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The Lelong Number of a Point of a Complex Analytic Set

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Contents

Introduction

Let V be an n-dimensional complex vector space with a hermitian product. Let M be a pure p-dimensional analytic set in an open set $G \subset V$, and suppose that $0 \in M$. Let $n(r, M)$ denote the function of $r \in \mathbf{R}^+$, the set of positive real numbers, defined by dividing the $2p$-dimensional area of M intersect the ball of radius r and center 0 by the area of the $2p$-dimensional ball of radius r. P. LELONG [3] and W. STOLL [8] have proven that $n(r, M)$ is monotonic increasing in r, and thus the limit as r tends to 0 exists. Let $n(0, M)$ denote this limit. In the case that $p = n - 1$, STOLL in [6] has shown that $n(0, M)$ is an integer. In fact, he proves that if f is a holomorphic function in a neighborhood of 0 such that the germ of f generates the ideal of function germs vanishing on M at 0, then $n(0, M)$ is simply the zero-multiplicity of f at 0 (defined in § 4A). However the proof is in the language of divisors and cannot be extended to an analytic set of arbitrary codimension. In the case of $p = 1$, $n(0, M)$ can be directly computed as M can be parameterized in a neighborhood of 0. If $\sum_{\lambda=1}^{n} f_\lambda \mathfrak{v}_\lambda$ is such a parameterization, where $(\mathfrak{v}_1, \ldots, \mathfrak{v}_n)$ is a base of V and where the f_λ's are holomorphic functions on an open set $U \subset \mathbf{C}$, the field of complex numbers, $0 \in U$, and $f_\lambda(0) = 0$, then it can be easily shown that $n(0, M)$ is equal to $\min_{1 \leq \lambda \leq n} \{\nu(0, 0, f_\lambda)\}$, where $\nu(0, 0, f_\lambda)$ is the zero multiplicity of f_λ at 0.

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The purpose of this paper is to prove that $n(0, M)$ is a positive integer for an analytic set M of arbitrary dimension. The proof is divided into three parts. In the first part, it is proven that $n(0, M)$ is an integer if M is an analytic cone with center 0 (defined in § 2). The second part relates $n(0, M)$ to the limit of the area of a family $\{N(w)\}$, $w \in \mathbf{C} - \{0\}$ of analytic sets. These sets have the property that they "tend to" T, the tangent cone to M at 0 (§ 3), as w tends to 0. In § 4, a theorem on the continuity of the area is proven. It is shown that the limit of the area of the $N(w)$'s as w goes to 0 is equal to the product of a positive integer and the area of T. Then this together with the result of § 2 applied to T yields the final result.

§ 1. Definitions

Let V be a complex vector space of dimension n. Let $(\cdot|\cdot)$ be a hermitian product on V, that is,

a) $(\mathfrak{z}|\mathfrak{w}) \in \mathbf{C}$ for $\mathfrak{z} \in V$, $\mathfrak{w} \in V$;
b) $(\mathfrak{z}|\mathfrak{w}) = \overline{(\mathfrak{w}|\mathfrak{z})}$;
c) $(\alpha_1 \mathfrak{z}_1 + \alpha_2 \mathfrak{z}_2 | \mathfrak{w}) = \alpha_1(\mathfrak{z}_1|\mathfrak{w}) + \alpha_2(\mathfrak{z}_2|\mathfrak{w})$ for $\alpha_1, \alpha_2 \in \mathbf{C}$
d) $(\mathfrak{z}|\mathfrak{z}) > 0$ if $\mathfrak{z} \neq 0$.

Then $|\mathfrak{z}| = \sqrt{(\mathfrak{z}|\mathfrak{z})}$ defines a norm on V. Let d be the exterior derivative on V. Consider $(\mathfrak{z}|\mathfrak{a})$ as a function of $\mathfrak{z}$ for fixed $\mathfrak{a}$. Define

$$(d\mathfrak{z}|\mathfrak{a}) = d(\mathfrak{z}|\mathfrak{a}),$$
$$(\mathfrak{a}|d\mathfrak{z}) = \overline{(d\mathfrak{z}|\mathfrak{a})} = d(\mathfrak{a}|\mathfrak{z}).$$

Then $(d\mathfrak{z}|\mathfrak{z})$ and $(\mathfrak{z}|d\mathfrak{z})$ are differentials on V. Define

$$(d\mathfrak{z}|d\mathfrak{z}) = d(\mathfrak{z}|d\mathfrak{z}) = -d(d\mathfrak{z}|\mathfrak{z}),$$
$$\eta = (i/4)\,[(\mathfrak{z}|d\mathfrak{z}) - (d\mathfrak{z}|\mathfrak{z})].$$

Then $d\eta = (i/2)(d\mathfrak{z}|d\mathfrak{z})$.

Define

$$\upsilon = d\eta, \quad \upsilon_p = \frac{1}{p!} \bigwedge_{\nu=1}^{p} \upsilon.$$

Let M be an analytic set of pure dimension $p > 0$ in an open subset G of V. The set $\dot{M}$ of simple points of M forms a smooth complex submanifold of dimension p of V. Let L be a subset of M such that $L \cap \dot{M}$ is measurable on M. If χ is an exterior differential form of degree $2p$ on M such that $\int_{L \cap \dot{M}} \chi$ exists, define

$$\int_L \chi = \int_{L \cap \dot{M}} \chi.$$

Let $\iota: \dot{M} \to V$ be the injection defined by $\iota(\mathfrak{z}) = \mathfrak{z}$. If ξ is a continuous exterior differential form of degree $2p$ on V with compact carrier in G, then $\int_{\dot{M} \cap L} \iota^* \xi$ exists ([3], [7]), and is denoted by $\int_L \xi$.

If $L \subseteqq M$ and $L \cap \dot{M}$ is measurable and if $\bar{L}$ is contained in G and compact, then $\int_L \upsilon_p$ exists and is non-negative. The integral is positive if $L \cap \dot{M}$ is not a set of measure zero. The integral $\int_L \upsilon_p$ is the Lebesgue area of $L \cap \dot{M}$.

Define

$$B_r = \{\mathfrak{z} \in V \mid |\mathfrak{z}| < r\}$$
$$M_0^r = M \cap B_r$$
$$W_p = \pi^p / p!$$

Suppose $0 \in M$ and $B_R \subset G$. For $0 < r < R$, define

$$0 \leqq n(r, M) = \frac{1}{W_p r^{2p}} \int_{M_0^r} \upsilon_p .$$

Then $n(r, M)$ is a monotonic increasing function ([3], [8]). The limit

$$n(0, M) = \lim_{r \to +0} n(r, M)$$

exists, and is called the *Lelong Number of M at* 0. It will be shown that the Lelong Number is always a positive integer.

§ 2. The Lelong number of an analytic cone

Again, let V be an n-dimensional complex vector space with a hermitian product. Let $T \subset V$ be a pure p-dimensional *analytic cone* with center 0, that is, a pure p-dimensional analytic set in V such that $\mathfrak{z} \in T$ implies $u\mathfrak{z} \in T$ for all $u \in \mathbf{C}$. In this section, it will be shown that $n(0, T)$ is a positive integer.

Define on V

$$\sigma = \frac{i}{4} [(\mathfrak{z} \mid d\mathfrak{z}) - (d\mathfrak{z} \mid \mathfrak{z})] \, |\mathfrak{z}|^{-2} = \frac{\eta}{|\mathfrak{z}|^2} \quad \text{for} \quad \mathfrak{z} \neq 0 .$$

Then

$$d\sigma = \frac{i}{2} \frac{(d\mathfrak{z} \mid d\mathfrak{z}) \, |\mathfrak{z}|^2 - (d\mathfrak{z} \mid \mathfrak{z}) \wedge (\mathfrak{z} \mid d\mathfrak{z})}{|\mathfrak{z}|^4} .$$

Define $\omega = d\sigma$, $\omega_p = \frac{1}{p!} \bigwedge_{\nu=1}^{p} \omega$ on $V - \{0\}$.

Let A be a pure p-dimensional analytic subset of an open subset G of V with $p > 0$. If L is a subset of A such that $L \cap \dot{A}$ is measurable on $\dot{A}$ and if $\bar{L}$ is compact and contained in $G - \{0\}$, then $\int_L \omega_p$ exists and is non-negative. If $L \subseteqq A$ and $L \cap \dot{A}$ is measurable and $\int_{L-\{0\}} \omega_p$ exists, define $\int_L \omega_p = \int_{L-\{0\}} \omega_p$.

Let $\iota : A \to V$ be the injection. Let ξ be a continuous exterior differential form of degree $2p$ on V with compact carrier in G. If $\xi = d\tau$, where τ is an exterior differential form of class C^1 and degree $2p - 1$ on G, and where τ

has a compact carrier in G, then [3, Theorem 7]

$$\int_A \xi = \int_A d\tau = 0 .$$

Define, for any subset L of V,

$$L_r^s = L \cap \{\mathfrak{z} \mid r \leqq |\mathfrak{z}| \leqq s\}, \quad 0 \leqq r < s \leqq \infty .$$

The following two propositions are a generalization of results of W. STOLL [8, Propositions 1 and 2].

Proposition 2.1. *Let A be a pure p-dimensional analytic set in $G = \{\mathfrak{z} \mid |\mathfrak{z}| < R\}$ where $p > 0$ and $0 < R \leqq \infty$. Let f be a function of class C^1 on G. Suppose that a number r_0 exists such that*

1) $0 < r_0 < R$,
2) $f(\mathfrak{z}) = 0 \quad$ for $\quad |\mathfrak{z}| \leqq r_0$.

Let q be an integer, $0 \leqq q \leqq p-1$. Let $b = p - q$. Then

$$\frac{p}{r^{2b}} \int_{A_0^r} f(\mathfrak{z})\, v_p(\mathfrak{z}) = \frac{b!\,q!}{(p-1)!} \int_{A_0^r} f(\mathfrak{z})\, v_q(\mathfrak{z}) \wedge \omega_b(\mathfrak{z}) +$$

$$+ \int_{A_0^r} \left[\frac{1}{|\mathfrak{z}|^{2b}} - \frac{1}{r^{2b}}\right] df \wedge \eta \wedge v_{p-1} . \quad (v_0 = 1)$$

Proof. Define

$$\psi = v_q \wedge \frac{\sigma}{b} \wedge \omega_{b-1} \quad (\omega_0 = 1)$$

$$\chi = \frac{(p-1)!}{b!\,q!} \frac{1}{r^{2b}} \eta \wedge v_{p-1} .$$

Then

$$d\psi = v_q \wedge \omega_b, \quad d\chi = \frac{p!}{b!\,q!} \frac{1}{r^{2b}} v_p ,$$

and

$$\frac{1}{b} \sigma \wedge \omega_{b-1} = \left(\frac{i}{2}\right)^{b-1} \frac{1}{b!} \frac{i}{4} \frac{1}{|\mathfrak{z}|^2} [(\mathfrak{z} \mid d\mathfrak{z}) - (d\mathfrak{z} \mid \mathfrak{z})] \wedge$$

$$\wedge \left[\frac{(d\mathfrak{z} \mid d\mathfrak{z})}{|\mathfrak{z}|^2} - \frac{(d\mathfrak{z} \mid \mathfrak{z}) \wedge (\mathfrak{z} \mid d\mathfrak{z})}{|\mathfrak{z}|^4}\right]^{b-1}$$

$$= \left(\frac{i}{2}\right)^{b-1} \frac{1}{b!} \frac{i}{4} \frac{(\mathfrak{z} \mid d\mathfrak{z}) - (d\mathfrak{z} \mid \mathfrak{z})}{|\mathfrak{z}|^2} \wedge$$

$$\wedge \left[\frac{(d\mathfrak{z} \mid d\mathfrak{z})^{b-1}}{|\mathfrak{z}|^{2b-2}} - (b-1) \frac{(d\mathfrak{z} \mid d\mathfrak{z})^{b-2}}{|\mathfrak{z}|^{2b-4}} \wedge \frac{(d\mathfrak{z} \mid \mathfrak{z}) \wedge (\mathfrak{z} \mid d\mathfrak{z})}{|\mathfrak{z}|^4}\right]$$

$$= \left(\frac{i}{2}\right)^{b-1} \frac{1}{b!} \frac{i}{4} \frac{(\mathfrak{z} \mid d\mathfrak{z}) - (d\mathfrak{z} \mid \mathfrak{z})}{|\mathfrak{z}|^{2b}} \wedge (d\mathfrak{z} \mid d\mathfrak{z})^{b-1}$$

$$= \frac{1}{b} \eta \wedge \frac{1}{|\mathfrak{z}|^{2b}} v_{b-1} .$$

Thus

$$\psi = \upsilon_q \wedge \frac{\eta}{b} \wedge \frac{\upsilon_{b-1}}{|\mathfrak{z}|^{2b}}$$

$$= \frac{(p-1)!}{q!b!} \frac{1}{|\mathfrak{z}|^{2b}} \eta \wedge \upsilon_{p-1},$$

$$\psi - \chi = \frac{(p-1)!}{q!b!} \left[\frac{1}{|\mathfrak{z}|^{2b}} - \frac{1}{r^{2b}}\right] \eta \wedge \upsilon_{p-1}.$$

Let α be a C^∞-function on the real line $\mathbf{R}$ such that $0 \leqq \alpha(x) \leqq 1$ for all x and $\alpha(x) = 1$ for $x \leqq 0$ and $\alpha(x) = 0$ for $x \geqq 1$. Define K by

$$K = \operatorname*{Max}_{x \in \mathbf{R}} |\alpha'(x)|.$$

Take any r in $r_0 < r < R$. Take s in $r/2 < s < r$. Define $t = (s+r)/2$. Then $t - s = (r-s)/2$. Define λ_s by $\lambda_s(x) = \alpha\left(\frac{x-s}{t-s}\right)$. Then

a) $0 \leqq \lambda_s(x) \leqq 1$ for all x.
b) $\lambda_s(x) = 1$ for all $x \leqq s$,
c) $\lambda_s(x) = 0$ for all $x \geqq t$,
d) $|\lambda_s'(x)| \leqq \dfrac{K}{t-s} = \dfrac{2K}{r-s}$ for all x,
e) $\lambda_s'(x) \neq 0$ implies $s < x < t$,
f) $\lambda_s(x) \to 1$ as $s \to r - 0$ if $x < r$,
g) $\lambda_s'(x) \to 0$ as $s \to r - 0$ if $x < r$.

And

$$d\lambda_s(|\mathfrak{z}|) \wedge \eta = \frac{i}{4} \lambda_s'(|\mathfrak{z}|) \frac{(d\mathfrak{z} \,|\, \mathfrak{z}) \wedge (\mathfrak{z} \,|\, d\mathfrak{z})}{|\mathfrak{z}|}.$$

For $s \leqq |\mathfrak{z}| \leqq r$,

$$|\lambda_s'(|\mathfrak{z}|)| \left|\frac{1}{|\mathfrak{z}|^{2b}} - \frac{1}{r^{2b}}\right| \leqq \frac{2K}{r-s} \frac{r^{2b} - |\mathfrak{z}|^{2b}}{r^{2b}|\mathfrak{z}|^{2b}} \leqq$$

$$\leqq \frac{2^{2b+1}K}{r^{4b}} \sum_{\mu=0}^{2b-1} r^\mu |\mathfrak{z}|^{2b-1-\mu} \leqq$$

$$\leqq \frac{2^{2b+2}Kb}{r^{2b+1}}.$$

Therefore

$$\int_{A\mathfrak{r}} f \, d\lambda_s \wedge (\psi - \chi) = \left(\frac{(p-1)!}{q!b!}\right) \int_{A\mathfrak{r}_0} f \left(\frac{1}{|\mathfrak{z}|^{2b}} - \frac{1}{r^{2b}}\right) d\lambda_s \wedge \eta \wedge \upsilon_{p-1}$$

$$= \frac{(p-1)!}{q!b!} \int_{A\mathfrak{r}_0} f \left(\frac{1}{|\mathfrak{z}|^{2b}} - \frac{1}{r^{2b}}\right) \frac{i}{4} \lambda_s'(|\mathfrak{z}|) \frac{(d\mathfrak{z} \,|\, \mathfrak{z}) \wedge (\mathfrak{z} \,|\, d\mathfrak{z})}{|\mathfrak{z}|} \wedge \upsilon_{p-1}$$

$$\to 0 \quad \text{as} \quad s \to r - 0.$$

Moreover

$$\int_{A_0^r} \lambda_s\, df \wedge (\psi - \chi) \to \int_{A_0^r} df \wedge (\psi - \chi) \quad \text{as} \quad s \to r - 0,$$

$$\int_{A_0^r} \lambda_s f\, d(\psi - \chi) \to \int f\, d(\psi - \chi) \quad \text{as} \quad s \to r - 0.$$

Therefore

$$0 = \int_{A_0^r} d(f \lambda_s (\psi - \chi))$$

$$= \int_{A_0^r} f\, d\lambda_s \wedge (\psi - \chi) + \int_{A_0^r} \lambda_s\, df \wedge (\psi - \chi) + \int_{A_0^r} \lambda_s f\, d(\psi - \chi)$$

implies that

$$0 = \int_{A_0^r} df \wedge (\psi - \chi) + \int_{A_0^r} f\, d(\psi - \chi),$$

that is,

$$\frac{p!}{b!\,q!} \frac{1}{r^{2b}} \int_{A_0^r} f\, \upsilon_p = \int_{A_0^r} f\, \upsilon_q \wedge \omega_b + \frac{(p-1)!}{q!\,b!} \int_{A_0^r} \left(\frac{1}{|\mathfrak{z}|^{2b}} - \frac{1}{r^{2b}} \right) df \wedge \eta \wedge \upsilon_{p-1}.$$

q.e.d.

Proposition 2.2. *Let A be an analytic set of pure dimension $p > 0$ in $G = \{\mathfrak{z} \mid |\mathfrak{z}| < R\}$ where $0 < R \leqq \infty$. Take r and s such that $0 < r < s < R$. Let q be an integer, $0 \leqq q \leqq p - 1$. Let $b = p - q$. Then*

$$\frac{b!\,q!}{p!} \int_{A_r^s} \upsilon_q \wedge \omega_b = \frac{1}{s^{2b}} \int_{A_0^s} \upsilon_p - \frac{1}{r^{2b}} \int_{A_0^r} \upsilon_p .$$

Proof. Let α be a C^∞-function on $\mathbf{R}$ such that $0 \leqq \alpha(x) \leqq 1$ for all x and $\alpha(x) = 1$ for $x \leqq 0$ and $\alpha(x) = 0$ for $x \geqq 1$. Take $0 < t < r < s < R$. Define

$$f(\mathfrak{z}) = \alpha\left(\frac{|\mathfrak{z}| - t}{r - t} \right).$$

The function f is of class C^∞ and $f(\mathfrak{z}) = 1$ for $|\mathfrak{z}| \leqq t$ and $f(\mathfrak{z}) = 0$ for $|\mathfrak{z}| \geqq r$. From Proposition 2.1,

$$\frac{b!\,q!}{(p-1)!} \int_{A_0^s} (1 - f)\, \upsilon_q \wedge \omega_b = \frac{p}{s^{2b}} \int_{A_0^s} (1 - f)\, \upsilon_p + \int_{A_0^s} \left[\frac{1}{|\mathfrak{z}|^{2b}} - \frac{1}{s^{2b}} \right] df \wedge \eta \wedge \upsilon_{p-1},$$

$$\frac{b!\,q!}{(p-1)!} \int_{A_0^r} (1 - f)\, \upsilon_q \wedge \omega_b = \frac{p}{r^{2b}} \int_{A_0^r} (1 - f)\, \upsilon_p + \int_{A_0^r} \left[\frac{1}{|\mathfrak{z}|^{2b}} - \frac{1}{r^{2b}} \right] df \wedge \eta \wedge \upsilon_{p-1}.$$

And

$$\int_{A_0^s} df \wedge \eta \wedge \upsilon_{p-1} = - \int_{A_0^s} f\, d\eta \wedge \upsilon_{p-1} = -p \int_{A_0^s} f\, \upsilon_p$$

$$\int_{A_0^r} df \wedge \eta \wedge \upsilon_{p-1} = -p \int_{A_0^r} f\, \upsilon_p .$$

Hence

$$\begin{aligned}
\frac{b!q!}{(p-1)!}\int_{A_r^s} \upsilon_q\wedge\omega_b &= \frac{b!q!}{(p-1)!}\int_{A_r^s}(1-f)\,\upsilon_q\wedge\omega_b \\
&= \frac{p}{s^{2b}}\int_{A_0^s}(1-f)\,\upsilon_p - \frac{p}{r^{2b}}\int_{A_0^r}(1-f)\,\upsilon_p + \\
&\quad + \int_{A_r^s}\frac{1}{|\mathfrak{z}|^{2b}}\,df\wedge\eta\wedge\upsilon_{p-1} - \\
&\quad - \frac{1}{s^{2b}}\int_{A_0^s} df\wedge\eta\wedge\upsilon_{p-1} + \frac{1}{r^{2b}}\int_{A_0^r} df\wedge\eta\wedge\upsilon_{p-1} \\
&= \frac{p}{s^{2b}}\int_{A_0^s}(1-f)\,\upsilon_p - \frac{p}{r^{2b}}\int_{A_0^r}(1-f)\,\upsilon_p + 0 + \\
&\quad + \frac{p}{s^{2b}}\int_{A_0^s} f\,\upsilon_p - \frac{p}{r^{2b}}\int_{A_0^r} f\,\upsilon_p \\
&= \frac{p}{s^{2b}}\int_{A_0^s}\upsilon_p - \frac{p}{r^{2b}}\int_{A_0^r}\upsilon_p
\end{aligned}$$

q.e.d.

Note that by letting $q=0$, Proposition 2.2 gives

$$\int_{A_r^s}\omega_p = \frac{1}{s^{2p}}\int_{A_0^s}\upsilon_p - \frac{1}{r^{2p}}\int_{A_0^r}\upsilon_p .$$

Thus $n(r, A) = \frac{1}{W_p r^{2p}}\int_{A_0^r}\upsilon_p$ is monotonic increasing, and so $n(0, A) = \lim_{r\to 0} n(r, A)$ exists.

Assume now that $p \geqq 2$. Let $q=1$. Then

$$\int_{A_r^s}\upsilon\wedge\omega_{p-1} = \frac{p}{s^{2p-2}}\int_{A_0^s}\upsilon_p - \frac{p}{r^{2p-2}}\int_{A_0^r}\upsilon_p .$$

Since $\lim_{r\to 0}\frac{1}{r^{2p}}\int_{A_0^r}\upsilon_p$ exists,

$$\int_{A_0^s}\upsilon\wedge\omega_{p-1} = \frac{p}{s^{2p-2}}\int_{A_0^s}\upsilon_p .$$

In particular, if T is a pure p-dimensional analytic cone with center 0 and $p \geqq 2$, then

$$\frac{p}{r^{2p-2}}\int_{T_0^r}\upsilon_p = \int_{T_0^r}\upsilon\wedge\omega_{p-1} .$$

Fubini's Theorem shall now be applied to $\int_{T_0^r} \upsilon \wedge \omega_{p-1}$. A statement of the theorem follows. The theorem in a more general setting is stated and proved by W. STOLL in [6].

