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Inner Diameter, Perimeter, and Girth of Spheres

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1. Introduction

In some recent research on differential equations in a Banach space [17], the author was led to consider certain geometrical properties of such a space, and notably the following problem: to give a lower bound for the length of a curve joining antipodal points of the unit sphere and lying *on* the sphere. Although the original context was infinite-dimensional, this question does not appear to be trivial by any means even for three-dimensional normed spaces. It belongs to a family of little-explored geometrical problems dealing with the inner metric structure of the (surface of the) unit sphere. The inner metric geometry of convex surfaces has been extensively studied (e. g., [1], [4]); what is new and decisive about the questions we are referring to is that the inner metric is determined by the norm generated by the body bounded by the surface itself. They are, of course, problems "in the large".

The purpose of this paper is little more than to present a few parameters associated with a normed space that arise in this context, and to give a preliminary discussion of their values. The two-dimensional case has been discussed previously, even for "norms" induced by non-symmetric convex sets [8], and a strong result obtained recently [9], [10] (see also the Remark at the end of Section 4). In a sense (and in more than one technical device) this paper is related to [16]. It is hoped that it will help place some of the relevant questions in the proper light, even if it cannot claim to contribute significantly to answering them. The amazing amount of underbrush that has to be cleared away, especially in Sections 3, 4, 5, indicates, to this author at least, that the geometry of finite-dimensional convex sets is still quite imperfectly known. All the fundamental problems in our study are finite-dimensional; the infinite-dimensional case has been included mainly because the extra cost is slight, and almost all the results concerning it are obtained with the use of the finite-dimensional theory.

Section 3 discusses the inner metric of the unit sphere. Section 4 deals with two-dimensional spaces; this case can be analysed quite thoroughly, and the conclusions concerning it are required in most of the remaining work. Section 5 introduces the parameters: inner diameter, perimeter, girth, that are the main object of our study. Section 6 is concerned with the concept of "nearness" of isomorphic normed spaces, and of equivalence classes of such spaces under congruence. This concept is applied in Sections 7 and 8 to show that the parameters are continuous with respect to it, and attain, for every fixed finite dimension, their extreme values. Section 8 contains the main results concerning these extreme values, Section 9 includes some conjectures and concluding remarks.

2. Normed spaces

We shall be dealing throughout this paper with real normed spaces; the fact that the scalars are real numbers will always be understood. Completeness of infinite-dimensional spaces will play no rôle. If X is such a space, $\|\cdot\|_X$ denotes the corresponding norm, and we set $\Sigma(X)=\{x\in X:\|x\|\leqq 1\}$, $\Sigma_0(X)=\{x\in X:\|x\|<1\}$, $\partial\Sigma(X)=\{x\in X:\|x\|=1\}$. We write $\|\cdot\|$, Σ, Σ_0, $\partial\Sigma$, when no confusion is likely. If $p\in\partial\Sigma$, its *antipode* is $-p$, and the two points are *antipodal*. For any $A\subset X$, co A denotes the convex hull of A.

In particular, R denotes the real field with the norm $\|\cdot\|_R=|\cdot|$, so that $\Sigma(R)=[-1,1]$, etc.

$\dim X$ denotes the linear (Hamel) dimension of X or, rather, of the underlying linear space. X is *n-dimensional, finite-dimensional, countable-dimensional, infinite-dimensional* if $\dim X$ is, respectively, n, a finite cardinal (written $\dim X<<\infty$), finite or countable infinite, not finite (written $\dim X=\infty$).

A *subspace* Y of X is a linear manifold in X, provided with the norm of X (closedness is not assumed). Thus $\Sigma(Y)=\Sigma\cap Y$, $\Sigma_0(Y)=\Sigma_0\cap Y$, $\partial\Sigma(Y)=\partial\Sigma\cap Y$. A set $A\subset X$ is *finite-dimensional* if it is contained in a finite-dimensional subspace of X.

