To show this it is enough to show that $\nu_J(\pi_J(G) \setminus G_1) = 0$, for $\nu(H_1) = \nu_J(\pi_J(G))$ and $\nu(G) = \nu(G_0) = v_J(G_1)$. Now by Fubini's theorem

$$\nu(G) = \int_{\pi_J(G)} \mu_{A \setminus J}(G^t) \nu_J(dt)$$

and

$$\nu(G_0) = \int_{G_1} \mu_{A \setminus J}(G^t) \nu_J(dt) ,$$

where G^t denotes the section of G by $t \in X \times \prod_{i \in J} X_i$. But each G^t, being the section of an open set, is itself open, and $\mu_{A \setminus J}$, being the product of measures with full support, is itself with full support. So for all $t \in \pi_J(G) \setminus G_1$, $\mu_{A \setminus J}(G^t) > 0$, and so $\nu_J(\pi_J(G) \setminus G_1) = 0$ as desired.

Now, however $\prod_{i \in J} X_i$ is compact metric, so by case (a) there is a Baire set $E \supseteqq \pi_J(G)$ such that $\nu_J(E) = \nu_J(\pi_J(G))$. Put $H = \pi_J^{-1}(E)$, then H is Baire in $X \times \prod_{i \in A} X_i$, $H \supseteqq G$ and $\nu(H) = \nu(G)$ as required.

In the sequel we shall identify a cardinal α with the initial ordinal of cardinal α.

Proposition 2. *If for each ordinal $\xi < \alpha$, X_ξ is a compact metric space with at least two points, then for any λ such that $\tau(\alpha) \leqq \lambda \leqq \alpha$, we have a completion regular Radon measure on $X = \prod_{\xi < \alpha} X_\xi$, with homogeneous measure algebra of Maharam type λ.*

Proof. (a) $\lambda = \alpha$. There exists on each X_ξ a Radon probability measure μ_ξ with $\operatorname{supp}(\mu_\xi) = X_\xi$. We show that $\mu_\alpha = \bigotimes_{\xi < \alpha} \mu_\xi$ is homogeneous of type α. (It *is* completion regular, e.g. by the previous theorem.) If α were countably infinite, then μ_α would be separable, it would also be nonatomic since for each ξ, there exists B_ξ, Borel in X_ξ with $0 < \mu_\xi(B_\xi) < 1$ [because $\operatorname{supp}(\mu_\xi) = X_\xi$, and X_ξ has at least two points]. Hence if α were countably infinite, μ_α would be homogeneous of (Maharam) type $\aleph_0$. In general we may divide α into α disjoint countably infinite sets, and μ_α will then be the direct product of α probability measures, each homogeneous of type $\aleph_0$, i.e. μ_α is homogeneous of type α, as required.

(b) $\lambda = \tau(\alpha)$. By assumption, there exists a completion regular measure μ on X such that if $\mu = \oplus \mu_i$ is the Maharam decomposition of μ into homogeneous measures μ_i, then

$$\sup_i (\text{Maharam}(\mu_i)) = \tau(\alpha) .$$

But each μ_i is itself necessarily completion regular, and so Maharam$(\mu_i) \geqq \tau(\alpha)$. Hence the result.

(c) If $\tau(\alpha) < \lambda < \alpha$, divide α into disjoint sets, A and B, A of cardinal α, B of cardinal λ. Give $\prod_{\xi \in A} X_\xi$ a homeogeneous, completion regular measure of type $\tau(\alpha)$ as in (b) above; give $\prod_{\xi \in B} X_\xi$ a homogeneous, completion regular product measure of type λ as in (a) above; then the product measure is homogeneous of type λ, and by Theorem 3 is completion regular. This completes the proof of the proposition.

The following theorem is now obvious.

Theorem 4. *If for each ordinal $\xi < \alpha, X_\xi$ is a compact metric space of cardinal at least 2, then the measure algebras of completion regular measures on $X = \prod_{\xi < \alpha} X_\xi$ are precisely those which are decomposable into a direct sum of homogeneous measure algebras with Maharam types in the closed interval $[\tau(\alpha), \alpha]$.*

Note. In connection with Theorem 3, it is known (Fremlin [4]) that if (X, μ), (Y, ν) are arbitrary compact completion regular Radon measure spaces then $(X \times Y, \mu \otimes \nu)$ need not be completion regular. It is not however known what happens if X and Y are restricted to be products of unit intervals (or compact metric spaces).