Fubini's Theorem. *Let N and Q be pure dimensional complex manifolds with $\dim N = n$, $\dim Q = q < n$. Let $\sigma : N \to Q$ be a holomorphic map and suppose that σ has maximal rank. Define $N_y = \sigma^{-1}(y)$, a complex submanifold of N. Let φ be a differential form of bidegree (q, q) on Q. Let χ be a differential form of bidegree $(n-q, n-q)$ on the measurable set L in N. Suppose that $\chi \wedge \sigma^* \varphi$ is integrable over L. Let $\iota_y : N_y \to N$ be the injection. Then*

$$\int_L \chi \wedge \sigma^* \varphi = \int_Q \left(\int_{N_y \cap L} \iota_y^* \chi \right) \varphi .$$

In order to apply this theorem, the following is needed.

Let $\mathbf{P}(V)$ denote the complex projective space of the vector space V. Let $\varrho : V - \{0\} \to \mathbf{P}(V)$ be the residual map, which can be uniquely defined by requiring that $\varrho(\mathfrak{z}_1) = \varrho(\mathfrak{z}_2)$ if and only if $\mathfrak{z}_1 = u\mathfrak{z}_2$ for $u \in \mathbf{C} - \{0\}$. One and only one exterior differential form $\ddot{\omega}$ of bidegree $(1,1)$ exists on $\mathbf{P}(V)$ such that $\varrho^*(\ddot{\omega}) = \omega$. Define

$$\ddot{\omega}_q = \frac{1}{q!} \bigwedge_{\nu=1}^{q} \ddot{\omega} .$$

Then $\varrho^*(\ddot{\omega}_q) = \omega_q$. Let $T \subset V$ be a pure p-dimensional analytic cone with center 0 and $p \geqq 2$.

Define $\varrho(T - \{0\}) = \ddot{T}$. Then $\ddot{T}$ is a pure $(p-1)$-dimensional analytic set in $\mathbf{P}(V)$. Define $N = \dot{T} - \{0\}$, a pure p-dimensional smooth submanifold of $V - \{0\}$. Define $Q = \varrho(N)$, $\sigma = \varrho \,|\, N$. Then Q consists of all the simple points of $\ddot{T}$, and N is a cone, that is, $\mathfrak{z} \in N$, $u \in \mathbf{C} - \{0\}$ implies $u\mathfrak{z} \in N$. Hence $N = \sigma^{-1}(Q) = \varrho^{-1}(Q)$. And Q is a pure $(p-1)$-dimensional smooth submanifold of $\mathbf{P}(V)$. Let $\iota : N \to V - \{0\}$ and $j : Q \to \mathbf{P}(V)$ be the inclusions. Then

$$\begin{array}{ccc} N & \xrightarrow{\iota} & V - \{0\} \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \varrho} \\ Q & \xrightarrow{j} & \mathbf{P}(V) \end{array}$$

is commutative, and

$$\iota^* \omega_{p-1} = \iota^* \varrho^*(\ddot{\omega}_{p-1}) = \sigma^* j^*(\ddot{\omega}_{p-1}) .$$

Lemma 2.3. *The map $\sigma : N \to Q$ defined above has maximal rank.*

Proof. Identify V with $\mathbf{C}^n$ and denote $\varrho(\mathfrak{z}) = (z_1 : \ldots : z_n)$ if $\mathfrak{z} = (z_1, \ldots, z_n) \neq 0$. Let $\mathfrak{a} = (a_1, \ldots, a_n)$ be an arbitrary point of N. Define $a = \sigma(\mathfrak{a}) = (a_1 : \ldots : a_n) \in Q$. Then there exists $W' \subset \mathbf{C}^{p-1}$, $0 \in W'$ open, and $\alpha : W' \to \mathbf{P}(V)$ holomorphic such that $\alpha(0) = a$, $\alpha : W' \to \alpha(W') \subset Q$ topological, $\alpha(W')$ relatively open in Q, and $\operatorname{rank}_{\mathfrak{w}} \alpha = p-1$, $\mathfrak{w} \in W'$. There exists ν such that $a_\nu \neq 0$. Hence, if W' is small enough, $\tilde{\alpha} : W' \to V - \{0\}$ exists such that $\tilde{\alpha}$ is holomorphic and injective, and $\varrho \circ \tilde{\alpha} = \alpha$, $\tilde{\alpha}(0) = \mathfrak{a}$. Let $\tilde{\alpha}(\mathfrak{w}) = (\alpha_1(\mathfrak{w}), \ldots, \alpha_n(\mathfrak{w}))$ and, by choice of W', $\alpha_\nu(\mathfrak{w}) \neq 0$ for $\mathfrak{w} \in W'$. Define

$$f_\lambda(\mathfrak{w}) = \frac{\alpha_\lambda(\mathfrak{w})}{\alpha_\nu(\mathfrak{w})}, \quad \lambda = 1, \ldots, \nu-1, \nu+1, \ldots, n .$$

Then $\alpha(\mathfrak{w}) = (\alpha_1(\mathfrak{w}):\ldots:\alpha_n(\mathfrak{w})) = (f_1(\mathfrak{w}):\ldots:f_{\nu-1}(\mathfrak{w}):1:f_{\nu+1}(\mathfrak{w}):\ldots:f(\mathfrak{w}))$. Hence

$$\operatorname{rank}_{\mathfrak{w}} \frac{\partial(f_1,\ldots,f_{\nu-1},f_{\nu+1},\ldots,f_n)}{\partial(w_1,\ldots,w_{p-1})} = \operatorname{rank}_{\mathfrak{w}}\alpha = p-1$$

for $\mathfrak{w} \in W'$ using the coordinate system

$$\gamma(z_1:\ldots:z_n) = \left(\frac{z_1}{z_\nu},\ldots,\frac{z_{\nu-1}}{z_\nu},\frac{z_{\nu+1}}{z_\nu},\ldots,\frac{z_n}{z_\nu}\right)$$

in $\varrho\{\mathfrak{z}\,|\,\mathfrak{z}_\nu \neq 0\}$. Define $\beta: W' \times (\mathbf{C}-\{0\}) \to V-\{0\}$ by

$$\beta(\mathfrak{w},u) = \frac{u}{\alpha_\nu(\mathfrak{w})}\tilde{\alpha}(\mathfrak{w}) = (u\,f_1(\mathfrak{w}),\ldots,u\,f_{\nu-1}(\mathfrak{w}),u,$$
$$u\,f_{\nu+1}(\mathfrak{w}),\ldots,u\,f_n(\mathfrak{w}))\,.$$

Then β is holomorphic. If $\beta(\mathfrak{w}_1,u_1)=\beta(\mathfrak{w}_2,u_2)$, then $u_1=u_2$ and $\alpha(\mathfrak{w}_1) = \varrho(\beta(\mathfrak{w}_1,u_1)) = \varrho(\beta(\mathfrak{w}_2,u_2)) = \alpha(\mathfrak{w}_2)$. Hence $\mathfrak{w}_1=\mathfrak{w}_2$, and so β is injective. And $\beta(W'\times(\mathbf{C}-\{0\})) = \varrho^{-1}(\alpha(W'))$, for

$$\varrho(\beta(\mathfrak{w},u)) = \varrho(\tilde{\alpha}(\mathfrak{w})) = \alpha(\mathfrak{w}) \in \alpha(W')\,, \quad \text{or} \quad \beta(W'\times(\mathbf{C}-\{0\})) \subseteq \varrho^{-1}(\alpha(W'))\,.$$

And if $\mathfrak{z} \in \varrho^{-1}(\alpha(W'))$, then $\varrho(\mathfrak{z}) = \alpha(\mathfrak{w})$ for some $\mathfrak{w} \in W'$ and $\mathfrak{z} = v\,\tilde{\alpha}(\mathfrak{w})$ for some $v \in \mathbf{C}-\{0\}$. Then $u = v\cdot\alpha_\nu(\mathfrak{w}) \neq 0$. Hence $\beta(\mathfrak{w},u) = \frac{u}{\alpha_\nu(\mathfrak{w})}\tilde{\alpha}(\mathfrak{w}) = v\,\tilde{\alpha}(\mathfrak{w}) = \mathfrak{z}$, and so $\varrho^{-1}(\alpha(W')) \subseteq \beta(W'\times(\mathbf{C}-\{0\}))$. Thus $\beta: W'\times(\mathbf{C}-\{0\}) \to \varrho^{-1}(\alpha(W')) \subset N$ is bijective, holomorphic, and $\varrho^{-1}(\alpha(W')) = \sigma^{-1}(\alpha(W'))$ is open in N and $\beta(0,a_\nu) = \frac{a_\nu}{\alpha_\nu(0)}\tilde{\alpha}(0) = \mathfrak{a}$. Now

$$\begin{aligned}
\operatorname{rank}_{(\mathfrak{w},u)}\beta(\mathfrak{w},u) &= \operatorname{rank}_{(\mathfrak{w},u)} \frac{\partial(u\,f_1(\mathfrak{w}),\ldots,u\,f_{\nu-1}(\mathfrak{w}),u,u\,f_{\nu+1}(\mathfrak{w}),\ldots,u\,f_n(\mathfrak{w}))}{\partial(w_1,\ldots,w_{p-1},u)} \\
&= 1 + \operatorname{rank}_{\mathfrak{w}} \frac{\partial(u\,f_1(\mathfrak{w}),\ldots,u\,f_{\nu-1}(\mathfrak{w}),u\,f_{\nu+1}(\mathfrak{w}),\ldots,u\,f_n(\mathfrak{w}))}{\partial(w_1,\ldots,w_{p-1})} \\
&= p \quad \text{for} \quad (\mathfrak{w},u) \in W'\times(\mathbf{C}-\{0\})\,.
\end{aligned}$$

Thus β gives local coordinates of N at $\mathfrak{a}$. And $\sigma\circ\beta(\mathfrak{w},u) = \alpha(\mathfrak{w})$, or $\alpha^{-1}\circ\sigma\circ\beta(\mathfrak{w},u) = \mathfrak{w}$. Thus if $\tilde{\pi}: W'\times(\mathbf{C}-\{0\}) \to W'$ is the projection, $\operatorname{rank}_{\mathfrak{a}}\sigma = \operatorname{rank}_{\mathfrak{a}}\alpha^{-1}\circ\sigma\circ\beta = \operatorname{rank}_{\mathfrak{a}}\tilde{\pi} = p-1$. q.e.d.

Then Fubini's Theorem implies

$$\begin{aligned}
\int_{T_0^r} \upsilon\wedge\omega_{p-1} &= \int_{N\cap B_r} \iota^*\upsilon\wedge\iota^*\omega_{p-1} = \int_{N\cap B_r} \iota^*\upsilon\wedge\iota^*\varrho^*(\ddot{\omega}_{p-1}) \\
&= \int_{N\cap B_r} \iota^*\upsilon\wedge\sigma^*(j^*\ddot{\omega}_{p-1}) \\
&= \int_{a\in Q}\left(\int_{\sigma^{-1}(a)\cap B_r} \iota^*\upsilon\right) j^*\ddot{\omega}_{p-1} \\
&= \int_{a\in Q}\left(\int_{\sigma^{-1}(a)\cap B_r} \iota^*\upsilon\right) \ddot{\omega}_{p-1} \\
&= \int_{a\in \ddot{T}}\left(\int_{\sigma^{-1}(a)\cap B_r} \iota^*\upsilon\right) \ddot{\omega}_{p-1}
\end{aligned}$$

where $\sigma^{-1}(a)\cap B_r=\{z\mathfrak{a}\,|\,0<|z|<r\}$, $\mathfrak{a}$ chosen such that $\varrho(\mathfrak{a})=\sigma(\mathfrak{a})=a$ and $|\mathfrak{a}|=1$. Identify V with $\mathbf{C}^n$ by means of an orthonormal basis. Let $\mathfrak{a}=(a_1,\ldots,a_n)$. Define $j_{\mathfrak{a}}:\{z\,|\,0<|z|<r\}\to V-\{0\}$ by $j_{\mathfrak{a}}(z)=z\mathfrak{a}$. Then $\upsilon=\frac{i}{2}\sum_{\nu=1}^{n}dz_\nu\wedge d\overline{z_\nu}$, and

$$j_{\mathfrak{a}}^*\,\upsilon=\frac{i}{2}\sum_{\nu=1}^{n}a_\nu\overline{a_\nu}\,dz\wedge d\bar{z}=\frac{i}{2}\,dz\wedge d\bar{z}\,.$$

Thus

$$\int\limits_{\sigma^{-1}(a)\cap B_r}\iota^*\,\upsilon=\int\limits_{0<|z|<r}j_{\mathfrak{a}}^*\,\upsilon$$

$$=\int\limits_{0<|z|<r}\frac{i}{2}\,dz\wedge d\bar{z}=\pi r^2\,.$$

Hence

$$\int\limits_{T_0^r}\upsilon\wedge\omega_{p-1}=\pi r^2\int\limits_{\ddot{T}}\ddot{\omega}_{p-1}\,,$$

and

$$\frac{1}{W_p r^{2p}}\int\limits_{T_0^r}\upsilon_p=\frac{(p-1)!}{\pi^p r^2}\,\frac{p}{r^{2p-2}}\int\limits_{T_0^r}\upsilon_p$$

$$=\frac{(p-1)!}{\pi^p r^2}\int\limits_{T_0^r}\upsilon\wedge\omega_{p-1}$$

$$=\frac{(p-1)!}{\pi^{p-1}}\int\limits_{\ddot{T}}\ddot{\omega}_{p-1}\,.$$

Now $\ddot{T}$ is a pure $(p-1)$-dimensional analytic set in $\mathbf{P}(V)$, and so, from Chow's Theorem, $\ddot{T}$ is an algebraic set. From a result of G. DE RHAM, [4],

$$\frac{(p-1)!}{\pi^{p-1}}\int\limits_{\ddot{T}}\ddot{\omega}_{p-1}=m\,,$$

where m, a positive integer, is the degree of the algebraic set T. With the desire do make this paper as self-contained as possible, the fact that

$$\frac{(p-1)!}{\pi^{p-1}}\int\limits_{\ddot{T}}\ddot{\omega}_{p-1}$$

is a positive integer will also be proven here, by means of a method suggested by W. STOLL.

Proposition 2.4. *Let W be an $(n+1)$-dimensional complex vector space with a hermitian product. Let $\mathbf{P}(W)$ be the projective space. Let A be an analytic set in $\mathbf{P}(W)$ of pure dimension $q>0$. Then*

$$\frac{q!}{\pi^q}\int\limits_{A}\ddot{\omega}_q$$

is a positive integer.

Proof. Since A has only a finite number of branches A_λ, $\lambda = 1, \dots, k$, and because

$$\frac{q!}{\pi^q} \int\limits_A \ddot{\omega}_q = \sum_{\lambda=1}^{k} \frac{q!}{\pi^q} \int\limits_{A_\lambda} \ddot{\omega}_q$$

it is enough to prove the theorem for A irreducible. The proof is by induction on $d = n - q$. For $d = 0$, $A = \mathbf{P}(W)$, and

$$\frac{n!}{\pi^n} \int\limits_{\mathbf{P}(W)} \ddot{\omega}_n = 1 .$$

Now assume the proposition true for $n - q \leqq d - 1$, and let A be an irreducible, q-dimensional analytic set in $\mathbf{P}(W)$, where W is a vector space of dimension $n+1$, and where $n - q = d \geqq 1$. If $n = 1$, $q = 0$ and the proposition is trivial. Thus assume $n \geqq 2$. Choose a point $s \in \mathbf{P}(W)$, $s \notin A$. Choose an orthonormal basis of W in such a way that if W is identified with $\mathbf{C}^{n+1}$ and $\mathbf{P}(W)$ with $\mathbf{P}(\mathbf{C}^{n+1}) = \mathbf{P}^n$, and if $\varrho : \mathbf{C}^{n+1} - \{0\} \to \mathbf{P}^n$ is the residual map, then the point $\mathfrak{s} = (1, 0, \dots, 0) \in \mathbf{C}^{n+1}$ is in $\varrho^{-1}(s)$. Denote $\varrho(z_0, z_1, \dots, z_n) = (z_0 : z_1 : \dots : z_n) \in \mathbf{P}^n$ for $0 \neq \mathfrak{z} = (z_0, \dots, z_n) \in \mathbf{C}^{n+1}$. Let $\mathbf{P}^{n-1} = \mathbf{P}(\mathbf{C}^n)$, $\tilde{\varrho} : \mathbf{C}^n - \{0\} \to \mathbf{P}^{n-1}$ the residual map, $\tilde{\varrho}(z_1, \dots, z_n) = (z_1 : \dots : z_n)$ for $0 \neq (z_1, \dots, z_n) \in \mathbf{C}^n$. Define $\pi : \mathbf{P}^n - \{s\} \to \mathbf{P}^{n-1}$ by $\pi(z_0 : z_1 : \dots : z_n) = (z_1 : \dots : z_n)$. Let $a \in \mathbf{P}^{n-1}$. Then $\pi^{-1}(a) \cap A$ is analytic in the complex manifold $\pi^{-1}(a)$, and, if it contains an interior point, then $\pi^{-1}(a) \cap A = \pi^{-1}(a)$. But this would imply that $s \in A$, a contradiction. Hence $\pi^{-1}(a) \cap A$ consists of isolated points for every $a \in \mathbf{P}^{n-1}$. Clearly $\pi | A$ is a proper map. Hence $\pi(A) = B$ is an irreducible, q-dimensional analytic set in $\mathbf{P}^{n-1}$. Thus, from the induction assumption,

$$\frac{q!}{\pi^q} \int\limits_B \tilde{\omega}_q$$

is a positive integer, say m_1, where $\tilde{\omega}_q$ is the volume element in $\mathbf{P}^{n-1}$ associated to the hermitian product $(\mathfrak{z} | \mathfrak{w}) = \sum_{\nu=1}^{n} z_\nu \bar{w}_\nu$ on $\mathbf{C}^n$.

Let $S(A)$ be the set of non-simple points of A. Then $\pi(S(A))$ is an analytic set, thin in B. Let $B' = \dot{B} - \pi(S(A))$. Now B irreducible, $\pi(S(A))$ thin, implies that B' is a connected q-dimensional complex manifold. Let $A' = \pi^{-1}(B') \cap A = \pi^{-1}(B') \cap \dot{A}$, a q-dimensional complex manifold. Let $\tau = \pi | A'$. Then $\tau(A') = B'$. Let $N = \{a \in A' \,|\, \operatorname{rank}_a \tau < q\}$. Then N is a thin analytic set in A', and τ proper and $\tau^{-1}(b)$ discrete for $b \in B'$ implies that $\tau(N)$ is a thin analytic set in B'. Hence $B'' = B' - \tau(N)$ is connected. Let $A'' = \tau^{-1}(B'') = \pi^{-1}(B'') \cap A'$, and $\sigma = \tau | A''$. Then $\sigma : A'' \to B''$ is proper, and hence σ is an unrestricted or regular covering map of the complex manifold A'' onto the connected complex manifold B''. Therefore the number m_2 of points in $\sigma^{-1}(b)$ for $b \in B''$ is independent of b and finite. The map σ is of maximal rank with $\sigma(A'') = B''$. Hence from STOLL

[6, Satz 6],

$$\int_{B''} m_2 \tilde{\omega}_q = \int_{A''} \sigma^* \tilde{\omega}_q ,$$

and so,

$$\int_{B} m_2 \tilde{\omega}_q = \int_{A} \pi^* \tilde{\omega}_q .$$

Define the following operators on an n-dimensional complex manifold:

$$\partial = \sum_{\nu=1}^{n} \frac{\partial}{\partial z_\nu} dz_\nu \qquad \bar{\partial} = \sum_{\nu=1}^{n} \frac{\partial}{\partial \bar{z}_\nu} d\bar{z}_\nu .$$

Then $d = \partial + \bar{\partial}$.

Define $E_\lambda = \{\mathfrak{z} \in \mathbf{C}^{n+1} \mid \mathfrak{z} = (z_0, \ldots, z_n), z_\lambda \neq 0\}$ for $\lambda = 0, 1, \ldots, n$. Let $U_\lambda = \varrho(E_\lambda)$. Define, for

$$\zeta \in U_\lambda, \quad f_\lambda(\zeta) = \frac{|\mathfrak{z}|}{|z_\lambda|}, \quad g_\lambda(\zeta) = \frac{|z_1|^2 + \cdots + |z_n|^2}{|z_\lambda|^2}$$

where $\mathfrak{z} = (z_0, z_1, \ldots, z_n) \in \varrho^{-1}(\zeta)$. Note that f_λ and g_λ are independent of the choice of $\mathfrak{z} \in \varrho^{-1}(\zeta)$. Then, for any λ, $0 \leqq \lambda \leqq n$, it can be shown that $\ddot{\omega}(\zeta) = i\,\partial\bar{\partial} \log f_\lambda(\zeta)$ for $\zeta \in U_\lambda$, and similarly, $\pi^* \tilde{\omega}(\zeta) = (i/2)\,\partial\bar{\partial} \log g_\lambda(\zeta)$ for $\zeta \in U_\lambda - \{s\}$. Define, for $\zeta \in \mathbf{P}^n - \{s\}$, $h(\zeta) = \dfrac{|\mathfrak{z}|^2}{|z_1|^2 + \cdots + |z_n|^2}$, where $\mathfrak{z} = (z_0, \ldots, z_n) \in \varrho^{-1}(\zeta)$. Let $\theta(\zeta) = (i/2)\,\partial\bar{\partial} \log h(\zeta)$, $\zeta \in \mathbf{P}^n - \{s\}$. Now on $U_\lambda - \{s\}$, for any $0 \leqq \lambda \leqq n$,

$$\begin{aligned} \ddot{\omega} - \pi^* \tilde{\omega} &= \frac{i}{2} \partial\bar{\partial} \log f_\lambda^2 - \frac{i}{2} \partial\bar{\partial} \log g_\lambda \\ &= \frac{i}{2} \partial\bar{\partial} \log \frac{f_\lambda^2}{g_\lambda} \\ &= \frac{i}{2} \partial\bar{\partial} \log h = \theta . \end{aligned}$$

Now $\bigcup_{\lambda=0}^{n} (U_\lambda - \{s\}) = \mathbf{P}^n - \{s\}$, and so

$$\theta = \ddot{\omega} - \pi^* \tilde{\omega} \quad \text{on} \quad \mathbf{P}^n - \{s\} .$$

Define $\varphi(\zeta) = (i/2)\,\bar{\partial} \log h(\zeta)$ for $\zeta \in \mathbf{P}^n - \{s\}$. Then $d\varphi = (\partial + \bar{\partial})(\varphi) = \partial\varphi = \theta$, and

$$\begin{aligned} \ddot{\omega}^q &= (d\varphi + \pi^* \tilde{\omega})^q \\ &= \sum_{\mu=0}^{q} \binom{q}{\mu} (d\varphi)^{q-\mu} \wedge (\pi^* \tilde{\omega})^\mu , \\ \ddot{\omega}^q - \pi^* \tilde{\omega}^q &= \sum_{\mu=0}^{q-1} \binom{q}{\mu} (d\varphi)^{q-\mu} \wedge (\pi^* \tilde{\omega})^\mu . \end{aligned}$$

Define

$$\xi = \sum_{\mu=0}^{q-1} \binom{q}{\mu} (d\varphi)^{q-\mu-1} \wedge (\pi^* \tilde{\omega})^\mu \quad \text{on} \quad \mathbf{P}^n - \{s\}$$

Then $d\xi = 0$ $(d\pi^* \tilde{\omega} = \pi^* d\tilde{\omega} = 0)$, and $\ddot{\omega}^q - \pi^* \tilde{\omega}^q = d\varphi \wedge \xi = d(\varphi \wedge \xi)$. Let $\psi = \dfrac{\varphi \wedge \xi}{q!}$ on $\mathbf{P}^n - \{s\}$.