If X, Y are normed spaces, an *isomorphism from X to Y* is a linear homeomorphism $T:X\to Y$, i.e., a bounded bijective linear mapping with a bounded inverse. A *congruence* is an isometrical isomorphism; equivalently, an isomorphism T with $\|T\|=\|T^{-1}\|=1$. X, Y are *isomorphic [congruent]* if there exists an isomorphism [a congruence] from X to Y; these relations are equivalence relations. Isomorphic spaces have the same dimension; spaces with the same finite dimension are isomorphic.

If X, Y are normed spaces, we denote by $X\oplus Y$ the outer direct sum of X and Y (algebraically) with the norm $\|x\oplus y\|_{X\oplus Y}=\max\{\|x\|_X,\|y\|_Y\}$, so that $\Sigma(X\oplus Y)=\Sigma(X)\oplus\Sigma(Y)$. Obviously, $\dim(X\oplus Y)=\dim X+\dim Y$, and X, Y are canonically congruent to the subspace $X\oplus\{0\}$, $\{0\}\oplus Y$ of $X\oplus Y$, respectively.

2.1. Lemma. *If X, Y are normed spaces,* $\dim Y<\infty$, $\dim X\geqq\dim Y$, *there exists a normed space Z such that X and $Y\oplus Z$ are isomorphic.*

Proof. Set $\dim Y=n$, and let V be an n-dimensional subspace of X. There exists an isomorphism $T:V\to Y$. There further exists in X a bounded projection P with range V (e.g., $Px=\sum_1^n\langle x,e_i^*\rangle e_i$, where $\{e_i:i=1,\dots,n\}$ is a basis of V and the e_i^* are bounded linear functionals on X with $\langle e_i,e_j^*\rangle=\delta_{ij}$, $i,j=1,\dots,n$— these exist by the Hahn-Banach Theorem). Let Z be the null-space of P, a subspace of X. We consider the linear mapping $T':X\to Y\oplus Z$, defined by

$T'x = TPx \oplus (I-P)x$. T' is bounded: $\|T'\| \leqq \max\{\|TP\|, \|I-P\|\}$; and T' has the inverse $T'': Y \oplus Z \to X$ defined by $T''(y \oplus z) = T^{-1}y + z$, which is also bounded: $\|T''\| \leqq 1 + \|T^{-1}\|$. Therefore T' is an isomorphism from X to $Y \oplus Z$.

3. The inner metric

In this section we consider a fixed normed space X with $\dim X \geqq 2$. Our purpose is to examine the inner metric of $\partial\Sigma$, i.e., the distance given by the infimum of the lengths of curves. Concerning inner metrics in general, see [15]. Our first purpose is to show that the infimum of the lengths of curves with given endpoints in $\partial\Sigma$ cannot be decreased by allowing the curves to pass through the exterior of Σ. In an inner-product space this is obvious, since radial projection onto $\partial\Sigma$ of such a curve does not increase the length; but this argument fails in every other normed space (except for some two-dimensional ones: see [18]). The argument given below could be simplified somewhat in the special case when Σ is strictly convex.