We now prove two lemmas which will be used a couple of times in the sequel.

Lemma 2. *Let X_i, $i \in A$, be compact metric spaces each of cardinal at least 2, and let μ be a completion regular Radon measure on $\prod \{X_i : i \in A\}$. Then any supporting closed set F is determined by countably many coordinates, i.e. is Baire. [A measurable set F is supporting if $\mu(F \cap G) > 0$, whenever G is a nonempty open set meeting F.] In particular* supp(μ) *is Baire.*

Proof. There is a Baire set $E \subseteq F$ such that $\mu(F \backslash E) = 0$; let J be a countable subset of A which determines E. Then we claim that F is determined by J. For let $x \in \prod_{i \in A} X_i \backslash F$. Then there exists a basic open set of the form $H' \cap H''$, disjoint from F and containing x, where H' is open and depends only on coordinates in J, H'' is open and depends only on coordinates not in J. Since E depends only on coordinates in J and $E \cap H' \cap H'' = \emptyset$, we must have $E \cap H' = \emptyset$ and so

$$\mu(F \cap H') = \mu(E \cap H') = 0 .$$

Since F is supporting, this implies that $F \cap H' = \emptyset$. Thus x belongs to a set depending only on coordinates in J and disjoint from F. Since x is an arbitrary point of $\prod_{i \in A} X_i \backslash F$, it follows that F depends only on coordinates in J.

Note. The above property of products of metric spaces is not shared by all compact spaces. For a counter-example glue the point ω_1 of the ordinal space $\{\xi : \xi \leqq \omega_1\}$ to any point of the closed interval $[0, 1]$, and give $[0, 1]$ Lebesgue measure; one thus obtains a completion regular measure whose support is not Baire.

Lemma 3. *If there exists a completion regular measure which is homogeneous of Maharam type γ on $\{0, 1\}^A$, A uncountable, then there exists such a measure which is, in addition, fully supported, i.e. has support the whole of $\{0,1\}^A$.*

Proof. Let μ be a completion regular measure on $\{0, 1\}^A$ which is homogeneous of type γ, and let $K = \operatorname{supp}(\mu)$. By Lemma 2, K is a compact Baire set and so is determined by some countable set of coordinates $J \subset A$. The result will follow at once if we can show that K is itself homeomorphic to $\{0,1\}^A$. Let I be a countably infinite subset of A disjoint from J. Now $K = K_0 \times \{0,1\}^{A \setminus J}$ for some compact $K_0 \subsetneqq \{0,1\}^J$. But $K_0 \times \{0,1\}^I$ is a compact perfect, totally disconnected metric space and so ([5], p. 100, Corollary 2.98) is homeomorphic to the Cantor set $\{0,1\}^{\aleph_0}$. Hence since A and $A \setminus (I \cup J)$ have the same cardinal, K is homeomorphic to $\{0,1\}^A$.

Note. The conclusion of Lemma 3 holds for any product $\prod\{X_i : i \in A\}$ of compact metric spaces X_i, in place of $\{0,1\}^A$. However the proof is considerably more complicated, and since we do not need the result, we omit it.

We now show that for a large, in fact cofinal, class of cardinals, $\tau(A) = A$, and so our conjecture is true. For any cardinal α, α^+ denotes its successor.

Definition. Let $\mathscr{K}$ denote the class of cardinals α,

$$\mathscr{K} = \{\alpha : \beta < \alpha \text{ implies } \beta^{\aleph_0} < \alpha\} .$$

$\mathscr{K}$ is cofinal in the class of all cardinals: for any cardinal λ, observe that $(\lambda^{\aleph_0})^+ \in \mathscr{K}$. Thus $\mathbf{c}^+ \in \mathscr{K}$, and if GCH is assumed, many cardinals are in $\mathscr{K}$; however, regardless of whether we assume CH, neither $\mathbf{c}$ nor $\aleph_1$ is in $\mathscr{K}$.