Then $\ddot{\omega}_q - \pi^* \tilde{\omega}_q = d\psi$. Hence, from a previously quoted theorem of LELONG [3, Theoreme 7],

$$\int_A (\ddot{\omega}_q - \pi^* \tilde{\omega}_q) = \int_A d\psi = 0 \quad (s \notin A).$$

Consequently,

$$\frac{\pi^q}{q!} \int_A \ddot{\omega}_q = \frac{\pi^q}{q!} \int_A \pi^* \tilde{\omega}_q = \frac{\pi^q}{q!} \int_B m_2 \tilde{\omega}_q = m_1 m_2, \text{ a}$$

positive integer. q.e.d.

The results of this section are summarized in the following

Theorem 2.5. *Let V be an n-dimensional complex vector space with a hermitian product. Let $T \subset V$ be a pure p-dimensional analytic cone with center* 0. *Suppose $p > 0$. Then*

$$\frac{1}{W_p r^{2p}} \int_{T_0^r} v_p$$

is a positive integer independent of r.

Proof. For $p = n$, the theorem is trivial, and for $2 \leqq p \leqq n-1$, the theorem has already been proven. If $p = 1$ and T is irreducible, then, for any $0 \neq \mathfrak{a} \in T$, $T = \{u\mathfrak{a} \mid u \in \mathbf{C}\}$, and so $\frac{1}{\pi r^2} \int_{T_0^r} v = 1$. Thus for $p = 1$ and T arbitrary, $\frac{1}{\pi r^2} \int_{T_0^r} v$ equals the number of irreducible branches of T, a finite integer. q.e.d.

§ 3. The tangent cone

Let V be now a fixed n-dimensional complex vector space with a hermitian product. Let M be a pure p-dimensional analytic set in an open subset G of V such that $0 \in M$. Then $\mathfrak{t}$ is said to be a *tangent vector to M at* 0 if there exists a sequence $\{\mathfrak{z}_\lambda\}$, $\mathfrak{z}_\lambda \in M$, $\mathfrak{z}_\lambda \neq 0$, such that $\mathfrak{z}_\lambda \to 0$ and $\frac{\mathfrak{z}_\lambda}{|\mathfrak{z}_\lambda|} \to \mathfrak{t}$ as $\lambda \to \infty$. The set $T = \{u\mathfrak{t} \mid u \in \mathbf{C}$, $\mathfrak{t}$ a tangent vector to M at $0\}$, is called the *tangent cone to M at* 0. It will be shown that T is a pure p-dimensional analytic set in V. This has also recently been proven by H. WHITNEY in [10]. However the proof given here uses a natural geometrical construction which is essential to the remainder of this work.

Define

$$H = \{(\mathfrak{z}, w) \mid w\mathfrak{z} \in G, \mathfrak{z} \in V, w \in \mathbf{C}\}$$
$$N^* = \{(\mathfrak{z}, w) \mid w\mathfrak{z} \in M, \mathfrak{z} \in V, w \in \mathbf{C}\} \subset H$$
$$\pi: V \oplus \mathbf{C} \to V, \text{ projection}$$
$$\tau: V \oplus \mathbf{C} \to \mathbf{C}, \text{ projection}$$
$$E = V \times \{0\} = \tau^{-1}(0)$$
$$N = \overline{(N^* - E)} \cap H$$
$$N(w) = \tau^{-1}(w) \cap N.$$

Extend the hermitian product on V to a product on $V\oplus\mathbf{C}$ by defining, for $(\mathfrak{z}, w)$ and $(\mathfrak{z}', w')\in V\oplus\mathbf{C}$, $((\mathfrak{z}, w)|(\mathfrak{z}', w'))=(\mathfrak{z}|\mathfrak{z}')+w\overline{w}'$, where $(|)$ is the given hermitian product on V.

Proposition 3.1. *N is a pure $(p+1)$-dimensional analytic set in H, and $\pi(N(0))=\pi(N\cap E)=T$ is a pure p-dimensional analytic set in V.*

Proof. Define $\gamma: V\oplus\mathbf{C}\to V$ by $\gamma(\mathfrak{z}, w)=w\mathfrak{z}$. Then γ is holomorphic, $\gamma^{-1}(G)=H$, and $\gamma^{-1}(M)=N^*$. Hence N^* is analytic in H. Define $\alpha: H-E\to \to G\times(\mathbf{C}-\{0\})$ by $\alpha(\mathfrak{z}, w)=(\mathfrak{z}w, w)$. Then α is biholomorphic, and $\alpha(N^*-E)=M\times(\mathbf{C}-\{0\})$. Hence, for $w\neq 0$,

$$\dim_{(\mathfrak{z},w)}N^*=\dim_{(w\mathfrak{z},w)}M\times(\mathbf{C}-\{0\})=1+\dim_{w\mathfrak{z}}M.$$

Therefore M pure p-dimensional implies that N^*-E is pure $(p+1)$-dimensional in $V\times(\mathbf{C}-\{0\})$. Now, from general theory, $H\cap\overline{(N^*-E)}=N$ is analytic in H, and, for points in $E\cap N$, N can be expressed locally as the union of the irreducible branches of N^* not contained in E. Hence N is pure $(p+1)$-dimensional and $N\cap E=N(0)=N\cap\{(\mathfrak{z}, w)|w=0\}$ is p-dimensional.

Finally, $\pi(N\cap E)=T$: Since $(0, w)\in N^*$ for any w, $0\in\pi(N\cap E)$. Let $z\mathfrak{t}\in T$, $z\mathfrak{t}\neq 0$. There exists a sequence $\{\mathfrak{z}_\lambda\}$, $\mathfrak{z}_\lambda\in M-\{0\}$, such that $\mathfrak{z}_\lambda\to 0$ and $\frac{\mathfrak{z}_\lambda}{|\mathfrak{z}_\lambda|}\to\mathfrak{t}$ as $\lambda\to\infty$. Then $\left(\frac{z\mathfrak{z}_\lambda}{|\mathfrak{z}_\lambda|}, \frac{|\mathfrak{z}_\lambda|}{z}\right)\in N^*-E$, and $\left(\frac{z\mathfrak{z}_\lambda}{|\mathfrak{z}_\lambda|}, \frac{|\mathfrak{z}_\lambda|}{z}\right)\to(z\mathfrak{t}, 0)$. Thus $T\subset\pi(N\cap E)$. Conversely, let $\mathfrak{z}\in\pi(N\cap E)$ and assume that $\mathfrak{z}\neq 0$. There exists a sequence $\{(\mathfrak{z}_\lambda, w_\lambda)\}$, $(\mathfrak{z}_\lambda, w_\lambda)\in N^*-E$ such that $\mathfrak{z}_\lambda\to\mathfrak{z}$, $w_\lambda\to 0$, and $\mathfrak{z}_\lambda\neq 0$. Then $\mathfrak{z}_\lambda w_\lambda\in M-\{0\}$, and $\mathfrak{z}_\lambda w_\lambda\to 0$ as $\lambda\to\infty$. There exists a subsequence of $\{w_\lambda\}$, say $\{w_{\lambda_\nu}\}$, such that $\frac{w_{\lambda_\nu}}{|w_{\lambda_\nu}|}$ converges, say $\frac{w_{\lambda_\nu}}{|w_{\lambda_\nu}|}\to u$, as $\nu\to\infty$. Let $\mathfrak{t}=\lim_{\nu\to\infty}\frac{\mathfrak{z}_{\lambda_\nu}w_{\lambda_\nu}}{|\mathfrak{z}_{\lambda_\nu}w_{\lambda_\nu}|}$. Then $\mathfrak{z}=\frac{|\mathfrak{z}|}{u}\mathfrak{t}\in T$. Thus $\pi(N\cap E)\subset T$. q.e.d.

Define $I(w, r)=\int_{\pi(N(w))\cap B_r} v_p$ for $0\leqq r<\frac{R}{|w|}$, where $B_R\subset G$, and $\pi(N(w))\cap B_r=\{\mathfrak{z}\in V|(\mathfrak{z}, w)\in N(w),\ |\mathfrak{z}|<r\}$. Note that $I(w, r)/r^{2p}$ is monotonic increasing in r, and that $n(r, m)=\frac{1}{W_p r^{2p}}I(1, r)$.

Define $W=\{w|0<|w|\leqq 1\}$. For $w\in W$ and $0<r<R$, define $g: M_0^r\to\pi(N(w))$ by $g(\mathfrak{z})=\frac{\mathfrak{z}}{w}$. Then $g(M_0^r)=\pi(N(w))\cap B_{r/|w|}$, and

$$\begin{aligned}
I\left(w, \frac{r}{|w|}\right) &= \int_{\pi(N(w))\cap B_{r/|w|}} v_p\\
&= \int_{M_0^r} g^*(v_p)\\
&= \int_{M_0^r}\frac{1}{|w|^{2p}}v_p=\frac{I(1, r)}{|w|^{2p}}.
\end{aligned}$$

Thus $I(1,r)=|w|^{2p}I(w,r/|w|)$, and

$$I(w,s)=\frac{1}{|w|^{2p}}I(1,|w|s),\quad \text{letting}\quad r=|w|s\,.$$

For $w,w'\in W$,

$$|w|^{2p}I(w,r/|w|)=I(1,r)=|w'|^{2p}I(w',r/|w'|)\,,$$

$$I\left(w,\frac{r}{|w|}\right)=\left|\frac{w'}{w}\right|^{2p}I\left(w',\frac{r}{|w'|}\right)$$

$$I(w,s)=\left|\frac{w'}{w}\right|^{2p}I\left(w',\frac{s|w|}{|w'|}\right).$$

Define

$$\begin{aligned} l(w)&=\lim_{r\to0}\frac{I(w,r)}{r^{2p}}\\ &=\lim_{r\to0}\frac{I(1,|w|r)}{|w|^{2p}r^{2p}}\\ &=\lim_{s\to0}\frac{I(1,s)}{s^{2p}}=l(1)\end{aligned}$$

for all $w\in W$.

Lemma 3.2. $\dfrac{I(w,r)}{r^{2p}}\to l(w)$ *uniformly on* W *as* $r\to0$.

Proof.

$$0\leqq\frac{I(w,r)}{r^{2p}}-l(w)$$

$$=\frac{I(1,r|w|)}{(r|w|)^{2p}}-l(1)\leqq\frac{I(1,r)}{r^{2p}}-l(1)\,.$$

q.e.d.

Now if $|w|=|w'|$, then $I(w,r)=I(w',r)$.

And if

$$|w|<|w'|,\ I(w,r)=\left|\frac{w'}{w}\right|^{2p}I\left(w',\frac{r|w|}{|w'|}\right)$$

$$=\frac{I(w',r|w|/|w'|)}{(r|w|/|w'|)^{2p}}<I(w',r)\,.$$

Thus $\lim\limits_{w\to0}I(w,r)$ exists, $0<r<R$.

Hence, for $w\in W$,

$$n(r,M)=\frac{1}{W_pr^{2p}}I(1,r)=\frac{|w|^{2p}}{W_pr^{2p}}I\left(w,\frac{r}{|w|}\right),$$

$$\begin{aligned} n(0,M)&=\lim_{r\to0}\frac{1}{W_p}\frac{I(w,r/|w|)}{(r/|w|)^{2p}}\\ &=\lim_{s\to0}\frac{I(w,s)}{W_ps^{2p}}\,.\end{aligned}$$

Thus

$$n(0, M) = \lim_{w \to 0} \lim_{r \to 0} \frac{I(w, r)}{W_p r^{2p}}$$

$$= \lim_{r \to 0} \lim_{w \to 0} \frac{I(w, r)}{W_p r^{2p}}$$

$$= \lim_{r \to 0} \frac{1}{W_p r^{2p}} \lim_{w \to 0} I(w, r).$$

In the next section, $\lim_{w \to 0} I(w, r)$ will be related to $\int_{T_0^r} v_p$, and thus, the results of § 2 can be applied to determine $n(0, M)$.

§ 4. A continuity theorem

A. Multiplicity of a holomorphic map

It is necessary to introduce the concept of multiplicity of a holomorphic map as the multiplicity of $\tau | N$ must be considered in the proof of the continuity of the area. Let X and Y be complex spaces and let $\sigma: X \to Y$ be a holomorphic map. Then σ is said to be *non-degenerate* if the fibers $\sigma^{-1}(\sigma(x))$ consists of isolated points only.

Let X be a normal complex space, Y a complex space, and $\sigma: X \to Y$ a holomorphic, non-degenerate map. Take $a \in X$. Take any open neighborhood U of a such that $\bar{U}$ is compact and such that $\bar{U} \cap \sigma^{-1}(\sigma(a)) = \{a\}$. Such a neighborhood exists. Define

$$\mu_U(x, \sigma) = \# U \cap \sigma^{-1}(\sigma(x)) \quad \text{for} \quad x \in U,$$

where $\# A$ denotes the number of elements of A for a finite set A, defining $\# A$ to be 0 if A is empty and $\# A$ to be ∞ if A is infinite. The number $\nu_U(a, \sigma) = \limsup_{x \to a} \mu_U(x, \sigma)$ is independent of U [9, Lemma 2.1], and is denoted by $\nu(a, \sigma)$. Note that if $\varrho': X' \to X$ is a biholomorphic map from a normal complex space X', then, for $a' \in X'$, $\nu(a', \sigma \circ \varrho') = \nu(\varrho'(a'), \sigma)$.

Let X be now an arbitrary complex space and $\sigma: X \to Y$ be again a holomorphic, non-degenerate map. Let $\hat{X}$ be the normalization of X, and $\varrho: \hat{X} \to X$ the normalization map (see for example S. ABHYANKAR [1]). Then $\sigma \circ \varrho: \hat{X} \to Y$ is a holomorphic, non-degenerate map, as $\varrho^{-1}(a)$ consists of only a finite number of points for each $a \in X$. Define $\nu(a, \sigma) = \sum_{\hat{a} \in \varrho^{-1}(a)} \nu(\hat{a}, \sigma \circ \varrho)$[1].

Let X be again normal, and $\sigma: X \to Y$ a holomorphic map such that $\sigma^{-1}(\sigma(x))$ is an analytic set of pure dimension q for every $x \in X$. Suppose that X has pure dimension k. Take $a \in X$. Let Γ_a be the set of sets A satisfying the following conditions:

1. *An open neighborhood U_A of a exists such that $a \in A \subset U_A$ and such that A is analytic and of pure dimension $k - q$ in U_A.*
2. *The closure $\overline{U_A}$* is compact.
3. *The restriction $\sigma | A$ is non-degenerate.*

[1] Notice that the definition of multiplicity if X is normal does not require the fact that X is normal to be meaningful. Thus a multiplicity, not always equal to the one defined above, could be defined without passing to the normalization of X. See Section 4C.

Lemma 4.1. *Γ_a as defined above is non-empty.*

Proof. There exists an open, connected neighborhood $U \subset X$ of a and a proper, holomorphic map $\varphi: U \to D$ where D is an open set in $\mathbf{C}^k$ such that $\bar{U}$ is compact, $\varphi(U) = D$, $\varphi(a) = 0$, $\varphi^{-1}(0) = a$, $\varphi^{-1}(z)$ consists of isolated points for all $z \in D$, and, if S is an analytic set in an open set $U_1 \subset U$, then either S consists of isolated points or else there exists a sequence $\{x_\nu\}$ such that $x_\nu \in S$ and $x_\nu \to x_0 \in \bar{U}_1 - U_1$ as $\nu \to \infty$. Let $\sigma^{-1}\sigma(a) = L$ and $L' = \varphi(L \cap U)$, a q-dimensional analytic set in D. There exists an open neighborhood $D' \subset D$ of 0 and a set $A' \subset D'$ analytic in D' and of pure dimension $k-q$ such that $A' \cap L' = \{0\}$. Let $A'' = \varphi^{-1}(A')$, an analytic set of pure dimension $k-q$ in $\varphi^{-1}(D')$, an open neighborhood of a. Choose an open neighborhood Q of a such that $Q \subset \bar{Q} \subset \subset \varphi^{-1}(D')$. Now it is claimed that there exists an open neighborhood $W \subset Y$ of $\sigma(a)$ such that $x \in (\bar{Q} - Q) \cap A''$ implies that $\sigma(x) \notin W$. For suppose that there exists a sequence $x_\nu \in (\bar{Q} - Q) \cap A''$ such that $\sigma(x_\nu) \to \sigma(u)$, $\nu \to \infty$. Since $(\bar{Q} - Q) \cap A''$ is compact, $\{x_\nu\}$ contains a convergent subsequence. Without loss of generality, assume $x_\nu \to x_0 \in (\bar{Q} - Q) \cap A''$ as $\nu \to \infty$. Then $\sigma(x_0) = \sigma(a)$, and so $x_0 \in \sigma^{-1}\sigma(a) \cap U = L \cap U$. Thus $\varphi(x_0) \in L'$. And $x_0 \in A''$ implies $\varphi(x_0) \in A'$. Therefore $\varphi(x_0) \in L' \cap A' = \{0\}$, and so $\varphi(x_0) = 0$. Therefore $x_0 = a \in Q$, a contradiction, and so the claim is established. Choose such a W. Define

$$U_A = Q \cap \sigma^{-1}(W), \quad A = A'' \cap U_A.$$

Then U_A is an open neighborhood in X of a, $\bar{U}_A$ is compact, and A is a pure $(k-q)$-dimensional analytic set in U_A. Take any $b \in A$. Then $\sigma^{-1}\sigma(b) \cap A$ is an analytic set in U_A. Suppose that there exists a sequence $\{x_\nu\}$ such that $x_\nu \in \sigma^{-1}\sigma(b) \cap A$ and $x_\nu \to x_0 \in \bar{U}_A - U_A$ as $\nu \to \infty$. Then $x_\nu \in A \subset \bar{Q} \cap A''$ implies that $x_0 \in \bar{Q}$ and $x_0 \in A''$. And $x_\nu \in \sigma^{-1}\sigma(b)$ implies $x_0 \in \sigma^{-1}\sigma(b)$, and so $\sigma(x_0) = \sigma(b) \in W$. Thus $x_0 \in \sigma^{-1}(W)$. But $x_0 \notin U_A = Q \cap \sigma^{-1}(W)$, and so $x_0 \notin Q$. Hence $x_0 \in (\bar{Q} - Q) \cap A''$, and so $\sigma(x_0) \notin W$ by the choice of W, a contradiction. Consequently, $\sigma^{-1}\sigma(b) \cap A$ consists of isolated points only, that is, $\sigma | A$ is non-degenerate. q.e.d.

Thus, for $\sigma: X \to Y$ holomorphic, X normal, $\sigma^{-1}(\sigma(x))$ a pure q-dimensional analytic set for $x \in X$, define, for $a \in X$,

$$\nu(a, \sigma) = \operatorname*{Min}_{A \in \Gamma_a} \nu(a, \sigma | A).$$

Note again that if $\varrho': X' \to X$ is a biholomorphic map, then, for $a' \in X'$, $\nu(a', \sigma \circ \varrho') = \nu(a, \sigma)$ where $a = \varrho'(a')$. For if $A' \in \Gamma_{a'}$, then $\varrho'(A') = A \in \Gamma_a$ and $\varrho' | A': A' \to A$ is biholomorphic. Thus $\nu(a', \sigma \circ \varrho' | A') = \nu(a, \sigma | A)$ and so $\nu(a', \sigma \circ \varrho') \geqq \nu(a, \sigma)$. Similarly, if $A \in \Gamma_a$, then $(\varrho')^{-1}(A) \in \Gamma_{a'}$, and so $\nu(a, \sigma) \leqq \leqq (a', \sigma \circ \varrho')$. Hence $\nu(a, \sigma) = \nu(a', \sigma \circ \varrho')$.

Finally, let X and Y be arbitrary complex spaces, and let $\sigma: X \to Y$ be a holomorphic map such that $\sigma^{-1}(\sigma(x))$ is a pure q-dimensional analytic set for $x \in X$. Let $\hat{X}$ be the normalization of X and $\varrho: \hat{X} \to X$ the normalization map. Define, for $a \in X$,

$$\nu(a, \sigma) = \sum_{\hat{a} \in \hat{X}} \nu(\hat{a}, \sigma \circ \varrho).$$

The more common concept of the b-multiplicity of a holomorphic function is also needed. Let f be a holomorphic function on an open, connected set L contained in a complex vector space W, and let $\mathfrak{a} \in L$. Then $f(\mathfrak{z}) = \sum_{\lambda=0}^{\infty} P_\lambda(\mathfrak{z}-\mathfrak{a})$, where the series converges uniformly to f in an open neighborhood of $\mathfrak{a}$. The term P_λ is either identically zero or a homogeneous polynomial of degree λ, and the terms P_λ are uniquely defined by f. If $f \not\equiv 0$ on L, then the smallest index λ_0 such that $P_{\lambda_0} \not\equiv 0$ is called the *zero-multiplicity of f at* $\mathfrak{a}$, and denoted by $\nu(\mathfrak{a}, 0, f)$. For $b \in \mathbf{C}$, define the *b-multiplicity of f at* $\mathfrak{a}$, $\nu(\mathfrak{a}, b, f)$, tobe the zero-multiplicity of the function $f(\mathfrak{z}) - b$ at $\mathfrak{a}$.

Proposition 4.2. *Let $f \not\equiv 0$ be a holomorphic function on an open, connected set $L \subset \mathbf{C}^m$. Let $\mathfrak{a} \in L$. Then $\nu(\mathfrak{a}, f) = \nu(\mathfrak{a}, f(\mathfrak{a}), f)$.*

Proof (see Stoll [9], Lemma 2.3). For $n=1$, the proposition has been proven by W. Stoll [9, Lemma 2.2]. Assume $n \geqq 2$. The fiber $f^{-1}(f(\mathfrak{z}))$ is analytic and has pure dimension $n-1$. In an open neighborhood $U \subset L$ of $\mathfrak{a}$,

$$f(\mathfrak{z}) = f(\mathfrak{a}) + \sum_{\lambda=q}^{\infty} P_\lambda(\mathfrak{z}-\mathfrak{a}),$$

where P_λ is a homogeneous polynomial of degree λ or identically zero, and where $P_q \not\equiv 0$. Take any $A \in \Gamma_{\mathfrak{a}}$. Let $\hat{A}$ be the normalization of A, $\varrho: \hat{A} \to A$ the associated map. Let $\hat{a}_1 \in \varrho^{-1}(\mathfrak{a})$. An open neighborhood $\hat{U}_1$ of $\hat{a}_1$ and a biholomorphic map $g: L_1 \to \hat{U}_1$ of an open neighborhood L_1 of $0 \in \mathbf{C}$ exists such that $g(0) = \hat{a}_1$ and $\varrho(g(L_1)) = \varrho(\hat{U}_1) \subset U \cap A$. Then $\nu(0, f|A \circ \varrho \circ g) = (\hat{a}_1, f|A \circ \varrho)$. But, for $t \in L_1$,

$$\begin{aligned} f|A \circ \varrho \circ g(t) &= f(\varrho(g(0))) + \sum_{\lambda=q}^{\infty} P_\lambda(\varrho(g(t)) - \varrho(g(0))) \\ &= f(\mathfrak{a}) + \sum_{\lambda=q}^{\infty} c_\lambda t^\lambda . \end{aligned}$$

Therefore $\nu(\hat{a}_1, f|A \circ \varrho) = \nu(0, f|A \circ \varrho \circ g) \geqq q$. Therefore $\nu(\mathfrak{a}, f|A) = \sum_{\hat{a} \in \varrho^{-1}(\mathfrak{a})} \nu(\hat{a}, f|A \circ \varrho) \geqq q$. Therefore $\nu(\mathfrak{a}, f) \geqq q$. Take $\mathfrak{c}$ such that $P_q(\mathfrak{c}) \neq 0$, and define $A = \{\mathfrak{a} + t\mathfrak{c} \,|\, |t| < \varepsilon\}$, a one dimensional analytic set consisting only of normal points. Define $g(t) = \mathfrak{a} + t\mathfrak{c}$. Then

$$f(g(t)) = f(\mathfrak{a}) + \sum_{\lambda=q}^{\infty} P_\lambda(\mathfrak{c})\, t^\lambda \quad (P_\lambda(\mathfrak{c}) \neq 0).$$

Hence $A \in \Gamma_{\mathfrak{a}}$ if $\varepsilon > 0$ is small enough, and $\nu(\mathfrak{a}, f|A) = q$. Hence $\nu(\mathfrak{a}, f) = q$. q.e.d.