We are concerned with rectifiable curves in X or in subsets of X. All curves considered will be rectifiable: *curve* means "rectifiable geometric curve", as defined in [3; pp. 23—26], i.e., as the equivalence class of all *parametrizations* (continuous functions from a compact interval of real numbers into the space) with the same standard representation in terms of arc-length. Following common usage, however, and without danger of confusion, a curve $\mathfrak{c}$ often stands for the common range of its parametrizations — a compact set — as, e.g., in "a point of $\mathfrak{c}$", "co $\mathfrak{c}$"; we set $\varrho(\mathfrak{c}) = \max\{\|x\| : x \in \mathfrak{c}\}$. If $\mathfrak{c}$ is a curve in the subset A of X, say, we denote by $l(\mathfrak{c})$ its (finite) length, and by $g_{\mathfrak{c}}: [0, l(\mathfrak{c})] \to A$ its standard representation in terms of arc-length. Thus $g_{\mathfrak{c}}(0)$, $g_{\mathfrak{c}}(l(\mathfrak{c}))$ are the *initial point* and the *final point* of $\mathfrak{c}$, and $\mathfrak{c}$ is a curve *from* its initial point *to* its final point. If $a \in X$, λ is a real number, and $\mathfrak{c}$ is a curve in X, $a + \lambda\mathfrak{c}$ denotes the curve obtained from $\mathfrak{c}$ by the mapping $x \to a + \lambda x$. Thus $l(a + \lambda\mathfrak{c}) = \lambda l(\mathfrak{c})$, $g_{a+\lambda\mathfrak{c}}(s) = a + \lambda g_{\mathfrak{c}}(\lambda^{-1}s)$, $0 \leqq s \leqq l(a + \lambda\mathfrak{c})$. A curve $\mathfrak{c}$ is *simple* if it has no multiple points, i.e., if $g_{\mathfrak{c}}$ is injective.

3.1. Lemma. *If $p, q \in \partial\Sigma$ and $\mathfrak{c}$ is a curve from p to q in $X \backslash \Sigma_0$ with $\varrho(\mathfrak{c}) > 1$, there exists a curve $\mathfrak{c}_1$ from p to q in* co $\mathfrak{c}\backslash\Sigma_0$ *such that $l(\mathfrak{c}_1) \leqq l(\mathfrak{c})$ and $\varrho(\mathfrak{c}_1) < \varrho(\mathfrak{c})$.*

Proof. Let $\mathfrak{C}$ be the class of all curves $\mathfrak{c}'$ from p to q in co $\mathfrak{c}\backslash\Sigma_0$ such that $l(\mathfrak{c}') \leqq l(\mathfrak{c})$; this class contains $\mathfrak{c}$, hence is not empty. Set $\varrho = \varrho(\mathfrak{c})$. Suppose the conclusion to be false: then every $\mathfrak{c}' \in \mathfrak{C}$ satisfies $\varrho \leqq \varrho(\mathfrak{c}') \leqq \sup\{\|x\| : x \in \text{co}\,\mathfrak{c}\} \leqq$ $\leqq \varrho$, so that $\varrho(\mathfrak{c}') = \varrho > 1$. We may then set $\sigma(\mathfrak{c}') = \max\{s : \|g_{\mathfrak{c}'}(s)\| = \varrho\}$ for each $\mathfrak{c}' \in \mathfrak{C}$, since the set of numbers on which the maximum is taken is compact and not empty.

We claim that we can construct, for every $\mathfrak{c}' \in \mathfrak{C}$, some $\mathfrak{c}'' \in \mathfrak{C}$ such that $\sigma(\mathfrak{c}'') \leqq \sigma(\mathfrak{c}') - (\varrho - 1)$; repeated application of this construction will yield a contradiction, since $\varrho - 1 > 0$ and obviously $\sigma(\mathfrak{c}') \geqq 0$ for every $\mathfrak{c}' \in \mathfrak{C}$.

We proceed to establish our claim. We set $u = g_{\mathfrak{c}'}(\sigma(\mathfrak{c}'))$; now

$$\varrho - 1 = \|u\| - \|p\| \leqq \|u - p\| \leqq \sigma(\mathfrak{c}') \leqq l(\mathfrak{c}') - \|q - u\| \leqq l(\mathfrak{c}') - (\|u\| - \|q\|) = l(\mathfrak{c}') - (\varrho - 1);$$