Theorem 5. *If $A \in \mathscr{K}$, then $\tau(A) = A$, hence every completion regular Radon measure on $\prod_{i \in A} X_i$, (X_i compact metric of cardinal at least 2) is homogeneous of type A, and any two such measures of the same total mass are Baire isomorphic.*

Proof. Let μ be a completion regular measure of type β on $\{0,1\}^A$, by Lemma 3 we may assume that $\operatorname{supp}(\mu) = \{0,1\}^A$. Let $(\mathscr{E}, \mu)$ denote the measure algebra of $(\{0,1\}^A, \mu)$. For each $j \in A$, $k \in \{0,1\}$, let

$$G_{j,k} = \{0,1\}^{A \setminus \{j\}} \times \{k\} .$$

Since each $G_{j,k}$ is open, and the symmetric difference of any two of them has non-empty interior, the corresponding elements $\hat{G}_{j,k}$ of the measure algebra $\mathscr{E}$ (considered in the usual way as a metric space) are all at positive distance from each other: further they generate $\mathscr{E}$ and

$$\operatorname{cardinal}(\{\hat{G}_{j,k}\} : j \in A, k \in \{0,1\}) = A .$$

However if $\beta < A$, then, since $A \in \mathscr{K}$, $\beta^{\aleph_0} < A$, and $\mathscr{E}$ has at most $\beta^{\aleph_0}$ elements, which gives a contradiction. Thus $\beta = A$ (as remarked earlier, we certainly have $\beta \leqq A$) and so $\tau(A) = A$. The remaining assertions follow by the remarks following Proposition 1.

Remark. The proof of Theorem 5 shows that for any A we have

$$\tau(A) \leqq A \leqq \tau(A)^{\aleph_0} .$$

4. Results with Additional Set-Theoretic Hypotheses

We have noted above that neither $\aleph_1$ nor $\mathbf{c}$ belong to $\mathscr{K}$. The rest of the paper is devoted to showing that for these cardinals our conjecture is actually undecidable.

Definition. Let $\mathscr{A}$ be a Boolean algebra. Two sub-algebras $\mathscr{B}_1$ and $\mathscr{B}_2$ are *weakly independent* if $b_1 \cap b_2 \neq 0$ whenever b_1, b_2 are non-zero elements of $\mathscr{B}_1$ respectively $\mathscr{B}_2$. A family $\langle b_i \rangle_{i \in I}$ in $\mathscr{A}$ is *weakly independent* if $\left(\inf_{i \in J} b_i\right) \cap \left(\inf_{k \in K} (1 \setminus b_k)\right) \neq 0$, whenever J, K are disjoint, finite subsets of I.

Lemma 4. *Let $(\mathscr{A}, \mu)$ be a semi-finite, non-atomic (diffuse) measure algebra, $A \subset \mathscr{A} \setminus \{0\}$ a countable set. Then there exists $b \in \mathscr{A}$ such that $a \cap b \neq 0$ and $a \setminus b \neq 0$ for all $a \in A$.*

Proof. Suppose first that μ has finite magnitude. $(\mathscr{A}, \mu)$ is then a complete metric space under the metric $\varrho(a, b) = \mu(a \triangle b)$. If $a \neq 0$, $\{b : a \cap b = 0\}$ and $\{b : a \setminus b = 0\}$ are closed sets which, since $\mathscr{A}$ includes elements of arbitrarily small measure, have empty interior. Since A is countable it follows by Baire's theorem that

$$\bigcup_{a \in A} \{b : a \cap b = 0\} \cup \{b : a \setminus b = 0\}$$

cannot be the whole of $\mathscr{A}$. The general case, when $(\mathscr{A}, \mu)$ is only semi-finite, can be reduced to the above by taking an element of finite measure included in each element of A, and noting that the ideal $\mathscr{A}_f$ of elements of finite measure is again a complete metric space.

Lemma 5. *Let α be any cardinal, and $\mathscr{C}$ the algebra of clopen subsets of $X = \{0, 1\}^\alpha$. Let $(\mathscr{A}, \mu)$ be a measure algebra of finite magnitude and $\theta : \mathscr{C} \to \mathscr{A}$ a Boolean ring homomorphism which preserves the unit elements. Then there is an extension of θ to a sequentially order continuous ring homomorphism (which we also call θ) of the σ-algebra $\mathscr{B}$ of Borel subsets of X into the measure algebra $\mathscr{A}$. If we require that*

$$\theta G = \sup \{\theta H : H \in \mathscr{C}, H \subseteq G\}$$

for all open sets $G \subseteq X$, then the extension is unique. Further if $\nu = \mu\theta$, then ν is a Radon measure on X and θ gives an isometric embedding $\hat{\theta}$ of the measure algebra $(\mathscr{E}, \nu)$ of (X, ν) into $(\mathscr{A}, \mu)$.

