Math. Ann. 184, 163—168 (1970)

Reflexivity and the Girth of Spheres

JUAN JORGE SCHÄFFER and KONDAGUNTA SUNDARESAN

1. Introduction

James [4] introduced the concept of a uniformly non-square unit ball and certain other geometric properties of Banach spaces, all related to reflexivity. The purpose of this paper is to focus attention on a similar property, a generalized (negation of) uniform non-squareness, and show that it can be expressed in terms of the "girth" of the unit ball, as introduced by Schäffer [5], i.e., the infimum of the lengths of centrally symmetric simple closed rectifiable curves in its boundary. Specifically, it is shown that a Banach space is reflexive if the girth of its unit ball is not 4.

2. Geometric Properties

Let X be a given real non-trivial normed linear space with norm $\| \ \|$; let Σ denote its unit ball. For each positive integer n and each real ϱ, $0<\varrho<1$, we consider the following property of X:

($J_{n,\varrho}$). *There exist* $x_k \in \Sigma$, $k=1,\ldots,n$, *such that*

$$\left\|\sum_1^n \varepsilon_k x_k\right\| > \varrho n \tag{2.1}$$

for all sequences (ε_k), $\varepsilon_k = \pm 1$, $k=1,\ldots,n$, *in which every* -1, *if any, precedes each* $+1$, *if any.*

We say that X satisfies (J_n) if it satisfies ($J_{n,\varrho}$) for all sufficiently large ϱ, $0<\varrho<1$; and that X satisfies (J) if it satisfies (J_n) for every positive integer n.

We remark that (J_1) is always satisfied; that the negation of (J_2) is the property of Σ being uniformly non-square; and that, if ($J_{n,\varrho}$) were modified so that (2.1) held for *every* sequence (ε_k), $\varepsilon_k=\pm 1$, $k=1,\ldots,n$, the negation of (J_n) would become the property of X being *uniformly non-*l_n^1 (see [4]), and the negation of (J) the property of being a *B-space* (see [1, 3]), all provided X is a Banach space.

James proved that a non-reflexive Banach space satisfies (J_2), (J_3) [4; Theorems 1.1, 2.1]; extending his method, we shall show that such a space indeed satisfies (J).

Let m be a positive integer, $(p_1,\ldots,p_{2m})$ a strictly increasing sequence of positive integers, and $f=(f_j)$ an infinite sequence in X^*, the dual space of X, with $\|f_j\|=1$, $j=1,2,\ldots$. Set

$$S(p_1,\ldots,p_{2m}; f) = \{x\in X : \tfrac{3}{4} \leqq (-1)^{i-1} f_j(x) \leqq 1 \quad \text{for all} \quad j\in[p_{2i-1},p_{2i}], \quad i=1,\ldots,m\} \tag{2.2}$$

(where $[p,q]=\{j: p \leqq j \leqq q\}$). This set is convex. We define $R(p_1, \ldots, p_{2m}; f) = \inf\{\|x\| : x \in S(p_1, \ldots, p_{2m}; f)\}$. Observe that, if m, f, and all p_i save one, say p_l, are fixed, $S(p_1, \ldots, p_l, \ldots, p_{2m}; f)$, and hence $R(p_1, \ldots, p_l, \ldots, p_{2m}; f)$, is a monotone function of p_l, the sense depending on the parity of l. We may therefore define

$$K_m(f) = \lim_{p_1 \to \infty} \lim_{p_2 \to \infty} \cdots \lim_{p_{2m} \to \infty} R(p_1, \ldots, p_{2m}; f),$$
$$K_m = \inf\{K_m(f) : f = (f_j),\ f_j \in X^*,\ \|f_j\| = 1,\ j = 1, 2, \ldots\}$$

(some of these numbers may be infinite). For fixed f, the sequence $(K_m(f))$ is non-decreasing; the same is therefore true of (K_m).

In terms of these definitions, this is the fundamental result of James.

2.1. Lemma ([4; pp. 543–544]). *If X is a non-reflexive Banach space, $K_m \leqq 2m$, $m = 1, 2, \ldots$, and therefore* $\limsup_{m \to \infty} K_{m-1}/K_m = 1$.