Recall now the definition of V, M, N, τ, π, etc. given in the beginning of § 3.

Lemma 4.3. *Let $(\mathfrak{a}, b) \in \dot{N}(b)$, where $\dot{N}(b)$ is the set of simple points of the analytic set $N(b)$. Assume that $b \neq 0$. Then $\nu(\mathfrak{a}, b), \tau|N) = 1$.*

Proof. An open neighborhood U' of $0 \in \mathbf{C}^p$ and $\alpha: U' \to \underset{\sim}{U}$ biholomorphic exists where $\underset{\sim}{U}$ is relative open in $N(b)$ and $\alpha(0) = (\mathfrak{a}, b)$. It is $\alpha: U' \to V \oplus \mathbf{C}$ and $\operatorname{rank}_x \alpha = p$ for each $x \in U'$. Define $\beta = \pi \circ \alpha$. Then $\alpha(x) = (\beta(x), b)$ and so $\operatorname{rank}_x \beta = p$. Take $r > 0$ such that

$$\{(\mathfrak{z}, b) \,|\, |\mathfrak{z} - \mathfrak{a}| \leqq r\} \cap N(b) \subset \underset{\sim}{U} .$$

Define

$$U = \{\mathfrak{z} \mid \mathfrak{z} \in V, |\mathfrak{z} - \mathfrak{a}| < r\}$$
$$U'' = \alpha^{-1}((U \times \{b\}) \cap N(b)) = \alpha^{-1}(\pi^{-1}(U) \cap N(b)) \subset U'$$
$$W' = \{\lambda \mid |\lambda - | < 1/2, \lambda \in \mathbf{C}\}$$
$$Y = \left\{(\mathfrak{z}, w) \,\middle|\, \left|\frac{w}{b}\mathfrak{z} - \mathfrak{a}\right| < r, |w - b| < \frac{|b|}{2}\right\}$$
$$\tilde{\alpha}: U'' \times W' \to V \oplus \mathbf{C},$$

defined by $\tilde{\alpha}(x, \lambda) = (\lambda^{-1}\beta(x), \lambda b)$. It will be shown, by means of $\tilde{\alpha}$, that $N \cap Y$ contains only simple points of N. Obviously U'' is open in U' and $0 \in U''$. Take $(x, \lambda) \in U'' \times W'$. Then $\alpha(x) \in N(b), \beta(x) = \pi(\alpha(x)) \in U$, and $\alpha(x) = (\beta(x), b) \in N$ implies $\tilde{\alpha}(x, \lambda) = (\lambda^{-1}\beta(x), \lambda b) \in N$ as $\lambda^{-1}\beta(x) \cdot \lambda b = \beta(x) b \in M$. Now $|\beta(x) - \mathfrak{a}| < r$ as $\beta(x) \in U$. Hence $\left|\frac{\lambda b}{b}\frac{\beta(x)}{\lambda} - \mathfrak{a}\right| = |\beta(x) - \mathfrak{a}| < r$, and $|\lambda b - b| = |b||\lambda - 1| < |b|/2$. Hence $\tilde{\alpha}(x, \lambda) \in Y$. Therefore $\tilde{\alpha}: U'' \times W' \to N \cap Y$. Because β is one-one, $\tilde{\alpha}$ is also one-one. Let $x = (x_1, \ldots, x_p)$. Obviously $\tilde{\alpha}_{x_\nu}(x, \lambda) = (\lambda^{-1}\beta_{x_\nu}(x), 0)$, $\nu = 1, \ldots, p$, and $\tilde{\alpha}_\lambda(x, \lambda) = (-\lambda^2\beta(x), b)$, and so $\tilde{\alpha}_{x_1}, \ldots, \tilde{\alpha}_{x_p}, \tilde{\alpha}_\lambda$ are linearly independent over $\mathbf{C}$. Thus rank $_{(x,\lambda)}\tilde{\alpha}(x, \lambda) = p + 1$. Define now $\hat{\alpha}: N \cap Y \to U'' \times W'$ by

$$\hat{\alpha}(\mathfrak{z}, w) = \left(\alpha^{-1}\left(\frac{w\mathfrak{z}}{b}, b\right), \frac{w}{b}\right).$$

If $(\mathfrak{z}, w) \in N \cap Y$, $\left|\frac{w\mathfrak{z}}{b} - \mathfrak{a}\right| < r$ and $\left(\frac{w\mathfrak{z}}{b}, b\right) = \left(\frac{\mathfrak{z}}{b/w}, \frac{b}{w} \cdot w\right) \in N(b)$. Thus $\left(\frac{w\mathfrak{z}}{b}, b\right) \in \underset{\sim}{U}$ and so $\hat{\alpha}$ is defined. And $\hat{\alpha}$ is holomorphic. It is $|b^{-1}w - 1| = |b|^{-1}|w - b| < 1/2$, and so $\hat{\alpha}(\mathfrak{z}, w) \in U'' \times W'$. Now

$$\begin{aligned}
\tilde{\alpha}(\hat{\alpha}(\mathfrak{z}, w)) &= \tilde{\alpha}\left(\alpha^{-1}\left(\frac{w\mathfrak{z}}{b}, b\right), \frac{w}{b}\right) \\
&= \left(\frac{b}{w}\beta\left(\alpha^{-1}\left(\frac{w\mathfrak{z}}{b}, b\right)\right), \frac{w}{b} \cdot b\right) \\
&= \left(\frac{b}{w} \cdot \frac{w\mathfrak{z}}{b}, w\right) = (\mathfrak{z}, w).
\end{aligned}$$

Therefore $\hat{\alpha}$ is surjective, and so, $\tilde{\alpha}$ is bijective. Thus $\tilde{\alpha}^{-1} = \hat{\alpha}$ and $\tilde{\alpha}: U'' \times \times W' \to N \cap Y$ is biholomorphic. Hence every point of $N \cap Y$ is a simple point, and so, considered as a complex space, $N \cap Y$ is normal. And $\tilde{\alpha}$ biholomorphic implies that $\nu((\mathfrak{a}, b), \tau | N) = \nu((0, 1), \tau | N \circ \tilde{\alpha})$, as $\tilde{\alpha}(0, 1) = (\mathfrak{a}, b)$. Define $f: U'' \times \times W' \to \mathbf{C}$ by $f(x, \lambda) = \lambda b$. Then $\tilde{\alpha}(x, \lambda) = (\lambda^{-1}\beta(x), f(x, \lambda))$, and $\tau | N \circ \tilde{\alpha} = f$. But $\nu((0, 1), f) = \nu((0, 1), b, f)$, by Proposition 4.2, and $\nu((0, 1), b, f) = 1$. Therefore $\nu((\mathfrak{a}, b), \tau | N) = \nu((0, 1), b, f) = 1$. q.e.d.

Let $\hat{N}$ be the normalization of N and $\varrho: \hat{N} \to N$ the normalization map. Let $\hat{S}$ be the set of non-simple or singular points of $\hat{N}$. Then $\hat{S}$ is an analytic set of dimension less than or equal $\dim \hat{N} - 2 = p - 1$, as $\hat{N}$ is normal [1, 45.15].

Let $S=\varrho(\hat{S})$. Then S is an analytic set in N of dimension less than or equal $p-1$.

Recall that $T=\pi(N(0))$ was the tangent cone of M at 0. Now T is an algebraic set in V and so T has only finitely many irreducible branches $T_1, \ldots, T_b$, each branch being an analytic cone with center 0 and dimension p.

Lemma 4.4. *For fixed* $\lambda, \nu((\mathfrak{z},0), \tau|N)$ *is constant on* $(\dot{T}\times\{0\})\cap(T_\lambda\times\{0\})\cap\cap(N-S)$.

Proof. Identify $V\times\{0\}=V$. Now $\dot{T}\cap T_\lambda$ is a smooth, connected submanifold of V containing $S\cap\dot{T}\cap T_\lambda$, a thin, analytic subset. Consequently $\dot{T}\cap T_\lambda\cap\cap(N-S)$ is connected. Thus it is sufficient to prove that $\nu((\mathfrak{z},0),\tau|N)$ is locally constant.

Let $\mathfrak{a}\in\dot{T}\cap T_\lambda\cap(N-S)$. Let $\{\hat{a}_1,\ldots,\hat{a}_q\}=\varrho^{-1}(\mathfrak{a})$. For each $i=1,\ldots,q$, there exist neighborhoods X_i^* of a_i and X_i'' of $0\in\mathbf{C}^{p+1}$ and a biholomorphic map $\sigma_i: X_i''\to\hat{X}_i^*$, $\sigma_i(0)=\hat{a}_i$. And there exist neighborhoods $U^*\subset N$ of $\mathfrak{a}$ and W'' of $0\in\mathbf{C}^p$ and a biholomorphic map $\alpha: W''\to\dot{T}\cap T_\lambda\cap U^*$, $\alpha(0)=\mathfrak{a}$. Then there exists pairwise disjoint neighborhoods $\hat{X}_1,\ldots,\hat{X}_q$ of $\hat{a}_1,\ldots,\hat{a}_q$ in $\hat{X}_1^*,\ldots,\hat{X}_q^*$ and analytic sets $Y_1,\ldots,Y_q$ in a neighborhood U of $\mathfrak{a}$ in U^* such that $\varrho^{-1}(U)=\bigcup_{i=1}^{q}\hat{X}_i$, $U=\bigcup_{i=1}^{q}Y_i$, and $\varrho(\hat{X}_i)=Y_i$ for each $i=1,\ldots,q$, [1, 46.15]. Define $X_i'=\sigma_i^{-1}(\hat{X}_i)\subset X_i''$, and $\varrho_i=\varrho|\hat{X}_i:\hat{X}_i\to Y_i$, $i=1,\ldots,q$, and

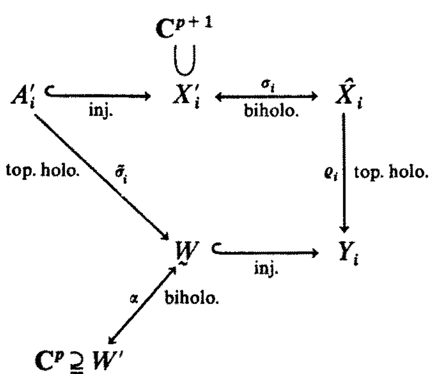
$$W'=\alpha^{-1}(U\cap\dot{T}\cap T_\lambda)\subset W'',\quad \underset{\sim}{W}=\alpha(W').$$

Each Y_i is locally irreducible, and so ϱ_i is a topological map [1, 46.10].

Define, for $i=1,\ldots,q$,

$$\begin{aligned} A_i' &= \{x\in X_i' \mid \tau\circ\varrho_i\circ\sigma_i(x)=0\}\\ &=\sigma_i^{-1}(\varrho_i^{-1}(Y_i\cap\underset{\sim}{W})),\\ \tilde{\sigma}_i &= \varrho_i\circ\sigma_i|A_i': A_i'\to Y_i\cap\underset{\sim}{W}, \end{aligned}$$

a topological, holomorphic map. Now $\underset{\sim}{W}\cap Y_i=U\cap T_\lambda\cap\dot{T}\cap Y_i=E\cap Y_i$, where $E=V\times\{0\}$. Thus $\dim\underset{\sim}{W}\cap Y_i=p$. But $\underset{\sim}{W}=U\cap T_\lambda\cap\dot{T}$ is an irreducible analytic set, and $Y_i\cap\underset{\sim}{W}$ is analytic in $\underset{\sim}{W}$. Therefore $Y_i\cap\underset{\sim}{W}=\underset{\sim}{W}$ for each $i=1,\ldots,q$. A diagram:



Now, for any i, $\alpha^{-1}\circ\tilde{\sigma}_i: A_i'\to W'$ is a holomorphic, topological map, and therefore, $\alpha^{-1}\circ\tilde{\sigma}_i$ is biholomorphic outside of a thin analytic set. Hence

$$\tilde{\sigma}_i^{-1}\circ\alpha: W'\to A_i'$$

is continuous on W' and holomorphic except on a thin analytic set. Then, by the Riemann Extension Theorem, $\tilde{\sigma}_i^{-1}\circ\alpha$ is holomorphic on W'. Hence $\alpha^{-1}\circ\tilde{\sigma}_i$ is a biholomorphic map, and so, A_i' consists of simple points only. Thus there exists a function f_i holomorphic in a neighborhood $Z_i'\subset X_i'$ of 0 such that

$$A_i'\cap Z_i' = \{x\in Z_i' \mid f_i(x)=0\}$$

and $\nu(x,0,f_i)=1$ for $x\in A_i'\cap Z_i'$, that is, $\dfrac{\partial f_i}{\partial x_j}(x)\neq 0$ for $x\in A_i'\cap Z_i'$ and at least one j, depending on x. Now $A_i'\cap Z_i'=\{x\in Z_i' \mid \tau\circ\varrho_i\circ\sigma_i(x)=0\}$, and so, in a neighborhood $Z_i\subset Z_i'$ of 0, $(\tau\circ\varrho_i\circ\sigma_i)^{m_i}=f_i$ for some natural number m_i. Let $W=\bigcap_{i=1}^{q}(\underset{\sim}{W}\cap\varrho_i(\sigma_i(Z_i)))$, a neighborhood in $T_\lambda\cap\dot{T}\cap(N-S)$ of $\mathfrak{a}$. For $\mathfrak{z}\in W$,

$$\begin{aligned}
\nu(\mathfrak{z},\tau\mid N) &= \sum_{\hat{z}\in\varrho^{-1}(\mathfrak{z})}\nu(\hat{\mathfrak{z}},\tau\mid N\circ\varrho)\\
&= \sum_{i=1}^{q}\nu(\varrho_i^{-1}(\mathfrak{z}),\tau\mid N\circ\varrho_i)\\
&= \sum_{i=1}^{q}\nu(\sigma_i^{-1}(\varrho_i^{-1}(\mathfrak{z})),\tau\mid N\circ\varrho_i\circ\sigma_i)\\
&= \sum_{i=1}^{q}\nu(\sigma_i^{-1}(\varrho_i^{-1}(\mathfrak{z})),f_i^{m_i})\\
&= \sum_{i=1}^{q} m_i\,.
\end{aligned}$$

q.e.d.

B. Local continuity

In this section, it will be shown that almost every point in $N(0)$ has a system of neighborhoods such that, in any one of these neighborhoods, the area of $N(w)$ tends to the area of $N(0)$ modulo $\nu(\cdot,\tau\mid N)$ as w tends to zero.

Lemma 4.5. *Let* $(\mathfrak{a},0)\in(\dot{T}\times\{0\})\cap(N-S)$. *Let* $U^*\subseteq V\oplus\mathbf{C}$ *be an open neighborhood of* $(\mathfrak{a},0)$. *Let* θ *be a real valued* C^∞*-function on* H. *Then there exists an open neighborhood* $U\subset U^*\cap H$ *of* $(\mathfrak{a},0)$ *such that*

$$\int_{U\cap N(w)}\theta(\mathfrak{z},w)\,\nu((\mathfrak{z},w),\tau\mid N)\,\upsilon_p\to\int_{U\cap N(0)}\theta(\mathfrak{z},0)\,\nu((\mathfrak{z},0),\tau\mid N)\,\upsilon_p\quad\text{as}\quad w\to 0\,.$$

Proof. Let $\hat{N}$ be the normalization of N, and $\varrho:\hat{N}\to N$ the associated map. Let $\{a_1,\dots,a_q\}=\varrho^{-1}((\mathfrak{a},0))$. There exists a unique λ such that $\mathfrak{a}\in\dot{T}_\lambda$. As in the proof of Lemma 4.4, there exist pairwise disjoint neighborhoods $\hat{X}_1,\dots,\hat{X}_q$ of $\hat{a}_1,\dots,\hat{a}_q$ and analytic sets $Y_1',\dots,Y_q'$ in a neighborhood $\underset{\sim}{U}\subset U^*\cap$

$\cap N \subset H$ of $(\mathfrak{a}, 0)$ such that:

i) $\underset{\sim}{U} \cap E \subsetneqq \dot{T}_\lambda \times \{0\}$,

ii) $\varrho^{-1}(\underset{\sim}{U}) = \bigcup_{i=1}^{q} \hat{X}_i$,

iii) $\underset{\sim}{U} = \bigcup_{i=1}^{q} Y_i'$,

iv) $\varrho(\hat{X}_i) = Y_i'$ for each $i = 1, \ldots, q$,

v) there exist an open neighborhood X_i' of $0 \in \mathbf{C}^{p+1}$ and $\sigma_i': X_i' \to \hat{X}_i$ biholomorphic, $\sigma_i'(0) = a_i$, for each $i = 1, \ldots, q$.

For each $i = 1, \ldots, q$, it has been shown that 0 is a simple point of $A_i' = \{t \in X' \mid \tau \circ \varrho \circ \sigma_i'(t) = 0\}$. Hence there exist an open neighborhood X_i of $0 \in \mathbf{C}^{p+1}$ and a biholomorphic map $\sigma_i'': X_i \to \sigma_i''(X_i) \subset X_i'$ such that $\sigma''(X_i \cap \{x' \in X_i \mid x_{p+1} = 0\}) = A_i' \cap \sigma_i''(X_i)$, $\sigma_i''(0) = 0$, and $X_i \cap \{x' \mid x_{p+1} = 0\}$ is connected, where $x' = (x_1, \ldots, x_p, x_{p+1})$. Define

$$\sigma_i = \varrho \circ \sigma_i' \circ \sigma_i'': X_i \to \sigma(X_i) \subset Y_i'.$$

Then σ_i is holomorphic and topological, $\sigma_i(X_i)$ is open in Y_i', and $\sigma_i(0) = (\mathfrak{a}, 0)$. Let $(\mathfrak{v}_1, \ldots, \mathfrak{v}_n)$ be an orthonormal base of V and $\mathfrak{v}_{n+1} = (0, 1) \in V \oplus \mathbf{C}$. Then

$$\sigma_i(x') = \sum_{\nu=1}^{n+1} \sigma_\nu^{(i)}(x')\, \mathfrak{v}_\nu .$$

Let $\eta_i(w) = \{x' \in X_i \mid \sigma_{n+1}^{(i)}(x') = w\}$. Then $\sigma_i(\eta_i(w)) = N(w) \cap \sigma_i(X_i)$, and $\eta_i(0) = \{x' \in X_i \mid x_{p+1} = 0\}$. Now there exist an open neighborhood $R_i \subset X_i$ of 0 and g_i, a holomorphic function on R_i, such that

$$\sigma_{n+1}^{(i)}(x') = x_{p+1}^{m_i}\, g_i(x'), \quad x' \in R_i ,$$

with $g_i(x') \neq 0$ for $x' \in R_i$, and where

$$m_i = \nu(0, 0, \sigma_{n+1}^{(i)}) .$$

Choose $\gamma_i' > 0$, $\delta_i' > 0$ such that, if

$$Q_i = \left\{ (x_1, \ldots, x_p) \,\middle|\, \sum_{\nu=1}^{p} |x_\nu|^2 < (\gamma_i')^2 \right\}$$

$$Q_i' = Q_i \times \{x_{p+1} \mid |x_{p+1}| < \delta_i'\} ,$$

then

$$\overline{Q_i'} \subsetneqq R_i .$$

Hence there exists $0 < \delta_i'' \leqq \delta_i'$ such that

$$m_i g_i(x') + x_{p+1} \frac{\partial g_i}{\partial x_{p+1}}(x') \neq 0$$

for $x' \in Q_i \times \{x_{p+1} \mid |x_{p+1}| \leqq \delta_i''\}$. Now define $f_i: Q_i' \times \mathbf{C} \to \mathbf{C}$ by

$$f_i(x', w) = x_{p+1}^{m_i}\, g_i(x') - w .$$

Then $f_i(0, \ldots, 0, x_{p+1}, 0) = x_{p+1}^{m_i}\, g(0, \ldots, 0, x_{p+1}) \not\equiv 0$, and so there exists a Weierstrass polynomial

$$\omega_i(x_{p+1}, x, w) = x_{p+1}^{m_i} + \sum_{\nu=0}^{m_i - 1} a_{i,\nu}(x, w)\, x_{p+1}^{\nu}$$

where $x=(x_1,\ldots,x_p)$ and the $a_{i,\nu}$'s are functions holomorphic in neighborhood

$$\left\{(x_1,\ldots,x_p,w)\,\middle|\,\sum_{\nu=1}^{p}|x_\nu|^2<(\gamma_i'')^2,\quad |w|<\varepsilon_i'\right\}$$

of $(0,0)\in\mathbf{C}^p\oplus\mathbf{C}$ with $0<\gamma_i''<\gamma_i'$, $0<\varepsilon_i'$ and a function e_i holomorphic on

$$\left\{(x_1,\ldots,x_{p+1},w)\middle|\,\sum_{\nu=1}^{p}|x_\nu|^2<(\gamma_i'')^2,\quad |x_{p+1}|<\delta_i\,,\quad |w|<\varepsilon_i'\right\}=L_i\,,$$

with $0<\delta_i\leqq\delta_i''$, such that

$$f_i=e_i\omega_i\,,\quad e_i\neq 0\quad\text{on}\quad L_i\,.$$

For $x=(x_1,\ldots,x_p)$, define $|x|=\left(\sum_{\nu=1}^{p}|x_\nu|^2\right)^{1/2}$. Then there exist γ_i, ε_i in $0<\gamma_i<\gamma_i''$, $0<\varepsilon_i<\varepsilon_i'$, such that $\omega_i(x_{p+1},x,w)=0$, $|x|<\gamma_i$, $|w|<\varepsilon_i$ imply $|x_{p+1}|<\delta_i$. Define

$$P_i=\{x\,|\,|x|<\gamma_i\}$$
$$P_i'=P_i\times\{x_{p+1}\,|\,|x_{p+1}|<\delta_i\}\,.$$

Then

1. $\sigma_i\colon P_i'\to Y_i'$ is holomorphic, $\sigma_i\colon P_i'\to\sigma_i(P_i')$ is topological and $\sigma_i(P_i')$ is open in Y_i', $\sigma_i(0)=(\mathfrak{a},0)$,
2. $x'\in P_i'$ implies $m_i\,g_i(x')+x_{p+1}\dfrac{\partial g_i}{\partial x_{p+1}}(x')\neq 0$,
3. $x\in P_i$, $|w|<\varepsilon_i$, $x'=(x,x_{p+1})$, $\omega(x_{p+1},x,w)=0$ imply $x'\in P_i'$.