Proof. On $\mathscr{C}$ define ν by $\nu = \mu\theta$. Then ν is finitely additive on the clopen subsets of the totally disconnected compact space X, and so is countably additive and extends to a measure ν on the Baire sets $\Sigma(\mathscr{C})$ and hence to a Radon measure, also called ν, on X.

Now consider the measure algebra $(\mathscr{E}, \nu)$ of (X, ν). As $\mathscr{C}$ is a base for the topology of X, $\hat{\mathscr{C}} = \{\hat{E} : E \in \mathscr{C}\}$ is dense in $\mathscr{E}$. ($\hat{E}$ denotes the element of $\mathscr{E}$ to which E belongs.) Now $\nu(\hat{E}) = \mu(\theta E)$ for every $E \in \mathscr{C}$, so there is a well defined ring homomorphism $\hat{\theta}$: $\hat{\mathscr{C}} \to \mathscr{A}$ such that $\hat{\theta}\hat{E} = \theta E$ for every $E \in \mathscr{C}$; further this ring homomorphism is measure preserving and therefore an isometric embedding of $\hat{\mathscr{C}}$ in $\mathscr{A}$. It therefore has a unique extension to an isometric (and so measure preserving) embedding, still called $\hat{\theta}$, of $(\mathscr{E}, \nu)$ in $(\mathscr{A}, \mu)$. Write $\theta E = \hat{\theta}\hat{E}$ for every $E \in \mathscr{B}$. θ is clearly an extension of our original map $\theta : \mathscr{C} \to \mathscr{A}$, and $\nu = \mu\theta$ on $\mathscr{B}$. This proves the assertions in the last sentence of the lemma concerning $(\mathscr{E}, \nu)$ and $\hat{\theta}$.

Since $E \to \hat{E}$ and $\hat{\theta}$ are sequentially order continuous, so is $\theta: \mathscr{B} \to \mathscr{A}$ ([3], 61 Db and 54 B). If G is open, then

$$\hat{G} = \sup \{\hat{H} : H \in \mathscr{C}, H \subseteq G\} ,$$

and $\hat{\theta}$ is order continuous ([3], 54 B), so that

$$\theta G = \sup \{\theta H : H \in \mathscr{C}, H \subseteq G\} .$$

Clearly θ is uniquely determined by this condition; this completes the proof of the lemma.

The following theorem gives the measure theoretic essence of what is behind our undecidability results.

Theorem 6. *Let λ, κ be infinite cardinals with $\lambda \leqq \kappa$. Let $(\mathscr{A}_\lambda, \mu)$ be the homogeneous measure algebra of Maharam type λ and magnitude 1. Then the following are equivalent:*

(i) *there is a completion regular Radon measure on $\{0,1\}^\kappa$ with measure algebra isomorphic to $(\mathscr{A}_\lambda, \mu)$ (i.e. homogeneous of Maharam type λ),*

(ii) *there is a family $\langle b_\xi \rangle_{\xi \in \kappa}$ in $\mathscr{A}_\lambda$, weakly independent, with the property that for every $a \in \mathscr{A}_\lambda$ there is a countable set $I_a \subseteq \kappa$ such that the sub-algebras generated by $\{a\} \cup \{b_\xi : \xi \in I_a\}$ and $\{b_\xi : \xi \in \kappa \setminus I_a\}$ are weakly independent.*

Proof. (i) *implies* (ii). Let ν be a completion regular Radon measure on $\{0,1\}^\kappa$ with measure algebra isomorphic to $(\mathscr{A}_\lambda, \mu)$, by Lemma 3 we may assume that ν has full support. For each $\xi \in \kappa$, let $E_\xi \subset \{0,1\}^\kappa$ be the set $\{t : t(\xi) = 0\}$ and let $b_\xi = \hat{E}_\xi$ in the measure algebra of ν, identified with $\mathscr{A}_\lambda$. Then (as in the proof of Theorem 5), $\langle b_\xi \rangle_{\xi \in \kappa}$ is weakly independent, because ν has full support.