2.2. Theorem. *If X is a non-reflexive Banach space, then X satisfies* (J).

Proof. It is sufficient to prove that X satisfies $(\mathrm{J}_{n,\varrho})$ for each fixed n and ϱ; let these be therefore given. By Lemma 2.1 there exists a positive integer m such that $K_{m-1}/K_m > \varrho$, and therefore a sequence $f = (f_j)$, etc., such that $K_{m-1}(f)/K_m(f) \geqq K_{m-1}/K_m(f) > \varrho$. Let m and f be thus fixed, and choose $\tau > 1$ so close to 1 that

$$K_{m-1}(f)/K_m(f) > \tau^2 \varrho\,. \tag{2.3}$$

By the definition of $K_m(f)$, there exists a strictly increasing sequence $(q_1, \ldots, q_{2mn})$ of positive integers with the following property: if $(q_{l(1)}, \ldots, q_{l(2m)})$, $(q_{l(1)}, \ldots, q_{l(2m-2)})$ are any subsequences, then

$$R(q_{l(1)}, \ldots, q_{l(2m)}; f) < \tau K_m(f), \tag{2.4}$$
$$R(q_{l(1)}, \ldots, q_{l(2m-2)}; f) \geqq \tau^{-1} K_{m-1}(f), \tag{2.5}$$

respectively. We fix such a sequence $(q_1, \ldots, q_{2mn})$ and relabel it as follows:

$$\begin{aligned}(p_1^1, p_1^2, \ldots, p_1^n, p_2^1, p_3^1, \ldots, p_2^n, p_3^n, \ldots, p_{2i}^1, p_{2i+1}^1,\\ \ldots, p_{2i}^n, p_{2i+1}^n, \ldots, p_{2m-2}^n, p_{2m-1}^n, p_{2m}^1, p_{2m}^2, \ldots, {}_{2m}^n)\,.\end{aligned} \tag{2.6}$$

From (2.4), (2.5), (2.6) we deduce in particular

$$R(p_1^k, \ldots, p_{2m}^k; f) < \tau K_m(f), \quad k = 1, \ldots, n, \tag{2.7}$$
$$R(p_1^n, p_2^1, p_3^n, \ldots, p_{2m-3}^n, p_{2m-2}^1; f) \geqq \tau^{-1} K_{m-1}(f), \tag{2.8}$$
$$R(p_3^k, p_2^{k+1}, p_5^k, p_4^{k+1}, \ldots, p_{2m-1}^k, p_{2m-2}^{k+1}; f) \geqq \tau^{-1} K_{m-1}(f),\ k = 1, \ldots, n. \tag{2.9}$$

Now (2.6) shows that $[p_{2i-1}^n, p_{2i}^1] \subset [p_{2i-1}^k, p_{2i}^k]$ for all $i = 1, \ldots, m-1$ and all $k = 1, \ldots, n$; therefore (2.2) implies

$$S(p_1^k, \ldots, p_{2m}^k; f) \subset S(p_1^n, p_2^1, p_3^n, \ldots, p_{2m-3}^n, p_{2m-2}^1; f), \quad k = 1, \ldots, n\,. \tag{2.10}$$

Similarly, if $1 \leqq l < k \leqq n$, (2.6) shows that $[p_{2i+1}^l, p_{2i}^{l+1}] \subset [p_{2i-1}^k, p_{2i}^k]$, $i = 1, \ldots, m-1$, and therefore

$$S(p_1^k, \ldots, p_{2m}^k; f) \subset S(p_3^l, p_2^{l+1}, p_5^l, p_4^{l+1}, \ldots, p_{2m-1}^l, p_{2m-2}^{l+1}; f), \quad 1 \leqq l < k \leqq n; \tag{2.11}$$

if $1 \leqq k \leqq l < n$, on the other hand, (2.6) shows that $[p_{2i+1}^l, p_{2i}^{l+1}] \subset [p_{2i+1}^k, p_{2i+2}^k]$, $i = 1, \ldots, m-1$, and therefore

$$-S(p_1^k, \ldots, p_{2m}^k; f) \subset S(p_3^l, p_2^{l+1}, p_5^l, p_4^{l+1}, \ldots, p_{2m-1}^l, p_{2m-2}^{l+1}; f), \quad 1 \leqq k \leqq l < n. \tag{2.12}$$