Recall that in the proof of Lemma 4.4 it was shown that $Y_i'\cap E=Y_j'\cap E$ for any $1\leqq i,\ j\leqq q$, where $E=V\times\{0\}$. Thus $D=\bigcap_{i=1}^{q}\sigma_i(P_i')\cap E$ is an open neighborhood in $N(0)$ of $\mathfrak{a}$, as $\sigma_i(P_i')$ is open in Y_i'. Take ξ such that if $\Omega=\{\mathfrak{a}+\mathfrak{z}\,|\,\mathfrak{z}\in V,\ |\mathfrak{z}|<\xi\}$, then $(\Omega\times\{0\})\cap N(0)\subseteqq\overline{(\Omega\times\{0\})}\cap N(0)\subset D$. Take $\zeta>0$, $\zeta\leqq\min_{i=1,\ldots,q}\varepsilon_i$ and such that

1. $(\Omega\times\{w\in\mathbf{C}\,|\,0\leqq|w|\leqq\zeta\})\cap N\subseteqq\bigcup_{i=1}^{q}Y_i'\subset\underset{\sim}{U}$,
2. $(\Omega\times\{w\in\mathbf{C}\,|\,0\leqq|w|\leqq\zeta\})\cap Y_i'\subseteqq\sigma_i(P_i')$, $\quad i=1,\ldots,q$.

There exists an open set $U\subset H$ such that $(\mathfrak{a},0)\in U\subset U^*$ and

$$(\Omega\times\{w\in\mathbf{C}\,|\,0\leqq|w|<\zeta\})\cap N=U\cap N\,.$$

Define $Y_i=U\cap Y_i'$, $i=1,\ldots,q$. Then $N\cap U=\bigcup_{i=1}^{q}Y_i$. From Lemma 4.3, $\nu((\mathfrak{z},w),\tau\,|\,N)=1$ for $(\mathfrak{z},w)\in\dot N(w)$, $w\neq 0$, and so $Y_i\cap Y_j\cap\dot N(w)=\Phi$ for any $i\neq j$, $1\leqq i,\ j\leqq q$, and $w\neq 0$. Now $N(0)\cap U=Y_i\cap N(0)$ for any $i=1,\ldots,q$, and for $(\mathfrak{z},0)\in N(0)\cap U$,

$$\begin{aligned}\nu((\mathfrak{z},0),\tau\,|\,N)&=\sum_{i=1}^{q}\nu(\sigma_i^{-1}(\mathfrak{z},0),\tau\,|\,N\circ\sigma_i)\\&=\sum_{i=1}^{q}\nu(\sigma_i^{-1}(\mathfrak{z},0),0,\sigma_{n+1}^{(i)})\\&=\sum_{i=1}^{q}m_i\,.\end{aligned}$$

Assume for the moment that

$$\int_{Y_i \cap N(w)} \theta(\mathfrak{z}, w)\, \mathfrak{v}_p \to m_i \int_{Y_i \cap N(0)} \theta(\mathfrak{z}, 0)\, \mathfrak{v}_p \quad \text{as} \quad w \to 0$$

for each $i = 1, \ldots, q$. Then, as $w \to 0$,

$$\begin{aligned}
&\int_{N(w) \cap U} \nu((\mathfrak{z}, w), \tau \mid N)\, \theta(\mathfrak{z}, w)\, \mathfrak{v}_p(\mathfrak{z}, w) \\
&= \sum_{i=1}^{q} \int^{*}_{Y_i \cap N(w)} \theta\, \mathfrak{v}_p \to \sum_{i=1}^{q} m_i \int_{Y_i \cap N(0)} \theta\, \mathfrak{v}_p \\
&= \sum_{i=1}^{q} m_i \int_{U \cap N(0)} \theta\, \mathfrak{v}_p \\
&= \int_{U \cap N(0)} \nu((\mathfrak{z}, 0), \tau \mid N)\, \theta(\mathfrak{z}, 0)\, \mathfrak{v}_p(\mathfrak{z}, 0)\,.
\end{aligned}$$

Thus all that remains is to prove that for any i,

$$1 \leqq i \leqq q, \quad \int_{Y_i \cap N(w)} \theta\, \mathfrak{v}_p \to m_i \int_{Y_i \cap N(0)} \theta\, \mathfrak{v}_p \quad \text{as} \quad w \to 0\,.$$

Let i be fixed, $1 \leqq i \leqq q$. The index i shall henceforth be omitted. Thus, for example, $\sigma = \sum_{\nu=1}^{n+1} \sigma_\nu \mathfrak{v}_\nu = \sum_{\nu=1}^{n+1} \sigma_\nu^{(i)} \mathfrak{v}_\nu$. Define, for $x \in P$,

$$\Lambda_0(x) = \theta(\sigma(x, 0)) \sum_{1 \leqq \nu_1 < \cdots < \nu_p \leqq n} \left| \frac{\partial(\sigma_{\nu_1}, \ldots, \sigma_{\nu_p})}{\partial(x_1, \ldots, x_p)} \right|^2_{(x,0)}.$$

Take w in $0 < |w| < \zeta$ and $x \in P$. Then

$$\omega(x_{p+1}, x, w) = \prod_{\mu=1}^{m} (x_{p+1} - x^{\mu}_{p+1}(x, w))$$

where $|x^{\mu}_{p+1}(x, w)| < \delta$, that is, $(x, x^{\mu}_{p+1}(x, w)) \in P'$. Hence

$$\begin{aligned}
\eta(w) \cap P' &= \{x' \in P' \mid \sigma_{n+1}(x') = w\} \\
&= \{(x, x^{\mu}_{p+1}(x, w)) \mid x \in P, 1 \leqq \mu \leqq m\}\,.
\end{aligned}$$

as $\omega(x_{p+1}, x, w)\; e(x', w) = \sigma_{n+1}(x') - w$, $e(x', w) \neq 0$. Now $\omega(x', w)\, e(x', w) = f(x', w) = x^{m}_{p+1} g(x') - w$, and $\dfrac{\partial f}{\partial x_{p+1}}(x', w) = x^{m-1}_{p+1}\left(m g(x') + x_{p+1} \dfrac{\partial g}{\partial x_{p+1}}(x')\right)$.

Let $z_\mu = (x, x^{\mu}_{p+1}(x, w), w)$. Then $w \neq 0$ implies $x^{\mu}_{p+1}(x, w) \neq 0$ for any $x \in P$. Thus $\dfrac{\partial f}{\partial x_{p+1}}(z_\mu) \neq 0$. But $\dfrac{\partial f}{\partial x_{p+1}}(z_\mu) = \omega(z_\mu) \dfrac{\partial e}{\partial x_{p+1}}(z_\mu) + e(z_\mu) \dfrac{\partial \omega}{\partial x_{p+1}}(z_\mu)$ $= e(z_\mu) \dfrac{\partial \omega}{\partial x_{p+1}}(z_\mu)\,.$

Hence $\dfrac{\partial \omega}{\partial x_{p+1}}(z_\mu) \neq 0$, and so the $x^{\mu}_{p+1}(x, w)$, $\mu = 1, \ldots, m$, are distinct for any $0 < |w| < \zeta$ and $x \in P$. Now, keep w in $0 < |w| < \zeta$ fixed. Then

$$\omega(x_{p+1}, x, w) = \prod_{\mu=1}^{m} (x_{p+1} - x^{\mu}_{p+1}(x, w))\,,$$

where $x^{\mu}_{p+1}(x,w) \neq x^{\nu}_{p+1}(x,w)$ if $\mu \neq \nu$ for all $x \in P$, and so $\dfrac{\partial \omega}{\partial x_{p+1}}(x_{p+1}, x, w) \neq 0$ for all $x_{p+1} = x^{\mu}_{p+1}(x,w)$ and $x \in P$. Hence

$$\omega(x_{p+1}, x, w) = \prod_{\mu=1}^{m} (x_{p+1} - h_\mu(x,w)),$$

where $h_\mu(x,w)$ is a well-defined, holomorphic function of $x \in P$, with $h_\mu(x,w) \neq h_\nu(x,w)$ if $\mu \neq \nu$. Define

$$\Lambda_w(x) = \sum_{\mu=1}^{m} \Big(\theta\big(\sigma(x, h_\mu(x,w))\big)\Big) \times$$
$$\times \left(\sum_{1 \leqq \nu_1 < \cdots < \nu_p \leqq n} \left| \frac{\partial\big(\sigma_{\nu_1}(x, h_\mu(x,w)), \ldots, \sigma_{\nu_p}(x, h_\mu(x,w))\big)}{\partial(x_1, \ldots, x_p)} \right|_x^2 \right).$$

It is now claimed that $\Lambda_w(x) \to m\Lambda_0(x)$ as $w \to 0$ uniformly on P. There exists a constant K such that $|g(x')| > K$ for all $x' \in \bar{P}'$. Take $\alpha > 0$. Define $d(\alpha) = \min(K\alpha^m, \zeta)$. Take w in $0 < |w| < d(\alpha)$. For any $x \in P$,

$$h^m_\mu(x,w)\, g(x, h_\mu(x,w)) - w = 0,$$

and so $|h_\mu(x,w)| < \left(\dfrac{d(\alpha)}{K}\right)^{1/m} \leqq \alpha$. A constant $\kappa > 0$ exists such that, for all $x' \in \bar{P}'$,

$$\left| \frac{\partial g}{\partial x_t}(x') \right| < \kappa, \quad t = 1, \ldots, p+1.$$

For w fixed, $0 < |w| < d\left(\dfrac{mK}{2\kappa}\right)$,

$$|h_\mu(x,w)| < mK/2\kappa, \quad x \in P.$$

And from $h^m_\mu(x,w)\, g(x, h_\mu(x,w)) - w = 0$,

$$0 = m h^{m-1}_\mu(x,w)\, g(x, h_\mu(x,w)) \frac{\partial h_\mu(x,w)}{\partial x_t} +$$
$$+ h^m_\mu(x,w) \left(\frac{\partial g}{\partial x_t}(x, h_\mu(x,w)) + \frac{\partial g}{\partial x_{p+1}}(x, h_\mu(x,w)) \frac{\partial h_\mu}{\partial x_t}(x,w) \right).$$

Since $h_\mu(x,w) \neq 0$,

$$0 = mg \frac{\partial h_\mu}{\partial x_t} + h_\mu \left(\frac{\partial g}{\partial x_t} + \frac{\partial g}{\partial x_{p+1}} \frac{\partial h_\mu}{\partial x_t} \right),$$
$$\frac{\partial h_\mu}{\partial x_t} \left(mg + h_\mu \frac{\partial g}{\partial x_{p+1}} \right) = -h_\mu \frac{\partial g}{\partial x_t}.$$

Now

$$\left| mg + h_\mu \frac{\partial g}{\partial x_{p+1}} \right| \geqq |mg| - \left| h_\mu \frac{\partial g}{\partial x_{p+1}} \right| \geqq mK - \frac{mK}{2\kappa} \cdot \kappa = \frac{mK}{2},$$

and so

$$\left| \frac{\partial h_\mu}{\partial x_t} \right| \leqq \frac{2}{mK} \left| \frac{\partial g}{\partial x_t} \right| |h_\mu| \leqq \frac{2\kappa}{mK} |h_\mu|$$

for $t=1,\dots,p$, $\mu=1,\dots,m$. Thus define $d_1(\alpha)=\min\left(d(\alpha), d\left(\frac{mK}{2\kappa}\right), d\left(\frac{mK}{2\kappa}\alpha\right)\right)$.
Then, for $x\in P$, $0<|w|<d_1(\alpha)$, $t=1,\dots,p$, $\mu=1,\dots,m$, it is

$$|h_\mu(x,w)|<\alpha \quad \text{and} \quad \left|\frac{\partial h_\mu}{\partial x_t}(x,w)\right|<\alpha .$$

Now there exists a constant c_0 such that

$$\left|\frac{\partial\sigma_\nu}{\partial x_t}(x')\right|<c_0 \quad \text{for} \quad x'\in\overline{P'}, \quad \nu=1,\dots,n, \qquad t=1,\dots,p+1 .$$

And for any $\alpha>0$, there exists $\Delta_0(\alpha)$ such that for all

$$1\leqq\nu\leqq n, \quad 1\leqq t\leqq p+1,$$

$$\left|\frac{\partial\sigma_\nu}{\partial x_t}(x,x_{p+1})-\frac{\partial\sigma_\nu}{\partial x_t}(x,0)\right|<\alpha$$

if $x\in P$ and $|x_{p+1}|\leqq\Delta_0(\alpha)$. Also, there exists a constant c_1 such that

$$|\theta(\sigma(x'))|<c_1 \quad \text{for all} \quad x'\in P',$$

and for any $\alpha>0$, there exists $\Delta_1(\alpha)$ such that

$$|\theta(\sigma(x,x_{p+1}))-\theta(\sigma(x,0))|<\alpha \quad \text{for} \quad x\in\overline{P} \quad \text{and} \quad |x_{p+1}|\leqq\Delta_1(\alpha).$$

For every $\beta>0$, there exists $\Delta(\beta)>0$ such that, if

$$A=\begin{pmatrix}a_{11}\dots a_{1p}\\ a_{p1}\dots a_{pp}\end{pmatrix}, \quad B=\begin{pmatrix}b_{11}\dots b_{1p}\\ b_{p1}\dots b_{pp}\end{pmatrix}$$

with $|a_{ij}|\leqq 2c_0, |b_{ij}|\leqq 2c_0, |a_{ij}-b_{ij}|\leqq\Delta(\beta)$ for $1\leqq i,j\leqq p$, then

$$\big||\det A|^2-|\det B|^2\big|<\beta .$$

Moreover there exists a constant c_2 such that

$$|\det A|^2<c_2 \quad \text{if} \quad |a_{ij}|<2c_0 .$$

Now take any $\beta>0$. Take $\alpha=\min\left(1,\frac{\Delta(\beta)}{1+c_0}\right)$. Take $d_2(\beta)=\min(d_1(\alpha), d_1(\Delta_0(\alpha)), d_1(\Delta_1(\alpha)))$. Take any w in $0<|w|<d_2(\beta)$ and any $x\in P$. Take μ in $1\leqq\mu\leqq m$. Then

$$|h_\mu(x,w)|\leqq\min(\alpha,\Delta_0(\alpha), \quad \Delta_1(\alpha))$$

and

$$\left|\frac{\partial h_\mu}{\partial x_t}(x,w)\right|\leqq\min(\alpha,\Delta_0(\alpha),\Delta_1(\alpha)), \quad t=1,\dots,p .$$

And for $1\leqq\nu\leqq n$, $1\leqq t\leqq p$,

$$\left|\frac{\partial\sigma_\nu}{\partial x_t}(x,h_\mu(x,w))-\frac{\partial\sigma_\nu}{\partial x_t}(x,0)\right|<\alpha .$$

Hence

$$\left|\frac{\partial}{\partial x_t}(\sigma_\nu(x, h_\mu(x, w))) - \frac{\partial \sigma_\nu}{\partial x_t}(x, 0)\right|$$

$$= \left|\frac{\partial \sigma_\nu}{\partial x_t}(x, h_\mu(x, w)) - \frac{\partial \sigma_\nu}{\partial x_t}(x, 0) + \right.$$

$$\left. + \frac{\partial \sigma_\nu}{\partial x_{p+1}}(x, h_\mu(x, w)) \frac{\partial h_\mu}{\partial x_t}(x, w)\right| \leqq$$

$$\leqq \alpha + c_0 \alpha = \alpha(1+c) \leqq \Delta(\beta).$$

For $1 \leqq \nu_1 < \cdots < \nu_p \leqq n$, define

$$A^\mu_{w, \nu_1, \ldots, \nu_p}(x) = \frac{\partial(\sigma_{\nu_1}(x, h_\mu(x, w)), \ldots, \sigma_{\nu_p}(x, h_\mu(x, w))}{\partial(x_1, \ldots, x_p)}$$

$$A_{\nu_1, \ldots, \nu_p}(x) = \frac{\partial(\sigma_{\nu_1}(x, 0), \ldots, \sigma_{\nu_p}(x, 0))}{\partial(x_1, \ldots, x_p)}$$

Then $\left||A^\mu_{w, \nu_1, \ldots, \nu_p}(x)|^2 - |A_{\nu_1, \ldots, \nu_p}(x)|^2\right| < \beta$, and $|A_{\nu_1, \ldots, \nu_p}(x)|^2 \leqq c_2$. Now $|\theta(\sigma(x, h_\mu(x, w))) - \theta(\sigma(x, 0))| < \beta$. Hence

$$|\Lambda_w(x) - m\Lambda_0(x)|$$

$$= \left|\sum_{\mu=1}^m \left\{\theta(\sigma(x, h_\mu(x, w))) \sum_{1 \leqq \nu_1 < \cdots < \nu_p \leqq n} |A^\mu_{w, \nu_1, \ldots, \nu_p}(x)|^2\right\} - \right.$$

$$\left. - \sum_{\mu=1}^m \left\{\theta(\sigma(x, 0)) \sum_{1 \leqq \nu_1 < \cdots < \nu_p \leqq n} |A_{\nu_1, \ldots, \nu_p}(x)|^2\right\}\right|$$

$$\leqq \sum_{\mu=1}^m |\theta(\sigma(x, h_\mu(x, w)))| \sum_{1 \leqq \nu_1 < \cdots < \nu_p \leqq n} \left||A^\mu_{w, \nu_1, \ldots, \nu_p}(x)|^2 - |A_{\nu_1, \ldots, \nu_p}(x)|^2\right| +$$

$$+ \sum_{\mu=1}^m |\theta(\sigma(x, h_\mu(x, w))) - \theta(\sigma(x, 0))| \sum_{1 \leqq \nu_1 < \cdots < \nu_p \leqq n} |A_{\nu_1, \ldots, \nu_p}(x)|^2 \leqq$$

$$\leqq m c_1 n^p \beta + m c_2 n^p \beta = c_3 \beta$$

where $c_3 = m(c_1 + c_2)n^p$ is independent of β, x, w. Thus $\Lambda_w(x) \to m\Lambda_0(x)$ as $w \to 0$ uniformly on P.

Now let W be any open set in P. Define $W' = W \times \{x_{p+1} \mid |x_{p+1}| < \delta\}$. Then $\sigma: W \times \{0\} \to \sigma(W') \cap N(0)$ is topological and holomorphic, and so

$$\int_{\sigma(W') \cap N(0)} \theta(\mathfrak{z}, 0)\, \upsilon_p = \int_W \theta(\sigma(x, 0)) \left(\frac{i}{2}\right)^p \times$$

$$\times \sum_{1 \leqq \nu_1 < \cdots < \nu_p \leqq n+1} \left|\frac{\partial(\sigma_{\nu_1}, \ldots, \sigma_{\nu_p})}{\partial(x_1, \ldots, x_p)}\right|^2 dx_1 \wedge d\bar{x}_1 \wedge \cdots \wedge dx_p \wedge d\bar{x}_p$$

$$= \int_W \theta(\sigma(x, 0)) \left(\frac{i}{2}\right)^p \times$$

$$\times \sum_{1 \leqq \nu_1 < \cdots < \nu_p \leqq n} \left|\frac{\partial(\sigma_{\nu_1}(x, 0), \ldots, \sigma_{\nu_p}(x, 0))}{\partial(x_1, \ldots, x_p)}\right|^2 dx_1 \wedge d\bar{x}_1 \wedge \cdots \wedge dx_p \wedge d\bar{x}_p$$

$$= \int_W \Lambda_0(x) \left(\frac{i}{2}\right)^p dx_1 \wedge d\bar{x}_1 \wedge \cdots \wedge dx_p \wedge d\bar{x}_p.$$

Take w fixed, $0<|w|<\zeta$. Then

$$\sigma(\eta(w)\cap W')=\sigma(W')\cap N(w).$$

Let $\iota_w:\eta(w)\cap W'\to P'$ be the inclusion. Then $\sigma\circ\iota_w:\eta(w)\cap W'\to\sigma(W')\cap N(w)$ is topological and holomorphic, and so

$$\int\limits_{\sigma(W')\cap N(w)} \theta(\mathfrak{z},w)\,\upsilon_p = \int\limits_{\eta(w)\cap W'} \theta(\sigma(x'))\left(\frac{i}{2}\right)^p \sum_{1\leqq\nu_1<\cdots<\nu_p\leqq n} d\sigma_{\nu_1}\wedge d\overline{\sigma_{\nu_1}}\wedge\cdots\wedge d\sigma_{\nu_p}\wedge d\overline{\sigma_{\nu_p}}.$$

Define $h'_\mu: W\to h'_\mu(W)\subset\eta(w)\cap W'$ by $h'_\mu(x)=(x,h_\mu(x,w))$ for $\mu=1,\ldots,m$. Then h'_μ is biholomorphic, and

$$\eta(w)\cap W'=\bigcup_{\mu=1}^{m} h'_\mu(W),\quad h'_\mu(W)\cap h'_\nu(W)=\Phi,\quad \mu\neq\nu.$$

Thus

$$\begin{aligned}\int\limits_{\sigma(W')\cap N(w)} \theta(\mathfrak{z},w)\,\upsilon_p &= \sum_{\mu=1}^{m}\int\limits_W \theta(\sigma(x,h_\mu(x,w)))\times\\ &\times\sum_{1\leqq\nu_1<\cdots<\nu_p\leqq n}\left|\frac{\partial(\sigma_{\nu_1}(x,h_\mu(x,w)),\ldots,\sigma_{\nu_p}(x,h_\mu(x,w)))}{\partial(x_1,\ldots,x_p)}\right|^2\times\\ &\times\left(\frac{i}{2}\right)^p dx_1\wedge d\overline{x}_1\wedge\cdots\wedge dx_p\wedge d\overline{x}_p\\ &=\int\limits_W \Lambda_w(x)\left(\frac{i}{2}\right)^p dx_1\wedge d\overline{x}_1\wedge\cdots\wedge dx_p\wedge d\overline{x}_p.\end{aligned}$$

Hence

$$\int\limits_{\sigma(W')\cap N(w)}\theta\,\upsilon_p\to m\int\limits_{\sigma(W')\cap N(0)}\theta\,\upsilon_p,\quad w\to 0.$$

Define

$$\psi:\mathbf{C}^{p+1}\to\mathbf{C}^p,\quad \psi(x_1,\ldots,x_{p+1})=(x_1,\ldots,x_p)$$
$$W_0=\psi(\sigma^{-1}(Y\cap E))\subset\overline{W}_0\subset P.$$

Take any open set $W\subset P$ such that $W\subset\overline{W}\subset W_0$. Define as before $W'=W\times\{x_{p+1}\in\mathbf{C}\,|\,|x_{p+1}|<\delta\}$. It shall be shown that there exists $\alpha>0$ such that for $|w|<\alpha$, $\sigma(W')\cap N(w)\subset Y\cap N(w)$. For assume that there exists a sequence $\{(\mathfrak{z}_\nu,w_\nu)\}$ such that $w_\nu\to 0$ as $\nu\to\infty$ and $(\mathfrak{z}_\nu,w_\nu)\in\sigma(W')\cap N(w_\nu)$, $(\mathfrak{z}_\nu,w_\nu)\notin Y\cap N(w_\nu)$. Then $\{\sigma^{-1}(\mathfrak{z}_\nu,w_\nu)\}\subset\overline{W}'$, and so there exists a convergent subsequence, which will also be denoted by $\{\sigma^{-1}(\mathfrak{z}_\nu,w_\nu)\}$. Let $\sigma^{-1}(\mathfrak{z}_\nu,w_\nu)\to\to(x,x_{p+1})\subset\overline{W}'$ as $\nu\to\infty$, where $\psi(x,x_{p+1})=x$. Then $w_\nu\to 0$ implies $x_{p+1}=0$. Now $(x,0)\in\overline{W}'$, and so $x\in\overline{W}\subset W_0$. Therefore $(x,0)\in\sigma^{-1}(Y)$ open, and so, for ν large enough, $\sigma^{-1}(\mathfrak{z}_\nu,w_\nu)\in\sigma^{-1}(Y)$, that is, $(\mathfrak{z}_\nu,w_\nu)\in Y$, a contradiction.