To see the other property of the $\langle b_\xi \rangle_{\xi \in \kappa}$, let a be any member of $\mathscr{A}_\lambda$, and let $E \subseteq \{0,1\}^\kappa$ be a Baire set such that $\hat{E} = a$. Let $\langle F_n \rangle_{n \in \mathbb{N}}$, $\langle F'_n \rangle_{n \in \mathbb{N}}$ be sequences of compact sets which are supporting, and are such that

$$\bigcup_{n \in \mathbb{N}} F_n \subseteq E, \ \nu\Big(E \setminus \bigcup_{n \in \mathbb{N}} F_n\Big) = 0,$$

$$\bigcup_{n \in \mathbb{N}} F'_n \subseteq \{0,1\}^\kappa \setminus E, \ \nu\Big((\{0,1\}^\kappa \setminus E) \setminus \bigcup_{n \in \mathbb{N}} F'_n\Big) = 0 .$$

By Lemma 2 each F_n and F'_n is countably determined, hence there exists a countable set $I = I_a$, such that all the F_n, F'_n are determined by coordinates in I. Let b respectively c be non-zero elements of the subalgebras of $\mathscr{A}_\lambda$ generated by $\{a\} \cup \{b_\xi : \xi \in I\}$ respectively $\{b_\xi : \xi \in \kappa \setminus I\}$. Then one of $b \cap a$ and $b \setminus a$ is non-zero, suppose that $b \cap a \neq 0$. Now $b \cap a$ is of the form $(\hat{G} \hat{\cap} \hat{E})$, where G is a non-empty clopen set determined by coordinates in I. Since $\nu(G \cap E) = \mu(b \cap a) > 0$, there is an $n \in \mathbb{N}$, such that $\nu(G \cap F_n) > 0$. Next, c is of the form $\hat{H}$, where H is a non-empty clopen set determined by coordinates in $\kappa \setminus I$. As $G \cap F_n$ is determined by coordinates in I, $H \cap G \cap F_n \neq \emptyset$; as F_n is supporting $\nu(H \cap G \cap F_n) > 0$ and so $c \cap b \cap a \neq 0$, and $c \cap b \neq \emptyset$. If it was $b \setminus a$ that was non-zero, the same argument would apply using the F'_n instead of the F_n.

(ii) *implies* (i). Let $\langle b_\xi \rangle_{\xi \in \kappa}$ be a family in $\mathscr{A}_\lambda$ with the given two properties. Let $\mathscr{C}$ be the algebra of clopen sets of $\{0,1\}^\kappa$. Let again $E_\xi = \{t \in \{0,1\}^\kappa : t(\xi) = 0\}$. The weak independence of the $\langle b_\xi\rangle_{\xi\in\kappa}$ and the $\langle E_\xi\rangle_{\xi\in\kappa}$ implies that there exists a unique, unit element preserving, ring homomorphism $\theta:\mathscr{C}\to\mathscr{A}_\lambda$ such that $\theta(E_\xi)=b_\xi$ for every $\xi\in\kappa$. By Lemma 5, there is a Radon measure ν on $\{0,1\}^\kappa$, and a unique isometric embedding $\hat\theta$ of the measure algebra $(\mathscr{E},\nu)$ of $(\{0,1\}^\kappa,\nu)$ into $(\mathscr{A}_\lambda,\mu)$, such that $\hat\theta(\hat E)=\theta(E)$ for every $E\in\mathscr{C}$.