Using (2.7), we choose $u_k \in S(p_1^k, \ldots, p_{2m}^k; f)$ with $\|u_k\| \leqq \tau K_m(f)$, $k = 1, \ldots, n$. Since all the sets $S(\ldots; f)$ are convex, (2.10) and (2.8) imply

$$n^{-1} \left\| \sum_1^n u_k \right\| = n^{-1} \left\| -\sum_1^n u_k \right\| \geqq \tau^{-1} K_{m-1}(f);$$

similarly, (2.11), (2.12), and (2.9) imply

$$n^{-1} \left\| -\sum_1^l u_k + \sum_{l+1}^n u_k \right\| \geqq \tau^{-1} K_{m-1}(f), \qquad l = 1, \ldots, n-1.$$

We finally set $x_k = u_k/\tau K_m(f)$ and find $\|x_k\| \leqq 1$ and

$$\left\| \sum_1^n \varepsilon_k x_k \right\| \geqq n \tau^{-2} K_{m-1}(f)/K_m(f) > \varrho n$$

(using (2.3)) for all sequences (ε_k), $\varepsilon_k = \pm 1$, $k = 1, \ldots, n$, with all -1 preceding all $+1$. Thus $(J_{n,\varrho})$ holds, and the proof is concluded.

We have shown that (J) is necessary for non-reflexivity; it is, however, not sufficient, as the following result shows.

2.3. Theorem. *Let X be a separable non-reflexive Banach space, and let (X_n) be an increasing sequence of finite-dimensional subspaces of X such that $\bigcup_1^\infty X_n$ is dense in X. There exists a separable reflexive Banach space Y and a sequence (Y_n) of subspaces of Y such that Y_n is congruent to X_n for every n; and each such Y and every space isomorphic to it satisfies* (J).

Proof. To construct Y, choose p, $1 < p < \infty$, and let Y be the closed subspace of the Banach space $l^p(X)$ consisting of those sequences (x_n) that satisfy $x_n \in X_n$, $n = 1, 2, \ldots$. This space is separable and reflexive (see [2]). Now X satisfies $(J_{n,\varrho})$ for every n and every ϱ by Theorem 2.2; the fact that the same is true of every space isomorphic to Y then follows exactly as in the proof of [4; Lemma 1.1, Theorem 1.2], and we need not repeat the argument.

Remark. Suppose that X, in addition, is not a B-space (i.e., is not uniformly non-l_n^1 for any n); this is conjectured in [4] to be the case always, and shown to hold when X has an unconditional basis. Then the same proof shows that Y and all spaces isomorphic to it are separable and reflexive but are not

B-spaces. An example is obtained by taking $X = l^1$, $X_n = l_n^1$, $p = 2$, so that Y can be represented as the space of infinite lower-triangular matrices of real numbers with $\|(x_{mn})\| = \left(\sum_{m=1}^{\infty}\left(\sum_{n=1}^{m}|x_{mn}|\right)^2\right)^{1/2} < \infty$.

3. The Girth of Spheres

Let X be as before, with $\dim X \geqq 2$. Let Σ_0, $\partial\Sigma$ denote the interior and the boundary of the unit ball, respectively. The *inner metric* δ of $\partial\Sigma$ is defined as usual by $\delta(p, q) = \inf\{l(\mathfrak{c}) : \mathfrak{c}$ a curve from p to q in $\partial\Sigma\}$; here "curve" means "rectifiable geometric curve", and $l(\mathfrak{c})$ is the length of $\mathfrak{c}$; details of notation, terminology, and proofs may be found in [5]. Among other parameters of X defined in [5], we have $m(x) = \inf\{\delta(-p, p) : p \in \partial\Sigma\} = \inf\{l(\mathfrak{c}) : \mathfrak{c}$ a curve in $\partial\Sigma$ with antipodal endpoints$\}$; $2m(X)$ may be termed the *girth* of Σ, a term more clearly justified by the characterization mentioned in the introduction [5; Lemma 5.1].

It is obvious that $m(X) \geqq 2$; in [6] it was shown that, although $m(X) > 2$ for all finite-dimensional X, there exist spaces with $m(X) = 2$. We shall now show that these are precisely the spaces satisfying (J).

In the following lemma, $p, q \in X$ are *opposite* if $p + q = 0$.

3.1. Lemma. $m(X) = \inf\{l(\mathfrak{c}) : \mathfrak{c}$ *a curve in* $X \backslash \Sigma_0$ *with opposite endpoints*$\}$.