Hence there exists $\alpha>0$ such that for $|w|<\alpha$, $\sigma(W')\cap N(w)\subset Y\cap N(w)$. Thus

$$\int_{\sigma(W')\cap N(w)} \theta\, \upsilon_p \leqq \int_{Y\cap N(w)} \theta\, \upsilon_p, \quad |w|<\alpha\,.$$

Now

$$\int_{\sigma(W')\cap N(w)} \theta\, \upsilon_p \to m \int_{\sigma(W')\cap N(0)} \theta\, \upsilon_p \quad \text{as} \quad w\to 0\,,$$

and

$$\int_{\sigma(W')\cap N(0)} \theta\, \upsilon_p = \int_W \Lambda_0(x) \left(\frac{i}{2}\right)^p dx_1\wedge d\bar{x}_1\wedge\cdots\wedge dx_p\wedge d\bar{x}_p\,.$$

Thus for any open set $W\subset\bar{W}\subset W_0$,

$$m\int_W \Lambda_0(x)\left(\frac{i}{2}\right)^p dx_1\wedge d\bar{x}_1\wedge\cdots\wedge dx_p\wedge d\bar{x}_p \leqq \liminf_{w\to 0} \int_{Y\cap N(w)} \theta\, \upsilon_p\,.$$

Therefore,

$$m\int_{Y\cap N(0)} \theta\, \upsilon_p = m\int_{W_0} \Lambda_0(x)\left(\frac{i}{2}\right)^p dx_1\wedge d\bar{x}_1\wedge\cdots\wedge dx_p\wedge d\bar{x}_p \leqq \liminf_{w\to 0}\int_{Y\cap N(w)} \theta\, \upsilon_p.$$

Now define, for $0<s<\zeta$,

$$F(s)=V\times\{w\in\mathbf{C}\mid |w|<s\}\,,$$
$$W(s)=\psi(\sigma^{-1}(Y\cap F(s)))\,,$$
$$W'(s)=W(s)\times\{x_{p+1}\mid |x_{p+1}|<\delta\}\,.$$

Then $W(s)$ is open in P, and

$$Y\cap F(s)\subset\sigma(W'(s))\cap F(s)$$

as

$$\sigma^{-1}(Y\cap F(s))\subset W'(s)\,.$$

Therefore, for $|w|<s$,

$$\int_{Y\cap N(w)} \theta\, \upsilon_p \leqq \int_{\sigma(W'(s))\cap N(w)} \theta\, \upsilon_p\,.$$

But as $w\to 0$,

$$\int_{\sigma(W'(s))\cap N(w)} \theta\, \upsilon_p \to m\int_{\sigma(W'(s))\cap N(0)} \theta\, \upsilon_p$$

$$= m\int_{W(s)} \Lambda_0(x)\left(\frac{i}{2}\right)^p dx_1\wedge d\bar{x}_1\wedge\cdots\wedge dx_p\wedge d\bar{x}_p\,.$$

Hence, for any $0<s<\zeta$,

$$\limsup_{w\to 0}\int_{Y\cap N(w)} \theta\, \upsilon_p \leqq m\int_{W(s)} \Lambda_0(x)\left(\frac{i}{2}\right)^p dx_1\wedge d\bar{x}_1\wedge\cdots\wedge dx_p\wedge d\bar{x}_p\,.$$

Now if $0<s'<s$, then $W(s')\subset W(s)$, and

$$\bigcap_{0<s<\zeta} W(s)=W_0 .$$

Thus

$$\limsup_{w\to 0} \int_{Y\cap N(w)} \theta\, \upsilon_p \leqq m \int_{W_0} \Lambda_0(x) \left(\frac{i}{2}\right)^p dx_1\wedge d\bar{x}_1\wedge\cdots\wedge dx_p\wedge d\bar{x}_p = m \int_{Y\cap N(0)} \theta\, \upsilon_p.$$

Consequently,

$$m \int_{Y\cap N(0)} \theta\, \upsilon_p \leqq \liminf_{w\to 0} \int_{Y\cap N(w)} \theta\, \upsilon_p \leqq \limsup_{w\to 0} \int_{Y\cap N(w)} \theta\, \upsilon_p \leqq m \int_{Y\cap N(0)} \theta\, \upsilon_p ,$$

and so

$$\lim_{w\to 0} \int_{Y\cap N(w)} \theta\, \upsilon_p = m \int_{Y\cap N(0)} \theta\, \upsilon_p . \qquad \text{q.e.d.}$$

C. Local boundedness

In this section it will be shown that for every point of $N(0)$, there exists a neighborhood such that for any ball in this neighborhood, the product of $\nu(\cdot,\tau|N)$ and the area of $N(w)$ intersect the ball is bounded by a constant times the radius of the ball to the power $2p$, the constant independent of w for $|w|$ sufficiently small. This result essentially has been proven by W. STOLL in § 2 of [9]. However in [9], the normalization of a complex space is not considered when the multiplicity of a holomorphic map is defined. Thus the two definitions of multiplicity must be related. Here the symbol $\tilde{\nu}$ will be used to denote the multiplicity of a map in the sense of [9]. The definition of $\tilde{\nu}$, along with the definitions of a distinguished base and a distinguished polycylinder, will be given here for the convenience of the reader.

Let X and Y be complex spaces and $\sigma: X\to Y$ a holomorphic, non-degenerate map. Take $a\in X$. Take any open neighborhood U of a such that U is compact and such that $\bar{U}\cap\sigma^{-1}(\sigma(a))=\{a\}$. Define

$$\tilde{\nu}(a,\sigma)=\limsup_{x\to a} \mu_U(x,\sigma)$$

where $\mu_U(x,\sigma)$ is as defined in § 4 A.

Now let $\sigma: X\to Y$ be a holomorphic map such that $\sigma^{-1}(\sigma(x))$ is an analytic set of pure dimension q for every $x\in X$. Suppose that X has pure dimension k. Take $a\in X$ and let Γ_a be as in § 4 A. Define

$$\tilde{\nu}(a,\sigma)=\operatorname*{Min}_{A\in\Gamma_a} \tilde{\nu}(a,\sigma\mid A) .$$

Thus $\tilde{\nu}$ is defined.

Let D be an open subset of an m-dimensional complex vector space W. Let $\mathfrak{a}$ be a point of an analytic subset A of D. A base $C=(\mathfrak{c}_1,\ldots,\mathfrak{c}_m)$ of W is said to be *distinguished with respect to* $(A,\mathfrak{a},k)$ if and only if the intersection

$F \cap A$ of A with $F = \left\{\mathfrak{a} + \sum_{\nu=k+1}^{m} z_\nu \mathfrak{c}_\nu\right\}$ contains $\mathfrak{a}$ as an isolated point. And U is said to be a *distinguished polycylinder with respect to* $(A, C, \mathfrak{a}, k)$ if and only if

1. It is $1 \leqq k < m$.
2. Numbers $\varepsilon_\nu > 0$ exist such that

$$U = \left\{\mathfrak{a} + \sum_{\nu=1}^{m} z_\nu \mathfrak{c}_\nu \,\middle|\, |z_\nu| < \varepsilon_\nu \quad \text{for} \quad \nu = 1, \ldots, m\right\} \subseteqq \bar{U} \subseteqq D .$$

3. Define

$$Y = \left\{\mathfrak{a} + \sum_{\nu=1}^{k} z_\nu \mathfrak{c}_\nu \,\middle|\, |z_\nu| < \varepsilon_\nu \quad \text{for} \quad \nu = 1, \ldots, k\right\}$$

and $\sigma: U \to Y$ the projection given by

$$\sigma\left(\mathfrak{a} + \sum_{\nu=1}^{m} z_\nu \mathfrak{c}_\nu\right) = \mathfrak{a} + \sum_{\nu=1}^{k} z_\nu \mathfrak{c}_\nu .$$

Define

$$X_\mathfrak{y} = \sigma^{-1}(\mathfrak{y}) = \left\{\mathfrak{y} + \sum_{\nu=k+1}^{m} z_\nu \mathfrak{c}_\nu \,\middle|\, |z_\nu| < \varepsilon_\nu \quad \text{for} \quad \nu = k+1, \ldots, m\right\} \quad \text{for} \quad \mathfrak{y} \in Y .$$

Then

$$A \cap \bar{X}_\mathfrak{y} = A \cap X_\mathfrak{y} \quad \text{for all} \quad y \in Y$$

and

$$A \cap \bar{X}_\mathfrak{a} = \{\mathfrak{a}\}$$

is required.

Lemma 4.6. *Let* $\mathfrak{a} \in N(0)$. *Let* $\hat{N}$ *be the normalization of* N *and* $\varrho: \hat{N} \to N$ *the associated map. Let* $\{a_1, \ldots, a_q\} = \varrho^{-1}(\mathfrak{a})$. *Let* $\hat{X}_1, \ldots, \hat{X}_q$ *be pairwise disjoint neighborhoods of* $a_1, \ldots, a_q$ *and* $X_1, \ldots, X_q$ *analytic sets in a neighborhood* $X \subset N$ *of* $\mathfrak{a}$, *such that* $\varrho^{-1}(X) = \bigcup_{i=1}^{q} \hat{X}_i$, $X = \bigcup_{i=1}^{q} X_i$, *and* $\varrho(\hat{X}_i) = X_i$ *for each* $i = 1, \ldots, q$, *and such that* $X \subset K \cap N$ *for some compact set* $K \subset V \oplus \mathbf{C}$. *Let* $C = (\mathfrak{c}_1, \ldots, \mathfrak{c}_n)$ *be a base of* V, *and let* $\mathfrak{c} = (0, 1) \in V \oplus \mathbf{C}$. *Let* $C' = (\mathfrak{c}_1, \ldots, \mathfrak{c}_p, \mathfrak{c}, \mathfrak{c}_{p+1}, \ldots, \mathfrak{c}_n)$, *a base of* $V \oplus \mathbf{C}$. *Suppose that*

$$U = \left\{\mathfrak{a} + \sum_{\nu=1}^{n} z_\nu \mathfrak{c}_\nu + w\mathfrak{c} \,\middle|\, |z_\nu| < \varepsilon_\nu, \quad \nu = 1, \ldots, n, |w| < \varepsilon_{n+1}\right\}$$

is a distinguished polycylinder with respect to $(N, C', \mathfrak{a}, p+1)$ *and to* $(N(0), C, \mathfrak{a}, p)$. *Suppose* $U \cap N \subset X$. *Suppose that* η *in* $0 < \eta < 1$ *exists such that* $N(0) \cap \overline{U - U_\eta} = \Phi$, *where*

$$U_\eta = \left\{\mathfrak{a} + \sum_{\nu=1}^{n} z_\nu \mathfrak{c}_\nu + w\mathfrak{c} \,\middle|\, |z_\nu| < \varepsilon_\nu, \ \nu = 1, \ldots, p, \ |w| < \eta\varepsilon_{n+1}, \ |z_\nu| < \eta\varepsilon_\nu, \ \nu = p+1, \ldots, n\right\}.$$

Define $\tilde{\pi}: U \to \tilde{\pi}(U) = Y'$ *by*

$$\tilde{\pi}\left(\mathfrak{a} + \sum_{\nu=1}^{n} z_\nu \mathfrak{c}_\nu + w\mathfrak{c}\right) = \mathfrak{a} + \sum_{\nu=1}^{p} z_\nu \mathfrak{c}_\nu .$$

For $\mathfrak{y} \in Y'$, *define*

$$L(\mathfrak{y}, w) = U \cap N(w) \cap \tilde{\pi}^{-1}(\mathfrak{y}).$$

Then there exist constants $\delta > 0, \kappa > 0$ *such that*

$$\sum_{(\mathfrak{z}, w) \in L(\mathfrak{y}, w)} \nu((\mathfrak{z}, w), \tau \mid N) < \kappa \quad \textit{for} \quad |w| < \delta.$$

Proof. Define $L_i(\mathfrak{y}, w) = L(\mathfrak{y}, w) \cap X_i$ for $i = 1, \ldots, q$. Now $\tau | X_i$ is not constant on any irreducible branch of X_i, that is, no $N(w) \cap X_i$ contains an irreducible branch of X_i. Hence there exist constants κ_i and δ_i such that if $|w| < \delta_i$, then

$$\sum_{(\mathfrak{z}, w) \in L(\mathfrak{y}, w)} \tilde{\nu}((\mathfrak{z}, w), \tau | X_i) < \kappa_i \quad \text{for each} \quad i = 1, \ldots, q.$$

The proof of this is contained in the proof of Lemma 2.6 of [9]. Compare

There	V	$\mathfrak{C}$	M	G	k	$\mathfrak{a}$	U	n	f
Here	$V \oplus \mathbf{C}$	C'	X_i	H	$p+1$	$\mathfrak{a}$	U	$n+1$	τ

η	U_η	$\tilde{Y}$	$N(w)$	$L(\mathfrak{v}, w)$
η	U_η	Y'	$N(w) \cap X_i$	$L_i(\mathfrak{y}, w)$

Define $\varrho_i = \varrho | \hat{X}_i : \hat{X}_i \to X_i$, $i = 1, \ldots, q$. There exists a constant l such that $\# \varrho^{-1}(x) < l$ for all $x \in X$. It will be shown that $\nu(\hat{z}, \tau \circ \varrho_i) < l \tilde{\nu}((z, w), \tau | X_i)$ for any $\hat{z} \in \hat{X}_i$ such that $\varrho_i(\hat{z}) = (\mathfrak{z}, w)$.

Let i be fixed. Take $b \in \hat{X}_i$. It is claimed first that $\nu(b, \tau \circ \varrho_i) \leqq \tilde{\nu}(b, \tau \circ \varrho_i)$. Take any $A \in \Gamma_b$. Then A is a pure 1-dimensional analytic set in a neighborhood of b. Let $\{A_1, \ldots, A_t\}$ be representatives in a neighborhood of b of the irreducible components of the germ of A at b. Then $A_1 \in \Gamma_b$ and $\tilde{\nu}(b, \tau \circ \varrho_i | A_1) \leqq \tilde{\nu}(b, \tau \circ \varrho_i | A)$. Let $\hat{A}_1$ be the normalization of A_1 and $\hat{\varrho}$ the associated map. Now $\hat{A}_1$ is pure 1-dimensional, and so, consists only of simple points. Hence, A_1 irreducible at b implies $\hat{\varrho}: \hat{A}_1 \to A_1$ is topological in a neighborhood $\hat{Z} \subset \hat{A}_1$ of $\hat{b} = \hat{\varrho}^{-1}(b)$. Choose an open neighborhood $\hat{D}$ of $\hat{b}$ such that the closure of $\hat{D}$ is compact and contained in $\hat{Z}$, and such that $\bar{\hat{D}} \cap (\tau \circ \varrho_i \circ \hat{\varrho})^{-1} (\tau \circ \varrho_i \circ \hat{\varrho}(\hat{b})) = \{\hat{b}\}$. Let $D = \hat{\varrho}(\hat{D})$. Then $D \subset A_1$ is an open neighborhood in A_1 of b, $\bar{D}$ is compact, and $\bar{D} \cap (\tau \circ \varrho_i | A_1)^{-1} \; (\tau \circ \varrho_i | A_1(b)) = \{b\}$. Since $\hat{\varrho}$ is topological on $\hat{D}$, for any $\hat{z} \in \hat{D}$ with $\hat{\varrho}(\hat{z}) = z$, $\# \hat{D} \cap (\tau \circ \varrho_i \circ \hat{\varrho})^{-1} \; (\tau \circ \varrho_i \circ \hat{\varrho}(\hat{z})) \; = \; \# D \cap (\tau \circ \varrho_i | A_1)^{-1} \circ \circ (\tau \circ \varrho_i) (z)$. Hence $\tilde{\nu}(\hat{b}, \tau \circ \varrho_i \circ \hat{\varrho}) = \tilde{\nu}(b, \tau \circ \varrho_i | A_1)$. Since $\hat{A}_1$ is a normal, pure 1-dimensional analytic space,

$$\nu(\hat{b}, \tau \circ \varrho_i \circ \hat{\varrho}) = \tilde{\nu}(\hat{b}, \tau \circ \varrho_i \circ \hat{\varrho}).$$

Since $\hat{\varrho}^{-1}(b) = \hat{b}$,

$$\nu(b, \tau \circ \varrho_i | A_1) = \nu(\hat{b}, \tau \circ \varrho_i \circ \hat{\varrho}).$$

Since $A_1 \in \Gamma_b$,

$$\nu(b, \tau \circ \varrho_i) \leqq \tilde{\nu}(b, \tau \circ \varrho_i | A_1).$$

Hence $\nu(b, \tau \circ \varrho_i) \leqq \tilde{\nu}(b, \tau \circ \varrho_i | A)$ for any $A \in \Gamma_b$. Therefore

$$\nu(b, \tau \circ \varrho_i) \leqq \tilde{\nu}(b, \tau \circ \varrho_i).$$

Now let $\varrho_i(b) = \mathfrak{b} \in X_i$. It is claimed that $\tilde{\nu}(b, \tau \circ \varrho_i) < l\,\tilde{\nu}(\mathfrak{b}, \tau | X_i)$. Take $B \in \Gamma_\mathfrak{b}$, considering $\mathfrak{b}$ as a point in the analytic space X_i. Then there exists an open neighborhood $U_B \subset X_i$ of $\mathfrak{b}$ such that $\mathfrak{b} \in B \subset U_B$, B is a pure 1-dimensional analytic set in U_B, $\overline{U}_B$ is compact, and $\tau | B$ is non-degenerate. Let $\varrho_i^{-1}(\mathfrak{b}) = \{b_1, \ldots, b_t\}$, with $b_1 = b$. There exist pairwise disjoint neighborhoods $Y_1, \ldots, Y_t$ in X_i of $b_1, \ldots, b_t$ such that $\varrho_i(Y_j) \subset U_B, j = 1, \ldots, t$. Let $\hat{B} = \varrho_i^{-1}(B) \cap Y_1$, $U_{\hat{B}} = \varrho_i^{-1}(U_B) \cap Y_1$. Then $U_{\hat{B}}$ is an open neighborhood of b, and $\hat{B}$ is a pure 1-dimensional analytic set in $U_{\hat{B}}$. And $\overline{U}_{\hat{B}} \subseteqq \varrho_i^{-1}(\overline{U}_B) \cap \overline{Y}_1$ is compact as ϱ_i is proper. And $\tau | B$ non-degenerate implies $\tau \circ \varrho_i | \hat{B}$ non-degenerate as the fibers $\varrho_i^{-1}(x)$ consist of isolated points for $x \in X_i$. Thus $\hat{B} \in \Gamma_b$. Take now $W \subset B$, an open neighborhood in B of $\mathfrak{b}$ such that $\overline{W}$ is compact, $\mathfrak{b} \in W \subset \overline{W} \subset B$, and $\overline{W} \cap \{(\tau | B)^{-1}(\tau(\mathfrak{b}))\} = \{\mathfrak{b}\}$. Then

$$\tilde{\nu}(\mathfrak{b}, \tau | B) = \limsup_{\mathfrak{x} \to \mathfrak{b}, \mathfrak{x} \in B} \# W \cap (\tau | B)^{-1}(\tau(\mathfrak{x})).$$

Define $\hat{W} = \varrho_i^{-1}(W) \cap \hat{B}$. Then $\hat{W}$ is an open neighborhood in $\hat{B}$ of b, $\overline{\hat{W}}$ is compact, and $\overline{\hat{W}} \cap (\tau \circ \varrho_i | \hat{B})^{-1}(\tau \circ \varrho_i(b)) = \{b\}$. Thus

$$\tilde{\nu}(b, \tau \circ \varrho_i | \hat{B}) = \limsup_{z \to b, z \in \hat{B}} \# \hat{W} \cap (\tau \circ \varrho_i | \hat{B})^{-1}(\tau \circ \varrho_i(z)).$$

But

$$\# \hat{W} \cap (\tau \circ \varrho_i | \hat{B})^{-1}(\tau \circ \varrho_i(z)) < l \# W \cap (\tau | B)^{-1}(\tau \circ \varrho_i(z))$$

for all $z \in \hat{B}$. Thus

$$\tilde{\nu}(b, \tau \circ \varrho_i | \hat{B}) < l\,\tilde{\nu}(\mathfrak{b}, \tau | B).$$

Choose $B \in \Gamma_\mathfrak{b}$ such that $\tilde{\nu}(\mathfrak{b}, \tau | X_i) = \tilde{\nu}(\mathfrak{b}, \tau | B)$. The existence of $\hat{B} \in \Gamma_b$ such that

$$\tilde{\nu}(b, \tau \circ \varrho_i | \hat{B}) < l\,\tilde{\nu}(b, \tau | B)$$

implies

$$\tilde{\nu}(b, \tau \circ \varrho_i) < l\,\tilde{\nu}(\mathfrak{b}, \tau | X_i).$$

Combining these two results,

$$\nu(b, \tau \circ \varrho_i) < l\,\tilde{\nu}(\mathfrak{b}, \tau | X_i).$$

Consequently, for w such that $|w| < \delta = \operatorname{Min}_{i=1,\ldots,q} \delta_i$,

$$\begin{aligned}
&\sum_{(\mathfrak{z}, w) \in L(\mathfrak{y}, w)} \nu((\mathfrak{z}, w), \tau | N) \\
&= \sum_{(\mathfrak{z}, w) \in L(\mathfrak{y}, w)} \ \sum_{\hat{z} \in \varrho^{-1}(\mathfrak{z}, w)} \nu(\hat{z}, \tau \circ \varrho) \\
&= \sum_{i=1}^{q} \ \sum_{(\mathfrak{z}, w) \in L_i(\mathfrak{y}, w)} \ \sum_{\hat{z} \in \varrho_i^{-1}(\mathfrak{z}, w)} (\nu(\hat{z}, \tau \circ \varrho_i) < \\
&< \sum_{i=1}^{q} \ \sum_{(\mathfrak{z}, w) \in L_i(\mathfrak{y}, w)} \ \sum_{\hat{z} \in \varrho_i^{-1}(\mathfrak{z}, w)} l\,\tilde{\nu}((\mathfrak{z}, w), \tau | X_i) < \\
&< \sum_{i=1}^{q} \ \sum_{(\mathfrak{z}, w) \in L_i(\mathfrak{y}, w)} l^2\,\tilde{\nu}((\mathfrak{z}, w), \tau | X_i) < \\
&< l^2 \sum_{i=1}^{q} \kappa_i = \kappa.
\end{aligned}$$

q.e.d.