We next show that ν is completion regular. To show this it is sufficient to show that any compact set K in $\{0,1\}^\kappa$ has a Baire kernel for ν. Let $F=\operatorname{supp}(\nu|K)$. Then clearly $F\subseteq K$ and $\nu(K\backslash F)=0$, it suffices to show that F is determined by only countably many coordinates, and so is Baire. Let $a=\hat\theta(\hat F)\in\mathscr{A}_\lambda$, and let $I=I_a$ be the corresponding countable set. We claim that F is determined by coordinates in I. For let $t\in\{0,1\}^\kappa\backslash F$. Then there exists a basic open set H containing t and disjoint from F; express H as $\bigcap_{\xi\in J}H_\xi\cap\bigcap_{\xi\in N}H_\xi$, where $J\subseteq I$, $N\subseteq\kappa\backslash I$ are finite, and each H_ξ is either E_ξ or $\{0,1\}^\kappa\backslash E_\xi$. Writing $H'=\bigcap_{\xi\in J}H_\xi$, $H''=\bigcap_{\xi\in N}H_\xi$, $b=\hat\theta(\hat H')$, $c=\hat\theta(\hat H'')$, we see that b is in the subalgebra generated by $\{b_\xi:\xi\in I\}$, c is in the subalgebra generated by $\{b_\xi:\xi\in\kappa\backslash I\}$ and that $a\cap b\cap c=\hat\theta(\hat F\cap\hat H)=0$, and that $c\neq 0$ (because $\langle b_\xi\rangle_{\xi\in N}$ are weakly independent). The property of $I=I_a$ tells us therefore that $a\cap b=0$, i.e. $\nu(F\cap H')=0$; since F is supporting, it follows that $F\cap H'=\emptyset$. Thus any point of $\{0,1\}^\kappa\backslash F$ belongs to a set disjoint from F, depending only on coordinates in I; hence F itself depends only on coordinates in I. Thus ν is completion regular.

We thus have a completion regular measure on $\{0,1\}^\kappa$ with measure algebra isomorphic to a closed subalgebra of $\mathscr{A}_\lambda$, and thus of Maharam type $\leq\lambda$. It follows that $\tau(\kappa)\leq\lambda\leq\kappa$. By Proposition 2, there exists a completion regular measure on $\{0,1\}^\kappa$ whose measure algebra is homogeneous of type λ, i.e. isomorphic to $\mathscr{A}_\lambda$.

Theorem 7. *If the continuum hypothesis holds, then there exists a completion regular measure with separable measure algebra on* $\{0,1\}^{\mathfrak{c}}=\{0,1\}^{\aleph_1}$. *Thus* $\tau(\mathfrak{c})=\tau(\aleph_1)=\aleph_0$, *and the conjecture is false for* $\mathfrak{c}=\aleph_1$.

Proof. Let Ω be the first uncountable ordinal, and $(\mathscr{A},\mu)$ the measure algebra of Lebesgue measure on $[0,1]$. Since the continuum hypothesis is assumed, we can enumerate $\mathscr{A}$ as $\langle a_\xi\rangle_{\xi<\Omega}$ with $a_0=1$. Choose $\langle b_\xi\rangle_{\xi<\Omega}$ inductively by Lemma 4, so that $a\cap b_\xi\neq 0$, $a\backslash b_\xi\neq 0$ for every non-zero element a in the countable subalgebra $\mathscr{A}_\xi$ of $\mathscr{A}$ generated by $\{a_\eta:\eta\leq\xi\}\cup\{b_\eta:\eta<\xi\}$. The $\langle b_\xi\rangle_{\xi<\Omega}$ are clearly weakly independent. Further since every a in $\mathscr{A}$ belongs to some $\mathscr{A}_\xi$, and the set $\{\eta:\eta<\xi\}$ is countable, the second condition of Theorem 6 (ii) follows with $I_a=\{\eta:\eta<\xi\}$. The conclusion now follows immediately from Theorem 6.

It follows that the Oxtoby-Prasad result [12] on homeomorphic measures on $[0,1]^{\aleph_0}$, does not generalize to $[0,1]^{\mathfrak{c}}$, at least if one assumes the continuum hypothesis.

For our final results we shall assume that Martin's Axiom (see Jech [6], p. 99 et seq.) holds and that the continuum hypothesis is false. Our only use of Martin's Axiom is the following important consequence.

Lemma 6. *If Martin's Axiom is true and* (X,μ) *is any compact Radon measure space with separable measure algebra, and* κ *is a set with* $\kappa<\mathfrak{c}$, *then whenever* $\langle E_\xi\rangle_{\xi\in\kappa}$ *is an* *increasing directed family of measurable sets,* $\bigcup_{\xi\in\kappa} E_\xi$ *is measurable and*

$$\mu\Big(\bigcup_{\xi\in\kappa} E_\xi\Big)=\sup_{\xi\in\kappa}\mu(E_\xi)\,.$$

Proof. See D. Normann [11], p. 169.