we may therefore consider the points v, w of $\mathfrak{c}'$ defined by $v = g_{\mathfrak{c}'}(\sigma(\mathfrak{c}') - (\varrho - 1))$, $w = g_{\mathfrak{c}'}(\sigma(\mathfrak{c}') + (\varrho - 1))$. On account of the definition of $\sigma(\mathfrak{c}')$ we have $\|w\| < \varrho$. The curve $\mathfrak{c}''$ is now obtained from $\mathfrak{c}'$ by replacing the piece with arc lengths $s \in [\sigma(\mathfrak{c}') - (\varrho - 1), \sigma(\mathfrak{c}') + (\varrho - 1)]$ by the straight-line segment vw, traversed once from v to w. Clearly $\mathfrak{c}'' \subset \operatorname{co} \mathfrak{c}' \subset \operatorname{co} \mathfrak{c}$ and $l(\mathfrak{c}'') \leqq l(\mathfrak{c}') \leqq l(\mathfrak{c})$. Further, for any interior point $(1-\lambda) v + \lambda w, 0 < \lambda < 1$, of the segment

$$\begin{aligned} \varrho = (1-\lambda)\varrho + \lambda\varrho > (1-\lambda)\|v\| + \lambda\|w\| \geqq \|(1-\lambda)v + \lambda w\| \geqq \\ \geqq \|u\| - (1-\lambda)\|u - v\| - \lambda\|u - w\| \geqq \varrho - (1-\lambda)(\varrho - 1) - \lambda(\varrho - 1) = 1\,; \end{aligned}$$

thus $\mathfrak{c}'' \cap \Sigma_0 = \emptyset$, so that $\mathfrak{c}'' \in \mathfrak{C}$; and every point of $\mathfrak{c}''$ beyond the arc-length $\sigma(\mathfrak{c}') - (\varrho - 1)$ is either an interior point of the segment, or a point of $\mathfrak{c}'$ at or beyond the arc-length $\sigma(\mathfrak{c}') + (\varrho - 1)$, so that its norm is $< \varrho$ in either case. Therefore $\sigma(\mathfrak{c}'') \leqq \sigma(\mathfrak{c}') - (\varrho - 1)$, as claimed.

3.2. Lemma. *If $p, q \in \partial\Sigma$, and $\mathfrak{c}$ is a finite-dimensional curve from p to q in $X \setminus \Sigma_0$, there exists a (finite-dimensional) simple curve $\mathfrak{c}_0$ from p to q in* $\operatorname{co} \mathfrak{c} \cap \partial\Sigma$ *with $l(\mathfrak{c}_0) \leqq l(\mathfrak{c})$.*

Proof. 1. $\mathfrak{c}$ is finite-dimensional and compact, hence $\operatorname{co} \mathfrak{c} \setminus \Sigma_0$ is finite-dimensional and compact. Define $\mathfrak{C}$ as in the proof of Lemma 3.1, and set $\varrho = \inf\{\varrho(\mathfrak{c}') : \mathfrak{c}' \in \mathfrak{C}\} \geqq 1$. Let $(\mathfrak{c}_n)$ be a sequence in $\mathfrak{C}$ with $\lim_{n\to\infty} \varrho(\mathfrak{c}_n) = \varrho$. Since all $\mathfrak{c}_n$ have the same initial and final points, and lengths bounded by $l(\mathfrak{c})$, there exists a subsequence that converges uniformly to a curve $\mathfrak{c}'_0 \in \mathfrak{C}$ (i.e., a subsequence of suitable parametrizations converges uniformly to a parametrization of $\mathfrak{c}'_0$; see [3; Th. (5.16), pp. 24—25]); obviously, $\varrho(\mathfrak{c}'_0) = \varrho$. We claim that $\varrho(\mathfrak{c}'_0) = 1$, so that $\mathfrak{c}'_0$ is in the finite-dimensional compact set $(\operatorname{co} \mathfrak{c} \setminus \Sigma_0) \cap \Sigma = \operatorname{co} \mathfrak{c} \cap (\Sigma \setminus \Sigma_0) = \operatorname{co} \mathfrak{c} \cap \partial\Sigma$: indeed, if $\varrho(\mathfrak{c}'_0) > 1$, there would exist, by Lemma 3.1 applied to $\mathfrak{c}'_0$, a curve $\mathfrak{c}_1$ from p to q in $\operatorname{co} \mathfrak{c}'_0 \setminus \Sigma_0 \subset \operatorname{co} \mathfrak{c} \setminus \Sigma_0$ with $l(\mathfrak{c}_1) \leqq l(\mathfrak{c}'_0) \leqq l(\mathfrak{c})$ — so that $\mathfrak{c}_1 \in \mathfrak{C}$ — but with $\varrho(\mathfrak{c}_1) < \varrho(\mathfrak{c}'_0) = \varrho$, which contradicts the definition of ϱ.