Lemma 4.7. *Let* $\mathfrak{a} \in N(0)$. *For* $d>0$, *define* $B'_d(\mathfrak{a}) = \{(\mathfrak{z}, w) \mid |\mathfrak{z}-\mathfrak{a}|^2 + |w|^2 < d^2\}$. *Then there exist constants* $d>0$, $\kappa>0$, $\delta>0$ *such that for every* $\gamma>0$ *and for any ball* B' *of radius* γ *with* $B' \subset B'_d(a)$,

$$\int_{B' \cap N(w)} \nu((\mathfrak{z}, w), \tau \mid N)\, v_p < \kappa \gamma^{2p}$$

for all w *with* $|w| < \delta$.

Proof. Let $\hat{N}$ be the normalization, $\varrho: \hat{N} \to N$ the normalization map, and $\{a_1, \ldots, a_q\} = \varrho^{-1}(\mathfrak{a})$. Then there exist pairwise disjoint neighborhoods $\hat{X}_1, \ldots, \hat{X}_q$ of $a_1, \ldots, a_q$ and analytic sets $X_1, \ldots, X_q$ in a neighborhood $X \subset N$ of $\mathfrak{a}$ such that $\varrho^{-1}(X) = \bigcup_{i=1}^{q} \hat{X}_i$, $X = \bigcup_{i=1}^{q} X_i$, $\varrho(\hat{X}_i) = X_i$ for each $i = 1, \ldots, q$, and $X \subset K \cap N$ where K is a compact set in $V \oplus \mathbf{C}$. And it will be proven in the appendix of this paper that there exists a basis $C = (\mathfrak{c}_1, \ldots, \mathfrak{c}_n)$ of V such that $C_\mu = (\mathfrak{c}_{\mu(1)}, \ldots, \mathfrak{c}_{\mu(n)})$ is distinguished with respect to $(T, \mathfrak{a}, p)$ for each permutation μ of $\{1, \ldots, n\}$. Define $\mathfrak{c} = (0, 1) \in V \oplus \mathbf{C}$. Define $C'_\mu = (\mathfrak{c}_{\mu(1)}, \ldots, \mathfrak{c}_{\mu(p)}, \mathfrak{c}, \mathfrak{c}_{\mu(p+1)}, \ldots, \mathfrak{c}_{\mu(n)})$, a basis of $V \oplus \mathbf{C}$. Identify $V = V \times \{0\}$. Then $\mathfrak{a}$ is an isolated point of

$$T \cap \left\{\mathfrak{a} + \sum_{\nu=p+1}^{n} z_\nu \mathfrak{c}_{\mu(\nu)} \mid z_\nu \in \mathbf{C}\right\}$$

implies that $\mathfrak{a}$ is an isolated point of

$$N(0) \cap \left\{\mathfrak{a} + \sum_{\nu=p+1}^{n} z_\nu \mathfrak{c}_{\mu(\nu)} + w\, \mathfrak{c} \mid z_\nu \in \mathbf{C},\ w \in \mathbf{C}\right\}$$

and

$$N \cap \left\{\mathfrak{a} + \sum_{\nu=p+1}^{n} z_\nu \mathfrak{c}_{\mu(\nu)} \mid z_\nu \in \mathbf{C}\right\}.$$

Hence C'_μ is distinguished with respect to $(N, \mathfrak{a}, p+1)$ and with respect to $(N(0), \mathfrak{a}, p)$. Hence a polycylinder U_μ distinguished with respect to $(N, C'_\mu, \mathfrak{a}, p+1)$ and $(N(0), C'_\mu, \mathfrak{a}, p)$ exists such that $U_\mu \cap N \subset X$. It can be chosen such that η in $0 < \eta < 1$ exists such that if

$$U_{\mu,\eta} = \left\{\mathfrak{a} + \sum_{\nu=1}^{n} z_{\mu(\nu)} \mathfrak{c}_{\mu(\nu)} + w\mathfrak{c} \;\Big|\; |z_{\mu(\nu)}| < \varepsilon_\nu^{(\mu)},\ \nu = 1, \ldots, p;\ |z_{\mu(\nu)}| < \eta \varepsilon_\nu^{(\mu)},\right.$$
$$\left. \nu = p+1, \ldots, n;\ |w| < \eta\, \varepsilon_{n+1}^{(\mu)}\right\},$$

then $\overline{U_\mu - U_{\mu,\eta}} \cap N(0) = \Phi$. Define

$$\tilde{\pi}_\mu\left(\mathfrak{a} + \sum_{\nu=1}^{n} z_\nu \mathfrak{c}_\nu + w\mathfrak{c}\right) = \mathfrak{a} + \sum_{\nu=1}^{p} z_{\mu(\nu)} \mathfrak{c}_{\mu(\nu)},$$

$$\tilde{\pi}_\mu(U_\mu) = Y'_\mu,$$

$$L_\mu(\mathfrak{y}, w) = U_\mu \cap N(w) \cap \tilde{\pi}_\mu^{-1}(\mathfrak{y}) \quad \text{for} \quad \mathfrak{y} \in Y'_\mu.$$

According to Lemma 4.6, $\kappa_\mu > 0$ and $\delta_\mu > 0$ exist such that

$$\sum_{(\mathfrak{z}, w) \in L_\mu(\mathfrak{y}, w)} \nu((\mathfrak{z}, w), \tau \mid N) < \kappa_\mu$$

if $|w| < \delta_\mu$ and $\mathfrak{y} \in Y'_\mu$. Define

$$\kappa' = \operatorname{Max}\{\kappa_\mu \mid \mu \text{ is a permutation of } \{1, \ldots, n\}\},$$
$$\delta = \operatorname{Min}\{\delta_\mu \mid \mu \text{ is a permutation of } \{1, \ldots, n\}\}.$$

Take $d > 0$ such that $\overline{B'_d(\mathfrak{a})} \subset \bigcap_{(\mu)} U_\mu$. Define on $V \oplus \mathbf{C}$, for $\mathfrak{z} = \sum_{\nu=1}^{n} z_\nu \mathfrak{c}_\nu$,

$$\chi(\mathfrak{z}) = \frac{i}{2} \sum_{\nu=1}^{n} dz_\nu \wedge d\bar{z}_\nu$$

$$\chi_p = \frac{1}{p!} \chi^p.$$

A constant $l > 0$ exists such that

$$\iota_w^* \upsilon_p \leqq l\, \iota_w^* \chi_p$$

on $\overline{B'_d(\mathfrak{a})} \cap N(w)$, where $\iota_w : N(w) \to V \oplus \mathbf{C}$ is the inclusion map for each w. Take $\gamma > 0$ and let

$$B' = \{(\mathfrak{z}, w) \mid |\mathfrak{z} - \mathfrak{b}|^2 + |w - b|^2 < \gamma^2\} \subset B'_d(\mathfrak{a}).$$

Take w in $|w| < \delta$. Then

$$J(w) = \int_{B' \cap N(w)} \nu((\mathfrak{z}, w), \tau \mid N)\, \upsilon_p \leqq l \int_{B' \cap N(w)} \nu((\mathfrak{z}, w), \tau \mid N)\, \iota_w^*(\chi_p)$$

$$= l \sum_{1 \leqq \nu_1 < \cdots < \nu_p \leqq n} \int_{B' \cap N(w)} \nu((\mathfrak{z}, w), \tau \mid N) \left(\frac{i}{2}\right)^p dz_{\nu_1} \wedge d\bar{z}_{\nu_1} \wedge \cdots \wedge dz_{\nu_p} \wedge d\bar{z}_{\nu_p}$$

$$= l \sum_{1 \leqq \nu_1 < \cdots < \nu_p \leqq n} \int_{\tilde{\pi}_\mu(B' \cap N(w))} \sum_{(\mathfrak{z}, w) \in L_\mu(\mathfrak{y}, w) \cap B} \nu((\mathfrak{z}, w), \tau \mid N) \left(\frac{i}{2}\right)^p \times$$
$$\times\, dz_{\nu_1} \wedge d\bar{z}_{\nu_1} \wedge \cdots \wedge dz_{\nu_p} \wedge d\bar{z}_{\nu_p}$$

where the permutation is defined uniquely with respect to the $\nu_1, \ldots, \nu_p$ by requiring that

$$\mu(1) = \nu_1, \ldots, \mu(p) = \nu_p, \quad \mu(p+1) < \cdots < \mu(n).$$

Now define

$$\langle \mathfrak{z} \mid \mathfrak{z}' \rangle = \sum_{\nu=1}^{n} z_\nu \bar{z}'_\nu \quad \text{for} \quad \mathfrak{z} = \sum_{\nu=1}^{n} z_\nu \mathfrak{c}_\nu, \quad \mathfrak{z}' = \sum_{\nu=1}^{n} z'_\nu \mathfrak{c}_\nu.$$

Then $\|\mathfrak{z}\| = [\langle \mathfrak{z} \mid \mathfrak{z} \rangle]^{1/2}$ is another norm on V. A constant $A > 0$ exists such that

$$A|\mathfrak{z}| \leqq \|\mathfrak{z}\| \leqq A^{-1}|\mathfrak{z}| \quad \text{for all} \quad \mathfrak{z} \in V.$$

Define $B'' = \{(\mathfrak{z}, w) \mid \|\mathfrak{z} - \mathfrak{b}\| < \gamma/A,\ |w - b| < \gamma\}$. If $(\mathfrak{z}, w) \in B'$, then $|\mathfrak{z} - \mathfrak{b}| < \gamma$ and $|w - b| < \gamma$. Hence $A\|\mathfrak{z} - \mathfrak{b}\| < \gamma$, and so $(\mathfrak{z}, w) \in B''$. Thus

$$\tilde{\pi}_\mu(B') \subseteqq \tilde{\pi}_\mu(B'' \cap U_\mu) \subseteqq$$
$$\subseteqq \left\{ \mathfrak{e}_\mu + \sum_{\nu=1}^{p} z_\nu \mathfrak{c}_{\mu(\nu)} \,\middle|\, \sum_{\nu=1}^{p} |z_\nu|^2 \leqq \left(\frac{\gamma}{A}\right)^2 \right\}$$

with $\mathfrak{e}_\mu = \sum_{\nu=1}^{p} b_{\mu(\nu)} \mathfrak{c}_{\mu(\nu)} + \sum_{\nu=p+1}^{n} a_{\mu(\nu)} \mathfrak{c}_{\mu(\nu)}$ where

$$\mathfrak{a} = \sum_{\nu=1}^{n} a_\nu \mathfrak{c}_\nu + 0\mathfrak{c}, \quad \mathfrak{b} = \sum_{\nu=1}^{n} b_\nu \mathfrak{c}_\nu + 0\mathfrak{c}.$$

Hence

$$J(w) \leqq l \sum_{1 \leqq \nu_1 < \cdots < \nu_p \leqq n} \int_{\tilde{\pi}_\mu(B')} \sum_{(\mathfrak{z}, w) \in L_\mu(\mathfrak{y}, w)} \nu((\mathfrak{z}, w), \tau \mid N) \left(\frac{i}{2}\right)^p \times$$

$$\times\, dz_{\nu_1} \wedge d\bar{z}_{\nu_1} \wedge \cdots \wedge dz_{\nu_p} \wedge d\bar{z}_{\nu_p} \leqq$$

$$\leqq l\kappa' n! \frac{\pi^{2p}}{p!} \left(\frac{\gamma}{A}\right)^{2p} = \kappa \gamma^{2p}$$

if $|w| < \delta$, where

$$\kappa = l\kappa' \frac{n!}{p!} \left(\frac{\pi}{A}\right)^{2p}$$

is independent of γ. q.e.d.

D. The limit of $I(w, r)$

In this section, the two local results of sections 4 B and 4 C are used to compute $\lim_{w \to 0} \int_{\pi(N(w)) \cap B_r} \nu((\mathfrak{z}, w), \tau \mid N)\, \upsilon_p$. This limit along with the results of § 4 A will yield $\lim_{w \to 0} \int_{\pi(N(w)) \cap B_r} \upsilon_p$.

Recall $\pi: V \oplus \mathbf{C} \to V$, the projection

$$B_r = \{\mathfrak{z} \in V \mid |\mathfrak{z}| < r\}$$

$$\pi(N(w)) \cap B_r = \{\mathfrak{z} \mid (\mathfrak{z}, w) \in N(w), \ \ \mathfrak{z} \in B_r\}$$

$$I(w, r) = \int_{\pi(N(w) \cap B_r} \upsilon_p$$

$$\pi(N(0)) = T.$$

And $S = \varrho(\hat{S})$, where $\hat{S}$ was the set of singular points of the normalization $\hat{N}$ of N and $\varrho: \hat{N} \to N$ the normalization map. Define

$$Q = [\bar{B}_r \cap (T - \dot{T})] \cup [\bar{B}_r \cap \pi(S \cap N(0))] \cup [(\bar{B}_r - B_r) \cap T].$$

The s-dimensional Hausdorff outer measure in $\mathbf{R}^m$ is needed. Let $L \subset \mathbf{R}^m$. Define $\Omega_k = \{B(t) \mid B(t)$ a ball of radius $t < 1/k\}$, $d^s(B(t)) = W_s' t^s$, $W_s' =$ the volume of the unit ball in $\mathbf{R}^s$,

$$\Omega_k(L) = \left\{\{B_i\}_{i \in \mathbf{N}} \mid B_i \in \Omega_k, \ \bigcup_{i=1}^{\infty} B_i \supset L\right\}$$

$$\lambda_k(L) = \inf \left\{\sum_{i=1}^{\infty} d^s(B_i) \mid \{B_i\}_{i \in \mathbf{N}} \in \Omega_k(L)\right\}$$

$$\mu_s(L) = \lim_{k \to \infty} \lambda_k(L).$$

This limit exists, and is called the *s-dimensional Hausdorff outer measure* of L. Note that $\mu_s(L)=0$ implies that for $\varepsilon>0$, there exists $k_0(\varepsilon)$ such that $\lambda_k(L)\leqq \leqq W_s'\,\varepsilon/2$ for $k>k_0(\varepsilon)$. Hence for any $k>k_0$, there exists $\{B_i\}_{i\in\mathbf{N}}\in\Omega_k(L)$ such that, if the ball B_i is of radius $t_i<1/k$, then

$$\sum_{i=1}^{\infty} d^s(B_i)=\sum_{i=1}^{\infty} W_s'\,t_i^s<\varepsilon\, W_s',$$

that is,

$$\bigcup_{i=1}^{\infty} B_i\supseteqq L \quad\text{and}\quad \sum_{i=1}^{\infty} t_i^s<\varepsilon\,.$$

Identify $V=\mathbf{R}^{2n}$. Now the sets $T-\dot{T}$ and $\pi(S\cap N(0))$ lie thin and analytic in V, and so they may be expressed as the finite union of manifolds, each manifold of dimension less than or equal $2p-2$. Hence $\mu_{2p}(\overline{B}_r\cap(T-\dot{T}))=0 =\mu_{2p}(\overline{B}_r\cap\pi(S\cap N(0)))$ (see for example HUREWICZ and WALLMAN, [2]). Also, if A is a real analytic set in an open set of $\mathbf{R}^m$, and if A is without interior points, then A is a set of measure zero. This can be easily shown by induction on m with the use of Fubini's Theorem. Now $\dot{T}\cap(\overline{B}_r-B_r)$ is a real analytic set in $\dot{T}$. Suppose that $\mathfrak{a}$ is an interior point of $\dot{T}\cap(\overline{B}_r-B_r)$ with respect to $\dot{T}$. Then there exists an orthogonal coordinate system $(\mathfrak{v}_1,\ldots,\mathfrak{v}_n)$ of V and a biholomorphic map

$$\gamma: U\to\dot{T}$$

of an open set $U\subset\mathbf{C}^p$ such that

$$\mathfrak{a}\in\gamma(U)\subset(\overline{B}_r-B_r)\cap\dot{T},$$

$$\gamma(z_1,\ldots,z_p)=\sum_{\nu=1}^{p} z_\nu\mathfrak{v}_\nu+\sum_{\nu=p+1}^{n} f_\nu(z)\,\mathfrak{v}_\nu,$$

where $z=(z_1,\ldots,z_p)$ and $f_{p+1},\ldots,f_n$ are holomorphic on U. Then for $z\in U$,

$$r^2=|\gamma(z)|^2=\sum_{\nu=1}^{p}|z_\nu|^2+\sum_{\nu=p+1}^{n}|f_\nu(z)|^2\,.$$

For any λ, $1\leqq\lambda\leqq p$,

$$0=\frac{\partial}{\partial z_\lambda}|\gamma(z)|^2=\bar{z}_\lambda+\sum_{\nu=p+1}^{n}\frac{\partial f_\nu(z)}{\partial z_\lambda}\overline{f_\nu(z)},$$

$$0=\frac{\partial}{\partial\bar{z}_\lambda}\frac{\partial}{\partial z_\lambda}|\gamma(z)|^2=1+\sum_{\nu=p+1}^{n}\left|\frac{\partial f_\nu(z)}{\partial z_\lambda}\right|^2\geqq 1,$$

a contradiction. Thus $\dot{T}\cap(\overline{B}_r-B_r)$ is without interior points in $\dot{T}$, and so has measure zero in $\dot{T}$. Since T is the union of $\dot{T}$ and a finite number of manifolds of dimension less than $2p$, it follows that $\mu_{2p}(T\cap(\overline{B}_r-B_r))=0$. Thus $\mu_{2p}(Q)=0$.

Lemma 4.8. *Given any $\varepsilon>0$, then $\delta=\delta(\varepsilon)>0$ and an open set $W=W(\varepsilon)\subset H$ exist such that $Q\times\{0\}\subset W$ and*

$$\int_{N(w)\cap W} \nu((\mathfrak{z},w),\tau\,|\,N)\,\upsilon_p<\varepsilon\quad\text{if}\quad |w|<\delta\,.$$

Proof. Take $\mathfrak{a} \in Q$. Then, according to Lemma 4.7, $d_\mathfrak{a} > 0$, $\delta_\mathfrak{a} > 0$, $\kappa_\mathfrak{a}$ exist such that if

$$B'_{d_\mathfrak{a}}(\mathfrak{a}) = \{(\mathfrak{z}, w) \mid |\mathfrak{z} - \mathfrak{a}|^2 + |w|^2 < d_\mathfrak{a}^2\},$$

and if $B' \subset B'_{d_\mathfrak{a}}(\mathfrak{a})$ is a ball of radius γ, then

$$\int_{B' \cap N(w)} v((\mathfrak{z}, w), \tau | N)\, \upsilon_p < \kappa_\mathfrak{a} \gamma^{2p}$$

for all w with $|w| < \delta_\mathfrak{a}$. Then $Q \times \{0\} \subseteqq \bigcup_{\mathfrak{a} \in Q} B'_{\frac{1}{2} d_\mathfrak{a}}(\mathfrak{a})$, and so $\mathfrak{a}_1, \ldots, \mathfrak{a}_q$ in Q exist such that

$$Q \times \{0\} \subseteqq \bigcup_{j=1}^{q} B'_{\frac{1}{2} d_j}(\mathfrak{a}_j), \quad \text{where} \quad d_j = d_{\mathfrak{a}_j}.$$

Define $d_{q+1} > 0$ to be the distance between $\bar{H} - H$ and $Q \times \{0\}$, and

$$d = \operatorname*{Min}_{j=1,\ldots,q,q+1} d_j, \quad \delta = \operatorname*{Min}_{j=1,\ldots,q} \delta_{\mathfrak{a}_j},$$
$$\kappa = \operatorname*{Max}_{j=1,\ldots,q} \kappa_{\mathfrak{a}_j}.$$

Let B' be any ball of radius $\gamma < d/4$ and $B' \cap (Q \times \{0\}) \neq \Phi$. Then $(\mathfrak{b}, 0) \in B' \cap \cap B'_{\frac{1}{2} d_j}(\mathfrak{a}_j)$ for some index j exists. Take $(\mathfrak{z}, w) \in B'$. Then

$$[|\mathfrak{z} - \mathfrak{a}_j|^2 + |w|^2]^{1/2} = |(\mathfrak{z}, w) - (\mathfrak{a}_j, 0)| \leqq |(\mathfrak{z}, w) - (\mathfrak{b}, 0)| + |(\mathfrak{b}, 0) - (\mathfrak{a}_j, 0)| \leqq$$
$$\leqq 2\gamma + \frac{1}{2} d_j < d_j.$$

Hence $\overline{B'} \subseteqq B'_{d_j}(\mathfrak{a}_j)$, and so, for all $|w| < \delta$,

$$\int_{B' \cap N(w)} v((\mathfrak{z}, w), \tau | N)\, \upsilon_p < \kappa \gamma^{2p}.$$

Now $\mu_{2p}(Q \times \{0\}) = 0$ in $\mathbf{R}^{2n+2}$. Thus there exists $\{B'_i\}_{i \in \mathbf{N}}$ such that $B'_i \subset H$ is an open ball of radius $\gamma_i < d/4$, and such that

$$W = \bigcup_{i=1}^{\infty} B'_i \supset Q \times \{0\}, \quad \sum_{i=1}^{\infty} \gamma_i^{2p} < \frac{\varepsilon}{\kappa}.$$

It can be assumed that $B'_i \cap (Q \times \{0\}) \neq \Phi$, $i \in \mathbf{N}$. Hence $\int_{B'_i \cap N(w)} v((\mathfrak{z}, w), \tau | N) \times \times \upsilon_p < \kappa \gamma_i^{2p}$ for $|w| < \delta$. Hence

$$\int_{W \cap N(w)} v((\mathfrak{z}, w), \tau | N)\, \upsilon_p < \varepsilon$$

for $|w| < \delta$, where $W \subset H$ is an open neighborhood of $Q \times \{0\}$. q.e.d.

Lemma 4.9.

$$\int_{\pi(N(w) \cap B_r} v((\mathfrak{z}, w), \tau | N)\, \upsilon_p \to \int_{T \cap B_r} ((\mathfrak{z}, 0), \tau | N)\, \upsilon_p$$

as $w \to 0$.