Theorem 8. *If Martin's Axiom is true and the continuum hypothesis is false, then a family* $\langle b_\xi\rangle_{\xi\in\kappa}$ *of the type described in* (ii) *of Theorem 6, cannot exist with* $\lambda=\aleph_0$, $\kappa=\aleph_1$; *consequently* $\tau(\aleph_1)=\aleph_1$, *and our conjecture is true for* $\aleph_1$.

Proof. Suppose $\tau(\aleph_1)=\aleph_0$, then there exists a completion regular measure μ with separable measure algebra on $\{0,1\}^\Omega$, where Ω denotes the first uncountable ordinal. μ, being completion regular, is non-atomic and so homogeneous. By Lemma 3 we may assume that μ has full support. Let for each $\xi<\Omega$,

$$H_\xi=\{t:t\in\{0,1\}^\Omega, t(\eta)=t(\xi)\quad\text{for}\quad \xi\leqq\eta<\Omega\}\,.$$

Then each H_ξ is closed and therefore measurable. Since we are assuming that $\aleph_1<\mathbf{c}$, and that Martin's axiom is true, Lemma 6 tells us that $\bigcup_{\xi<\Omega} H_\xi$ is measurable.

Let E be any Baire set such that $E\supseteqq\bigcup_{\xi<\Omega} H_\xi$, and let J be a countable set of coordinates determining E. Let $\xi<\Omega$ be such that $J\cap]\xi,\Omega[=\emptyset$. Then for any $t\in\{0,1\}^\Omega$, we may define t' by

$$\begin{aligned} t'(\eta)&=t(\xi)\quad\text{if}\quad \xi<\eta<\Omega\\ &=t(\eta)\quad\text{otherwise}\,. \end{aligned}$$

Then $t'\in H_\xi\subset E$ and $t(\eta)=t'(\eta)$ for $\eta\in J$, so $t\in E$, and $E=\{0,1\}^\Omega$. Thus the only Baire set containing $\bigcup_{\xi<\Omega} H_\xi$ is $\{0,1\}^\Omega$, and so $\mu\Big(\bigcup_{\xi<\Omega} H_\xi\Big)=1$. Again by Lemma 6, $\sup_{\xi<\Omega}\mu(H_\xi)=1$, and so (since $\aleph_1$ has cofinality $\aleph_1$), $\mu(H_\xi)=1$ for some $\xi<\Omega$.

Now let $\eta\in]\xi,\Omega[$, and let, as usual, $E_\eta=\{t:t(\eta)=0\}$. From the definition of H_ξ we have $E_\eta\cap H_\xi=E_\xi\cap H_\xi$ and so $\mu(E_\eta\triangle E_\xi)=0$. But $E_\eta\triangle E_\xi$ is a non-empty open set, and by assumption, μ has full support which gives a contradiction.

Corollary. *If Martin's Axiom is true and the continuum hypothesis is false, then* $\tau(\kappa)>\aleph_0$ *for all* $\kappa>\aleph_0$.

Proof. This is immediate from the monotone property of τ, mentioned just after Proposition 1.

We denote the cofinality of a cardinal κ by $\mathrm{cf}(\kappa)$ (see Jech [6], p. 11 and 12). Recall ([6], p. 17, Lemma 18) that for $n\in\mathbb{N}$, $\mathrm{cf}(\aleph_n)=\aleph_n$, (this has already been used in the proof of Theorem 8 for $\aleph_1$) although, of course $\mathrm{cf}(\aleph_\omega)=\aleph_0$.

Lemma 7. *If* $\mathrm{cf}(\tau(\kappa))>\aleph_0$, *then* $\mathrm{cf}(\kappa)\leqq\mathrm{cf}(\tau(\kappa))$.