2. The curve $\mathfrak{c}'_0$ satisfies all requirements, except perhaps that of being simple; its existence shows that p, q are connected by a curve in the (finite-dimensional) compact set $\operatorname{co} \mathfrak{c} \cap \partial\Sigma$; hence there is a "shortest join", i.e., a curve $\mathfrak{c}_0$ from p to q in $\operatorname{co} \mathfrak{c} \cap \partial\Sigma$ with minimum length, so that $l(\mathfrak{c}_0) \leqq l(\mathfrak{c}'_0) \leqq l(\mathfrak{c})$; and such a "shortest join" is a simple curve [3; (5.18), (5.19), pp. 25—26].

Remark. If X is a Banach space and convex hulls are replaced by closed convex hulls, Lemma 3.2 remains valid with "finite-dimensional" deleted in the assumption and the conclusion, since the closed convex hull of a compact set in a complete locally convex space is compact.

3.3. Theorem. (a): *If $p, q \in \partial\Sigma$, then*

$$\begin{aligned} &\inf\{l(\mathfrak{c}) : \mathfrak{c} \textit{ a curve from } p \textit{ to } q \textit{ in } X \setminus \Sigma_0\} \\ &= \inf\{l(\mathfrak{c}) : \mathfrak{c} \textit{ a curve from } p \textit{ to } q \textit{ in } \partial\Sigma\} \\ &= \inf\{l(\mathfrak{c}) : \mathfrak{c} \textit{ a simple curve from } p \textit{ to } q \textit{ in } \partial\Sigma\} \\ &= \inf\{l(\mathfrak{c}) : \mathfrak{c} \textit{ a finite-dimensional simple curve from } p \textit{ to } q \\ &\quad \textit{in } \partial\Sigma\} < \infty. \end{aligned} \tag{3.1}$$

(b): *If* $\dim X < \infty$, *these infima are attained.*

Proof. Proof of (a). We may assume without loss that $p \neq q$. Let $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4$ be, in order, the four sets of curves on which the infima are taken in (3.1). Now $\mathfrak{C}_1 \supset \mathfrak{C}_2 \supset \mathfrak{C}_3 \supset \mathfrak{C}_4$, and $\mathfrak{C}_1$ is not empty since $\dim X \geqq 2$; (3.1) will therefore be established if we show that for every $\mathfrak{c} \in \mathfrak{C}_1$ and every number $\varepsilon > 0$ there exists $\mathfrak{c}_0 \in \mathfrak{C}_4$ such that $l(\mathfrak{c}_0) \leqq l(\mathfrak{c}) + \varepsilon$.