Proof. If $\mathfrak{C}$ is the set of curves in $X \backslash \Sigma_0$ with opposite endpoints and $\mathfrak{C}_0$ is the subset of those that lie in $\partial\Sigma$, this inclusion and the definition of $m(X)$ imply $\inf\{l(\mathfrak{c}) : \mathfrak{c} \in \mathfrak{C}\} \leqq \inf\{l(\mathfrak{c}) : \mathfrak{c} \in \mathfrak{C}_0\} = m(X)$. Let $\mathfrak{c} \in \mathfrak{C}$ be given; setting $\mu = \min\{\|x\| : x \in \mathfrak{c}\} \geqq 1$, we find that $\mu^{-1}\mathfrak{c} \in \mathfrak{C}$, but $\mu^{-1}\mathfrak{c}$ contains a point $q \in \partial\Sigma$. The symmetric closed curve $\mathfrak{s}$ (not necessarily simple) obtained by putting $\mu^{-1}\mathfrak{c}$ and $-\mu^{-1}\mathfrak{c}$ end-to-end can therefore also be obtained by putting end-to-end a curve $\mathfrak{d}$ from $-q$ to q in $X \backslash \Sigma_0$ and the curve $-\mathfrak{d}$. By [5; Theorem 3.3],

$$m(X) \leqq \delta(-q, q) \leqq l(\mathfrak{d}) = \tfrac{1}{2} l(\mathfrak{s}) = l(\mu^{-1}\mathfrak{c}) = \mu^{-1} l(\mathfrak{c}) \leqq l(\mathfrak{c}).$$

Since $\mathfrak{c} \in \mathfrak{C}$ was arbitrary, $m(X) \leqq \inf\{l(\mathfrak{c}) : \mathfrak{c} \in \mathfrak{C}\}$, and the conclusion follows.

3.2. Theorem. *For a given positive integer n and a given ϱ, $0 < \varrho < 1$, the space X satisfies* $(\mathrm{J}_{n,\varrho})$ *if* $m(X) < 2\varrho^{-1}$ *and only if* $m(X) \leqq 2(\varrho - n^{-1})^{-1}$ *(the latter provided $\varrho n > 1$). Therefore X satisfies* (J) *if and only if* $m(X) = 2$.

Proof. 1. Assume that $m(X) < 2\varrho^{-1}$. There exists, then, a curve $\mathfrak{c}$ in $\partial\Sigma$ with antipodal endpoints, say $-p, p$, such that $l = l(\mathfrak{c}) < 2\varrho^{-1}$. Let $g : [0, l] \to \partial\Sigma$ be the parametrization of $\mathfrak{c}$ in terms of arc-length, and set $p_k = g(kn^{-1}l) \in \partial\Sigma$, $k = 0, \ldots, n$, so that $p_0 + p_n = -p + p = 0$. Set $x_k = l^{-1}n(p_k - p_{k-1})$, $k = 1, \ldots, n$. Then $\|x_k\| = l^{-1}n\|g(kn^{-1}l) - g((k-1)n^{-1}l)\| \leqq 1$, so that $x_k \in \Sigma$, $k = 1, \ldots, n$. Further, $-\sum_{1}^{j} x_k + \sum_{j+1}^{n} x_k = l^{-1}n(p_0 - p_j + p_n - p_j) = -2l^{-1}np_j$. Therefore $\left\|-\sum_{1}^{j} x_k + \sum_{j+1}^{n} x_k\right\| = 2l^{-1}n > \varrho$, $j = 0, \ldots, n$, and $(\mathrm{J}_{n,\varrho})$ holds.

2. Assume that X satisfies $(\mathrm{J}_{n,\varrho})$ with $\varrho n > 1$, and set $\mu = (\varrho n - 1)^{-1}$. Let $x_k \in \Sigma$, $k = 1, \ldots, n$, be as specified in $(\mathrm{J}_{n,\varrho})$ and set $p_j = \mu\left(-\sum_1^{j-1} x_k + \sum_{j+1}^n x_k\right)$, $j = 1, \ldots, n$. We consider the polygon $\mathfrak{p}$ with consecutive vertices $p_0 = -p_n$, $p_1, \ldots, p_n$, and claim that it lies in $X \setminus \Sigma_0$. Indeed, a point on the edge $p_0 p_1$ is of the form $\lambda p_0 + (1-\lambda)p_1 = \mu\left(\sum_1^n x_k - (1-\lambda)x_1 - \lambda x_n\right)$, $0 \leqq \lambda \leqq 1$, while a point on the edge $p_{j-1} p_j$, $j = 2, \ldots, n$, is of the form

$$\lambda p_{j-1} + (1-\lambda)p_j = \mu\left(-\sum_1^{j-1} x_k + \sum_j^n x_k + \lambda x_{j-1} - (1-\lambda)x_j\right),$$