Proof. Take $\varepsilon > 0$. From Lemma 4.8, there exist $W = W(\varepsilon)$ open, $\delta_1 = \delta_1(\varepsilon) > 0$ such that $Q \times \{0\} \subseteqq W \subseteqq H$ and, for $|w| < \delta_1$,

$$\int_{N(w) \cap W} v((\mathfrak{z}, w), \tau | N)\, \upsilon_p < \frac{\varepsilon}{3}.$$

Now $T \cap (\bar{B}_r - B_r)$ is compact and contained in $Q \subset W$ open. Hence there exist $0 < r' < r < r''$, $\delta_2 > 0$ such that, for

$$L = (B_{r''} - \bar{B}_{r'}) \times \{w \mid |w| < \delta_2, \ w \in \mathbf{C}\},$$

it is $N \cap \bar{L} \subset W$ and $\bar{L} \subset H$. Define $K = \bar{B}_{r'} - \pi(W \cap E)$, where $E = V \times \{0\}$. Then K is compact, $K \subset B_r$, and $K \cap Q = \Phi$. Take $(\mathfrak{a}, 0) \in (K \times \{0\}) \cap N(0)$. Then $\mathfrak{a} \notin Q$, and so $(\mathfrak{a}, 0) \in (\dot{T} \times \{0\}) \cap (N - S)$. From Lemma 4.5, there exist $U_\mathfrak{a}$ open, $\mathfrak{a} \in U_\mathfrak{a} \subset \bar{U}_\mathfrak{a} \subset H$, $\bar{U}_\mathfrak{a}$ compact with $\pi(\bar{U}_\mathfrak{a}) \subset B_r$, such that for every C^∞-function θ on H,

$$(1) \qquad \int_{U_\mathfrak{a} \cap N(w)} \theta(\mathfrak{z}, w)\, v((\mathfrak{z}, w), \tau \mid N)\, \upsilon_p \to \int_{U_\mathfrak{a} \cap N(0)} \theta(\mathfrak{z}, 0)\, v((\mathfrak{z}, 0), \tau \mid N)\, \upsilon_p$$

as $w \to 0$. Define $\delta_\mathfrak{a} = 1$. Now if $\mathfrak{a} \in K$ and $(\mathfrak{a}, 0) \notin N(0)$, then a $\delta_\mathfrak{a} > 0$ and an open neighborhood $U_\mathfrak{a}$ of $(\mathfrak{a}, 0)$ with $\bar{U}_\mathfrak{a}$ compact and $\bar{U}_\mathfrak{a} \subset H$ exist such that $N(w) \cap U_\mathfrak{a} = \Phi$ if $|w| < \delta_\mathfrak{a}$. Then for any C^∞-function θ on H, (1) holds also for this $U_\mathfrak{a}$. Because $K \times \{0\} \subseteqq \bigcup_{\mathfrak{a} \in K} U_\mathfrak{a}$, $\mathfrak{a}_1, \ldots, \mathfrak{a}_q$ in K exist such that $K \times \{0\} \subseteqq \bigcup_{i=1}^{q} U_{\mathfrak{a}_i}$. Define

$$\delta_3 = \operatorname*{Min}_{i=1,\ldots,q} \delta_{\mathfrak{a}_i},$$

$$U = \bigcup_{i=1}^{q} U_{\mathfrak{a}_i} \supseteqq K \times \{0\}.$$

Since $L \cup W \cup U$ contains

$$[(\bar{B}_r - \bar{B}_{r'}) \times \{w \mid |w| < \delta_2\}] \cup [W \cap E] \cup [(\bar{B}_{r'} \times \{0\}) - (W \cap E)]$$

which contains $\bar{B}_r \times \{0\}$, and since $L \cup W \cup U$ is open and $\bar{B}_r \times \{0\}$ is compact, $\delta_4 > 0$ exists such that $0 < \delta_4 < \delta_3$, $0 < \delta_4 < \delta_2$, and $P = \bar{B}_r \times \{w \mid |w| < \delta_4\} \subseteqq \subseteqq L \cup W \cup U$. Then, for $|w| < \delta_4 < \delta_2$,

$$N(w) \cap L = N \cap L \cap N(w) \subseteqq W \cap N(w),$$

and so

$$(\bar{B}_r \times \{w\}) \cap N(w) \subseteqq W \cup U, \quad |w| < \delta_4.$$

Now $P \cap N \subseteqq (U \cup W) \cap N \subset U \cup W$, and so the compact set $P \cap N \subseteqq W \cup \bigcup_{i=1}^{q} U_{\mathfrak{a}_i}$. Hence a partition of unity $\{\theta_i\}_{i=0,\ldots,q}$ to this covering of $P \cap N$ exists such that

1. θ_i is of class C^∞ on H, $0 \leqq \theta_i \leqq 1$, for $i = 0, \ldots, q$.
2. $\theta_i(\mathfrak{z}, w) = 0$ if $(\mathfrak{z}, w) \in H - U_{\mathfrak{a}_i}$ for $i = 1, \ldots, q$.
3. $\theta_0(\mathfrak{z}, w) = 0$ if $(\mathfrak{z}, w) \in H - W$.
4. $0 \leqq \sum_{i=0}^{q} \theta_i(\mathfrak{z}, w) \leqq 1$ if $(\mathfrak{z}, w) \in H$.
5. $\sum_{i=0}^{q} \theta_i(\mathfrak{z}, w) = 1$ if $(\mathfrak{z}, w) \in P \cap N$.

Define $\theta(\mathfrak{z}, w) = \sum_{i=1}^{q} \theta_i(\mathfrak{z}, w)$. If $|w| < \delta_4$, then

$$\begin{aligned}\int_{N(w)\cap U} \theta(\mathfrak{z}, w)\, \nu((\mathfrak{z}, w), \tau | N)\, \upsilon_p &= \sum_{i=1}^{q} \int_{N(w)\cap U} \theta_i(\mathfrak{z}, w)\, \nu((\mathfrak{z}, w), \tau | N)\, \upsilon_p \\ &= \sum_{i=1}^{q} \int_{N(w)\cap U_{a_i}} \theta_i(\mathfrak{z}, w)\, \nu((\mathfrak{z}, w), \tau | N)\, \upsilon_p \to \\ &\to \sum_{i=1}^{q} \int_{N(0)\cap U_{a_i}} \theta_i(\mathfrak{z}, w)\, \nu((\mathfrak{z}, 0), \tau | N)\, \upsilon_p \\ &= \sum_{i=1}^{q} \int_{N(0)\cap U} \theta_i(\mathfrak{z}, 0)\, \nu((\mathfrak{z}, 0), \tau | N)\, \upsilon_p \\ &= \int_{N(0)\cap U} \theta(\mathfrak{z}, 0)\, \nu((\mathfrak{z}, 0), \tau | N)\, \upsilon_p \quad \text{as} \quad w \to 0 .\end{aligned}$$

Hence $\delta_5 > 0$ exists such that $0 < \delta_5 < \delta_4$, $0 < \delta_5 < \delta_1$, and

$$\left| \int_{N(w)\cap U} \theta(\mathfrak{z}, w)\, \nu((\mathfrak{z}, w), \tau | N)\, \upsilon_p - \int_{N(0)\cap U} \theta(\mathfrak{z}, 0)\, \nu((\mathfrak{z}, 0), \tau | N)\, \upsilon_p \right| < \frac{\varepsilon}{3} \quad \text{for all } w \text{ with } |w| < \delta_5 .$$

Now

$$N(w) \cap (\bar{B}_r \times \{w\}) = (N(w) \cap U) \cup (N(w) \cap (\bar{B}_r \times \{w\}) - U)$$

for any $w \in \mathbf{C}$, as $\pi(\bar{U}_{a_i}) \subset B_r$ for each i. But if $(\mathfrak{z}, w) \in N(w) \cap (\bar{B}_r \times \{w\}) - U$, then $\theta(\mathfrak{z}, w) = 0$. Thus, if $|w| < \delta_5$, then

$$\left| \int_{N(w)\cap(\bar{B}_r \times \{w\})} \theta(\mathfrak{z}, w)\, \nu((\mathfrak{z}, w), \tau | N)\, \upsilon_p - \int_{N(0)\cap(\bar{B}_r \times \{0\})} \theta(\mathfrak{z}, 0)\, \nu((\mathfrak{z}, 0), \tau | N)\, \upsilon_p \right| < \frac{\varepsilon}{3} .$$

And

$$\begin{aligned} 0 \leqq \int_{N(w)\cap(\bar{B}_r \times \{w\})} \theta_0(\mathfrak{z}, w)\, \nu((\mathfrak{z}, w), \tau | N)\, \upsilon_p &\leqq \\ \leqq \int_{N(w)\cap W} \theta_0(\mathfrak{z}, w)\, \nu((\mathfrak{z}, w), \tau | N)\, \upsilon_p &\leqq \\ \leqq \int_{N(w)\cap W} \nu((\mathfrak{z}, w), \tau | N)\, \upsilon_p &< \frac{\varepsilon}{3} \end{aligned}$$

if $|w| < \delta_5 < \delta_1$. Now $\theta_0(\mathfrak{z}, w) + \theta(\mathfrak{z}, w) = 1$ for $(\mathfrak{z}, w) \in N(w) \cap \bar{B}_r \times \{w\}$ and $|w| < \delta_5$. Consequently,

$$\left| \int_{\pi(N(w)) \cap B_r} \nu((\mathfrak{z}, w), \tau | N)\, \upsilon_p - \int_{T \cap B_r} \nu((\mathfrak{z}, 0), \tau | N)\, \upsilon_p \right|$$

$$= \left| \int_{N(w) \cap (\bar{B}_r \times \{w\})} \nu((\mathfrak{z}, w), \tau | N)\, \upsilon_p - \int_{N(0) \cap (\bar{B}_r \times \{0\})} \nu((\mathfrak{z}, w), \tau | N)\, \upsilon_p \right| \leqq$$

$$\leqq \left| \int_{N(w) \cap (\bar{B}_r \times \{w\})} \theta(\mathfrak{z}, w)\, \nu((\mathfrak{z}, w), \tau | N)\, \upsilon_p - \int_{N(0) \cap (\bar{B}_r \times \{0\})} \theta(\mathfrak{z}, 0)\, \nu((\mathfrak{z}, 0), \tau | N)\, \upsilon_p \right| +$$

$$+ \left| \int_{N(w) \cap (\bar{B}_r \times \{w\})} \theta_0(\mathfrak{z}, w)\, \nu((\mathfrak{z}, w), \tau | N)\, \upsilon_p \right| +$$

$$+ \left| \int_{N(0) \cap (\bar{B}_r \times \{0\})} \theta_0(\mathfrak{z}, 0)\, \nu((\mathfrak{z}, 0), \tau | N)\, \upsilon_p \right| <$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \text{if} \quad |w| < \delta_5 . \qquad \text{q.e.d.}$$

Let $\{T_1, \ldots, T_b\}$ be the irreducible branches of T. From Lemma 4.4, for each $\lambda = 1, \ldots, b$, there exists a constant $m_\lambda \in \mathbf{N}$ such that

$$\nu((\mathfrak{z}, 0), \tau | N) = m_\lambda \quad \text{if} \quad \mathfrak{z} \in \dot{T} \cap T_\lambda \cap \pi(N - S),$$

which is almost everywhere on T_λ. Thus

$$\int_{T \cap B_r} \nu((\mathfrak{z}, 0), \tau | N)\, \upsilon_p = \sum_{\lambda=1}^{b} \int_{T_\lambda \cap B_r} \nu((\mathfrak{z}, 0), \tau | N)\, \upsilon_p$$

$$= \sum_{\lambda=1}^{b} m_\lambda \int_{T_\lambda \cap B_r} \upsilon_p .$$

And, from Lemma 4.3, $\nu((\mathfrak{z}, w), \tau | N) = 1$ if $(\mathfrak{z}, w) \in \dot{N}(w)$ and $w \neq 0$. Thus

$$I(w, r) = \int_{\pi(N(w)) \cap B_r} \upsilon_p = \int_{\pi(N(w)) \cap B_r} \nu((\mathfrak{z}, w), \tau | N)\, \upsilon_p .$$

Hence Lemma 4.9 implies

Theorem 4.10. *Let $\{T_1, \ldots, T_b\}$ be the irreducible branches of T. Suppose $0 < r < R$. Then there exist positive integers m_λ, $\lambda = 1, \ldots, b$ such that*

$$I(w, r) \to \sum_{\lambda=1}^{b} m_\lambda \int_{T_\lambda \cap B_r} \upsilon_p \quad \text{as} \quad w \to 0 .$$

§ 5. The final result

Theorem 5.1. *Let V be a complex vector space of dimension $n > 0$. Let $(|)$ be a hermitian product on V. Let G be open in V, $0 \in G$. Define $B_r = \{\mathfrak{z} \in V \,|\, |\mathfrak{z}| < r\}$. Assume $B_R \subset G$, $0 < R \leqq \infty$. Let M be a pure p-dimensional analytic set in G with $0 \in M$ and $0 < p < n$. Define $W_p = \dfrac{\pi^p}{p!}$. Then*

$$n(0, M) = \lim_{r \to 0} \frac{1}{W_p r^{2p}} \int_{M \cap B_r} \upsilon_p$$

is a positive integer.

Proof. From § 3,

$$n(0, M) = \lim_{r\to 0} \frac{1}{W_p r^{2p}} \int_{M\cap B_r} v_p$$

$$= \lim_{r\to 0} \frac{1}{W_p r^{2p}} \lim_{w\to 0} I(w, r).$$

Let $T_1, \ldots, T_b$ be the irreducible branches of T. Take $0 < r < R$. From Theorem 4.10, there exist positive integers $m_1, \ldots, m_b$ such that

$$\lim_{w\to 0} I(w, r) = \sum_{\lambda=1}^{b} m_\lambda \int_{T_\lambda \cap B_r} v_p .$$

From Theorem 2.5, for each $\lambda = 1, \ldots, b$,

$$\frac{1}{W_p r^{2p}} \int_{T_\lambda \cap B_r} v_p = m'_\lambda ,$$

a positive integer independent of r. Thus

$$n(0, M) = \sum_{\lambda=1}^{b} m_\lambda m'_\lambda ,$$

a positive integer. q.e.d.

Appendix

Let M be a pure p-dimensional analytic set in an open neighborhood of the origin of an n-dimensional complex vector space V. Suppose $0 \in M$ and $0 < p < n$. Let $S = \{\mu \mid \mu$ a permutation of $\{1, \ldots, n\}\}$. A basis $(v_1, \ldots, v_n)$ of V is said to be *clear* if, for every $\mu \in S$, the basis $(v_{\mu(1)}, \ldots, v_{\mu(n)})$ is distinguished with respect to $(M, 0, p)$ (defined in § 4 C). The purpose of this appendix is to prove the existence of a clear basis. The proof is due to W. STOLL. See also DE RHAM [5].

Let $q = n - p$. Let $\Lambda^q V$ denote the space of exterior q vectors over V. Let $\mathbf{P}(\Lambda^q V)$ denote the complex projective space to $\Lambda^q V$, and

$$\sigma : \Lambda^q V - \{0\} \to \mathbf{P}(\Lambda^q V)$$

the residual map. Let

$$V'_q = \{a_1 \wedge \cdots \wedge a_q \mid a_1 \wedge \cdots \wedge a_q \neq 0,\ a_\nu \in V,\ \nu = 1, \ldots, q\} \subset \Lambda^q V - \{0\} .$$

Let $G = \sigma(V'_q)$. Then G is a smooth, connected, complex submanifold of $\mathbf{P}(\Lambda^q V)$, the *Grassman manifold* of q-planes in V.

Let $\mathbf{P}(V)$ denote the complex projective space to V, and

$$\varrho : V - \{0\} \to \mathbf{P}(V)$$

the residual map. Take $a_\nu \in V$, $\nu = 1, \ldots, q$. Define

$$E(a_1, \ldots, a_q) = \{z \in V \mid z \wedge a_1 \wedge \cdots \wedge a_q = 0\}$$

$$= \left\{\sum_{\nu=1}^{q} \lambda_\nu a_\nu \mid \lambda_\nu \in \mathbf{C},\ \nu = 1, \ldots, q\right\}.$$

Take $\alpha \in G$. Take any $a_1 \wedge \cdots \wedge a_q$ contained in $V'_q \cap \sigma^{-1}(\alpha)$. Define

$$E(\alpha) = \varrho(E(a_1, \ldots, a_q)).$$

This is well-defined, and, moreover, for α and β contained in G, $E(\alpha) = E(\beta)$ if and only if $\alpha = \beta$.

Lemma A.1. *Let N be an analytic set in $\mathbf{P}(V)$ of dimension $p-1$. Let*

$$A = \{\alpha \in G \mid E(\alpha) \cap N \neq \Phi\}.$$

Then A is a thin, analytic set in G.

Proof. From Lemma 3 of STOLL [8], $A \neq G$. Thus it remains to show only that A is analytic. Define $T = \varrho^{-1}(N) \cup \{0\}$. By Chow's Theorem, T is an analytic set in V of dimension p, and

$$T = \{z \in V \mid Q_1(z) = \cdots = Q_k(z) = 0\}$$

where Q_ν is a homogeneous polynomial, $\nu = 1, \ldots, k$. Let

$$\begin{aligned} L &= \{(a_1 \wedge \cdots \wedge a_q, z) \mid z \in T,\ a_1 \wedge \cdots \wedge a_q \wedge z = 0\} \\ &= \{(a_1 \wedge \cdots \wedge a_q, z) \mid a_1 \wedge \cdots \wedge a_q \wedge z = 0,\ Q_1(z) = \cdots = Q_k(z) = 0\} \subseteqq \Lambda^q V \oplus V. \end{aligned}$$

Then L is analytic, and for any λ_1 and λ_2 in $\mathbf{C}$, $(a_1 \wedge \cdots \wedge a_q, z) \in L$ implies $(\lambda_1(a_1 \wedge \cdots \wedge a_q), \lambda_2 z) \in L$. Let $L' = \cap\, [(\Lambda^q V - \{0\}) \times (V - \{0\})]$. Then

$$M = (\sigma \oplus \varrho)(L') \subseteqq G \times \mathbf{P}(V),$$

and in fact, M is analytic in $G \times \mathbf{P}(V)$. Define

$$\pi : G \times \mathbf{P}(V) \to G,$$

the projection. Then $\pi | M : M \to G$ is proper, and so $\pi(M)$ is analytic in G. But $\pi(M) = A$, for take $\alpha \in \pi(M)$. There exists $z \in T$ and $a_1 \wedge \cdots \wedge a_q \in \Lambda^q V$ such that $(a_1 \wedge \cdots \wedge a_q, z) \in L'$ and $\sigma(a_1 \wedge \cdots \wedge a_q) = \alpha$. Then $a_1 \wedge \cdots \wedge a_q \wedge z = 0$, $z \neq 0$, and so $\varrho(z) \in E(\alpha) \cap \varrho(T - \{0\}) = E(\alpha) \cap N$. Thus $\alpha \in A$. Conversely, let $\alpha \in A$. There exists $z \varepsilon T - \{0\}$ such that $\varrho(z) \in E(\alpha) \cap N$. Choose any $a_1 \wedge \cdots \wedge a_q \in V'_q$ such that $\sigma(a_1 \wedge \cdots \wedge a_q) = \alpha$. Then $z \in E(a_1, \ldots, a_q)$, and so $(a_1 \wedge \cdots \wedge a_q, z) \in L'$. And $\pi((\sigma \oplus \varrho)(a_1 \wedge \cdots \wedge a_q, z)) = \alpha$. Thus $\alpha \in \pi(M)$. q.e.d.

Denote the set of bases of V by

$$\Gamma = \left\{(v_1, \ldots, v_n) \in \bigoplus_{\nu=1}^{n} V \mid v_1 \wedge \cdots \wedge v_n \neq 0\right\}.$$

Then Γ is a connected complex manifold, the complement of an analytic set of codimension 1.

Theorem A.2. *Let M be a pure p-dimensional analytic set in an open neighborhood of the origin of an n-dimensional complex vector space V. Suppose $0 \in M$ and $0 < p < n$. Then there exists a thin, analytic set $\Delta \subset \Gamma$ such that $(v_1, \ldots, v_n) \in \Gamma - \Delta$ implies that $(v_1, \ldots, v_n)$ is a clear basis.*

Proof. Let T denote the tangent cone to M at 0. According to Proposition 3.1, T is a pure p-dimensional analytic set in V. Let $N = \varrho(T - \{0\})$. Then N is an analytic set in $\mathbf{P}(V)$ of dimension $p-1$. Let

$$A = \{\alpha \in G \mid E(\alpha) \cap N \neq \Phi\}.$$

From Lemma A.1, A is a thin analytic set in G. For $\mu \in S$, define $\tau_\mu: \Gamma \to G$ by

$$\tau_\mu((v_1, \ldots, v_n)) = \sigma(v_{\mu(p+1)} \wedge \cdots \wedge v_{\mu(n)}).$$

Then τ_μ is holomorphic. And τ_μ is onto, for take $\alpha \in G$, $\alpha = \sigma(a_1 \wedge \cdots \wedge a_q)$, $a_1 \wedge \cdots \wedge a_q \in V'_q$. Extend $(a_1, \ldots, a_q)$ to a basis $(a_1, \ldots, a_q, a_{q+1}, \ldots, a_n) \in \Gamma$ of V. Permute $(a_1, \ldots, a_n)$ to $(b_1, \ldots, b_n) \in \Gamma$ such that $a_\nu = b_{\mu(p+\nu)}$, $\nu = 1, \ldots, q$. Then $\tau_\mu((b_1, \ldots, b_n)) = \sigma(b_{\mu(p+1)} \wedge \cdots \wedge b_{\mu(n)}) = \sigma(a_1 \wedge \cdots \wedge a_q) = \alpha$. Define

$$\Delta = \bigcup_{\mu \in S} \tau_\mu^{-1}(A),$$

a thin analytic set in Γ as each $\tau_\mu^{-1}(A)$ is thin and analytic. Now take $(v_1, \ldots, v_n) \in \Gamma - \Delta$. Suppose that $(v_1, \ldots, v_n)$ is not a clear basis. Then there exists $\mu \in S$ such that $(v_{\mu(1)}, \ldots, v_{\mu(n)})$ is not distinguished with respect to $(M, 0, p)$, that is, 0 is not an isolated point of $E \cap M$, where $E = E(v_{\mu(p+1)}, \ldots, v_{\mu(n)})$. Thus there exists a sequence $\{z_\lambda\}$ such that $z_\lambda \to 0$ as $\lambda \to \infty$ and $z_\lambda \neq 0$, $z_\lambda \in E \cap M$. There exists a subsequence $\{z_{\lambda_\nu}\}$ such that $z_{\lambda_\nu}/|z_{\lambda_\nu}|$ converges, say, to t, as $\nu \to \infty$. Then t is a tangent vector to M at 0, and $t \in T$. And $z_\lambda \in E$ for all λ implies that $t \in E$. Let $\alpha = \sigma(v_{\mu(p+1)} \wedge \cdots \wedge v_{\mu(n)})$. Then $\varrho(t) \in \varrho(E) \cap \varrho(T - \{0\}) = E(\alpha) \cap N$. Thus $\alpha \in A$. But $\alpha = \tau_\mu((v_1, \ldots, v_n))$, and so $(v_1, \ldots, v_n) \in \tau_\mu^{-1}(A) \subset \Delta$, a contradicition. Consequently, every basis in $\Gamma - \Delta$ is clear. q.e.d.

Bibliography

1. ABHYANKAR, S.: Local analytic geometry. New York: Academic Press 1964.
2. HUREWICZ, W., and H. WALLMAN: Dimension theory. Princeton, New Jersey: Princeton University Press 1948.
3. LELONG, P.: Intégration sur un ensemble analytique complexe. Bull. Soc. Math. France **85**, 239—262 (1957).
4. DE RHAM, G.: Currents in an analytic complex manifold. Seminars on Analytic Functions, vol. **1**, pp. 54—64. Princeton, New Jersey: Institute for Advanced Study 1957.
5. — On the area of complex manifolds. Seminar on Several Complex Variables, Institute for Advanced Study 1957—1958 (unpublished).
6. STOLL, W.: Mehrfache Integrale auf komplexen Mannigfaltigkeiten. Math. Z. **57**, 116—154 (1952).
7. — Einige Bemerkungen zur Fortsetzbarkeit analytischer Mengen. Math. Z. **60**, 287—304 (1954).
8. — The growth of the area of a transcendental analytic set of dimension one. Math. Z. **81**, 76—98 (1963).
9. — The growth of the area of a transcendental analytic set. I. Math. Ann. **156**, 47—78 (1964).
10. WHITNEY, H.: Tangents to an analytic variety. Ann. Math. (2), **81**, 496—549 (1965).

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