Proof. Clearly we may assume $\tau(\kappa)<\kappa$. Let μ be a completion regular measure of Maharam type $\tau(\kappa)$ on $\{0,1\}^\kappa$, and let $\langle a_\xi\rangle_{\xi\in\tau(\kappa)}$ be a dense set in the measure algebra $(\mathscr{A},\mu)$ of μ. Let $B\subset\kappa$ be a set of cardinal $\tau(\kappa)$, such that each a_ξ is representable by a set in $\{0.1\}^\kappa$ determined by coordinates in B. We may assume κ well-ordered so that B corresponds to $\tau(\kappa)$. Each $a \in \mathscr{A}$ can then be represented by a set determined by coordinates in a countable subset B_a of B; as $\mathrm{cf}(\tau(\kappa)) > \aleph_0$, there is an $\eta < \tau(\kappa)$ such that $B_a \subseteq [0, \eta]$; let $\eta(a)$ be the least such η.

For $\xi < \kappa$, set, as usual, $E_\xi = \{t \in \{0,1\}^\kappa : t(\xi) = 0\}$, and consider $\eta(\hat{E}_\xi) < \tau(\kappa)$. For each $\zeta < \tau(\kappa)$, the set

$$C_\xi = \{\xi : \eta(\hat{E}_\xi) \leqq \zeta\}$$

must have cardinal $< \kappa$: for $\{0,1\}^{C_\zeta}$ carries a completion regular Radon measure of type $\leqq \max(\zeta, \aleph_0) < \tau(\kappa)$, so $\tau(\mathrm{card}(C_\zeta)) < \tau(\kappa)$, and τ is a monotone function.

Express $\tau(\kappa)$ as

$$\bigcup_{\xi < \mathrm{cf}(\tau(\kappa))} \lambda(\xi)$$

with each $\lambda(\xi) < \tau(\kappa)$. Then

$$\mathscr{A} = \mathscr{A}_{\tau(\kappa)} = \bigcup_{\xi < \mathrm{cf}(\tau(\kappa))} C_{\lambda(\xi)}\,.$$

Each $C_{\lambda(\xi)}$ has cardinal $< \kappa$, and $\mathscr{A}$ has cardinal (at least) κ, thus $\mathrm{cf}(\kappa) \leqq \mathrm{cf}(\tau(\kappa))$.

Theorem 9. (a) *If Martin's Axiom is true and the continuum hypothesis is false, then* $\tau(\aleph_\xi) = \aleph_\xi$ *for* $\xi \leqq \omega$.

(b) *If Martin's Axiom is true and* $\mathbf{c} = \aleph_n$, $n \geqq 2$ *(which is consistent, Solovay and Tennenbaum* [15]*) then* $\tau(\mathbf{c}) = \mathbf{c}$.

Proof. (a) For $n \in \mathbb{N}$, this is immediate from Theorem 8, Lemma 7 and the fact that $\mathrm{cf}(\aleph_n) = \aleph_n$. For $\aleph_\omega$ it follows since we clearly have

$$\aleph_n = \tau(\aleph_n) \leqq \tau(\aleph_\omega) \leqq \aleph_\omega$$

for all $n \in \mathbb{N}$.

(b) Is immediate from (a).

Theorem 10. *The conjecture* $\tau(\kappa) = \kappa$ *is undecidable for* $\kappa = \aleph_1$ *and* $\kappa = \mathbf{c}$.

Proof. Immediate from Theorems 7, 8, and 9.

Remarks. 1. Lemma 7 implies that for every $n \in \mathbb{N}$ (and also for $n = \omega$) either $\tau(\aleph_n) = \aleph_n$ or $\tau(\aleph_n) = \aleph_0$. Observe that the continuum hypothesis implies $\tau(\aleph_1) = \aleph_0$, but, by the remark following Theorem 5, $\tau(\aleph_2) = \aleph_2$, since we then have $\aleph_0^{\aleph_0} = \mathbf{c} = \aleph_1 < \aleph_2$. Observe also that if $\tau(\aleph_1) = \aleph_1$, then the monotone property of τ and Lemma 7 imply that $\tau(\aleph_n) = \aleph_n$ and $\tau(\aleph_\omega) = \aleph_\omega$.

2. We do not know whether Martin's Axiom and the negation of the continuum hypothesis imply $\tau(\mathbf{c}) = \mathbf{c}$ without any extra assumption.

3. We have been able to show, using Theorem 6, that there are models of set theory in which $\tau(\mathbf{c}) = \aleph_0$, without the continuum hypothesis being true.

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