Let $\mathfrak{c} \in \mathfrak{C}_1$ and $\varepsilon > 0$ be given; set $l = l(\mathfrak{c})$, $\eta = (l+2)^{-1}\varepsilon$, and choose an integer $m \geqq \frac{1}{2}(\eta^{-1}+1)\, l$. We define $\mathfrak{c}_1$ to be the broken line with consecutive vertices $p, u_0, u_1, \ldots, u_m, q$, where $u_i = (1+\eta)\, g_{\mathfrak{c}}(im^{-1} l)$, $i = 0, \ldots, m$; thus $\|u_i\| \geqq \geqq 1+\eta$ for all i, and $\|u_i - u_{i-1}\| \leqq (1+\eta)\, m^{-1} l$, $i = 1, \ldots, m$. Since $u_0 = (1+\eta)\, p$, $u_m = (1+\eta)\, q$, every point x of $\mathfrak{c}_1$ is of one of the forms $x = (1+\tau)\, p$ or $x = (1+\tau) q$, $0 \leqq \tau \leqq \eta$ — and $\|x\| = 1+\tau \geqq 1$ in either case —, or else $x = (1-\lambda)\, u_{i-1} + \lambda u_i$, $0 \leqq \lambda \leqq 1$, $1 \leqq i \leqq m$ — and then $\|x\| \geqq \|u_{i-1}\| - \lambda \|u_i - u_{i-1}\| \geqq (1+\eta)(1-\frac{1}{2}m^{-1} l) \geqq \geqq 1$ or $\|x\| \geqq \|u_i\| - (1-\lambda)\, \|u_i - u_{i-1}\| \geqq (1+\eta)\,(1-\frac{1}{2}m^{-1} l) \geqq 1$, according as $0 \leqq \lambda \leqq \frac{1}{2}$ or $\frac{1}{2} \leqq \lambda \leqq 1$. Thus $\mathfrak{c}_1$ is a finite-dimensional curve from p to q in $X \backslash \Sigma_0$, and $l(\mathfrak{c}_1) = \|u_0 - p\| + \|u_m - q\| + \sum_1^m \|u_i - u_{i-1}\| \leqq 2\eta + (1+\eta)\, l = l + \varepsilon$. We now apply Lemma 3.2 to $\mathfrak{c}_1$ and find $\mathfrak{c}_0 \in \mathfrak{C}_4$ with $l(\mathfrak{c}_0) \leqq l(\mathfrak{c}_1) \leqq l + \varepsilon$, as was to be shown.

Proof of (b). Since $\partial\Sigma$ is compact if $\dim X < \infty$, $\inf\{l(\mathfrak{c}) : \mathfrak{c} \in \mathfrak{C}_2\}$ is attained, and indeed at a simple curve, i.e., at some $\mathfrak{c}_0 \in \mathfrak{C}_4 = \mathfrak{C}_3$ [3; (5.18), (5.19), pp. 25—26].

For any $p, q \in \partial\Sigma$ we now define $\delta_X(p, q) = \inf\{l(\mathfrak{c}) : \mathfrak{c}$ a curve from p to q in $\partial\Sigma\}$, so that $\delta_X(p, q)$ — or $\delta(p, q)$ for short — is the common value of the infima in (3.1). Thus δ_X is the inner metric of $\partial\Sigma$ (cf. Theorem 3.5).

3.4. Theorem. *Let* $p, q \in \partial\Sigma$ *be given. Then:*

(a): *If* Y *is a subspace with* $\dim Y \geqq 2$ *and* $p, q \in Y$, *then* $\delta_Y(p, q) \geqq \delta_X(p, q)$.

(b): *For any integer* $n > 1 + \frac{1}{2}\delta_X(p, q)$ *and any* $\sigma > 0$ *there exists a subspace* Y *with* $2 \leqq \dim Y \leqq n$ *such that* $p, q \in Y$ *and*

$$\delta_Y(p, q) \leqq \delta_X(p, q)\,\bigl(1 - n^{-1}(1 + \tfrac{1}{2}\delta_X(p, q))\bigr)^{-1} + \sigma\,; \tag{3.2}$$

therefore

$$\delta_X(p, q) = \inf\{\delta_Y(p, q) : Y \text{ a subspace with } 2 \leqq \dim Y < \infty\}\,. \tag{3.3}$$

(c): *There exists a countable-dimensional subspace* Y *such that* $p, q \in Y$ *and* $\delta_Y(p, q) = \delta_X(p, q)$.