$0 \leqq \lambda \leqq 1$; and therefore

$$\|\lambda p_{j-1} + (1-\lambda)p_j\| > \mu(\varrho n - \lambda - (1-\lambda)) = \mu(\varrho n - 1) = 1, \qquad 0 \leqq \lambda \leqq 1,\ j = 1, \ldots, n,$$

as claimed. Also, $p_1 - p_0 = \mu(x_n - x_1)$, $p_j - p_{j-1} = -\mu(x_{j-1} + x_j)$, $j = 2, \ldots, n$; by Lemma 3.1,

$$m(X) \leqq l(\mathfrak{p}) = \sum_1^n \|p_j - p_{j-1}\| \leqq 2n\mu = 2(\varrho - n^{-1})^{-1}.$$

We can now combine Theorems 2.2 and 3.2 and obtain our main result.

3.3. Theorem. *If X is a Banach space and $m(X) > 2$, then X is reflexive.*

Remark. A slightly more general result is: *If $m(X) > 2$, then the completion of X is reflexive.* An easy proof is obtained from the following observation: if X is a dense subspace of Y and Y satisfies $(\mathrm{J}_{n,\varrho})$, then X satisfies $(\mathrm{J}_{n,\varrho'})$ for every ϱ', $0 < \varrho' < \varrho$; therefore, if X does not satisfy (J), neither does Y. It is in fact true that $m(X) = m(Y)$ whenever X is a dense subspace of Y.

It follows from Theorem 2.3 that the converse of Theorem 3.3 does not hold. To put this remark into clearer perspective, we recall some concepts from [5]. An *isomorphism class* $\boldsymbol{X}$ is the class of all normed spaces isomorphic to some one of them. Obviously, either all spaces in an isomorphism class are Banach spaces, or reflexive Banach spaces, or none is. Also, all spaces in $\boldsymbol{X}$ have the same (linear) dimension $\dim \boldsymbol{X}$. We set $m_*(\boldsymbol{X}) = \inf\{m(X) : X \in \boldsymbol{X}\}$, $m^*(\boldsymbol{X}) = \sup\{m(X) : X \in \boldsymbol{X}\}$. It was shown in [5; Theorem 8.3], [6; Theorem 7] that $m_*(\boldsymbol{X}) = 2$ if and only if $\dim \boldsymbol{X}$ is infinite. In contrast to the fact that $m^*(\boldsymbol{X}) \geqq \pi$ for all $\boldsymbol{X}$ with finite, and for some with infinite, dimension [5; Theorem 8.4], our present Theorems 3.3, 3.2, and 2.3 imply the following result.

3.4. Theorem. *$m^*(\boldsymbol{X}) = 2$ for every isomorphism class of non-reflexive Banach spaces, and for some isomorphism classes of (separable) reflexive Banach spaces.*

Remark. By combining the Remark to Theorem 3.3 with the fact that $m(X) \leqq m(Y)$ if X is a subspace of Y, we find, in contrast to [5; Theorem 8.4, (c)]: *For every infinite cardinal* $\aleph$ *there exists an isomorphism class* X *with* $\dim X = \aleph$ *and* $m^*(X) = 2$.

References

1. Beck, A.: A convexity condition in Banach spaces and the strong law of large numbers. Proc. Amer. Math. Soc. **13**, 329—334 (1962).
2. Day, M. M.: Reflexive spaces not isomorphic to uniformly convex spaces. Bull. Amer. Math. Soc. **47**, 313—317 (1941).
3. Giesy, D. P.: A convexity condition on Banach spaces invariant under conjugation. Amer. Math. Soc. Notices **10**, 665, abstract 607—15 (1963).
4. James, R. C.: Uniformly non-square Banach spaces. Ann. of Math. **80**, 542—550 (1964).
5. Schäffer, J. J.: Inner diameter, perimeter, and girth of spheres. Math. Ann. **173**, 59—79 (1967).
6. — Addendum: Inner diameter, perimeter, and girth of spheres. Math. Ann. **173**, 79—82 (1967).

Professor J. J. Schäffer
Professor K. Sundaresan
Carnegie-Mellon-University
Department of Mathematics
Schenley Park
Pittsburgh, Pennsylvania 15213, USA

(Received December 22, 1968)