Proof. (a) is trivial, since $\partial\Sigma(Y) \subset \partial\Sigma$. We may assume $p \neq q$. Set $\delta_X(p, q) = \delta$, and let n, σ be as assumed. There exists a curve $\mathfrak{c}$ from p to q in $X \backslash \Sigma_0$ (or even in $\partial\Sigma$) with $l(\mathfrak{c}) = l$ so close to δ that $n > 1 + \frac{1}{2} l$ and $l(1 - n^{-1}(1+\frac{1}{2}l))^{-1} \leqq \leqq \delta(1 - n^{-1}(1+\frac{1}{2}\delta))^{-1} + \sigma$. We now carry out the construction in the proof of Theorem 3.3, (a), with $\varepsilon = (n(1+\frac{1}{2}l)^{-1} - 1)^{-1} l > 0$, $\eta = (l+2)^{-1}\varepsilon = \frac{1}{2}(n - (1+\frac{1}{2}l))^{-1} l$, and $m = \frac{1}{2}(\eta^{-1}+1) = n-1$. The $m+1 = n$ points $u_0, \ldots, u_m$ are contained in some subspace Y with $2 \leqq \dim Y \leqq n$, which also contains $p = (1+\eta)^{-1} u_0$ and $q = (1+\eta)^{-1} u_m$. The broken line $\mathfrak{c}_1$ therefore lies in Y, and so does the curve $\mathfrak{c}_0$. Thus

$$\delta_Y(p,q) \leqq l(\mathfrak{c}_0) \leqq l+\varepsilon = l(1-n^{-1}(1+\tfrac{1}{2}l))^{-1} \leqq \delta(1-n^{-1}(1+\tfrac{1}{2}\delta))^{-1}+\sigma,$$

and (3.2) holds for this Y. (3.3) is an immediate consequence. If $n_0 > 1+\frac{1}{2}\delta_X(p,q)$ and, for every $n \geqq n_0$, Y_n is a subspace with $2 \leqq \dim Y_n \leqq n$, $p, q \in Y_n$, that satisfies (3.2) with $\sigma = n^{-1}$, and if $Y = \sum_{n_0}^{\infty} Y_n$ is the countable-dimensional space spanned by the Y_n, (a) and (b) yield

$$\delta_X(p,q) \leqq \delta_Y(p,q) \leqq \delta_{Y_n}(p,q) \leqq \delta_X(p,q)\,(1-n^{-1}(1+\tfrac{1}{2}\delta_X p,q)))^{-1}+n^{-1} \to$$
$$\to \delta_X(p,q) \quad \text{as} \quad n \to \infty,$$

so that (c) holds.

Remark. If $\dim X < \infty$, we may choose $\mathfrak{c}$ in such a way that $l = \delta$ (Theorem 3.3, (b)); therefore (b) holds with $\sigma = 0$ in this case.

3.5. Theorem. *If* $p, q \in \partial\Sigma$, *then*

$$\|q-p\| \leqq \delta(p,q) \leqq 2\|q-p\|. \tag{3.4}$$

δ is a metric on $\partial\Sigma$, and is (uniformly) equivalent to the metric induced on $\partial\Sigma$ by the norm of X. If $\mathfrak{c}$ is a curve in $\partial\Sigma$, its length with respect to δ is equal to $l(\mathfrak{c})$.

Proof. The fact that $\|q-p\| \leqq \delta(p,q)$, that δ is a metric, and that curves have the same length in both metrics follows at once from the definitions. Since p, q certainly lie in some 2-dimensional subspace, Theorem 3.4, (a) shows that, in order to prove $\delta(p,q) \leqq 2\|q-p\|$ — from which the equivalence of the metrics follows — there is no loss in assuming that $\dim X = 2$. The proof of this special case will be given in Theorem 4.4.

We record a well-known fact about inner-product spaces.

3.6. Lemma. *If X is an inner-product space with the inner product $(\cdot,\cdot)$, then* $(p,q) = \cos\delta(p,q)$, $0 \leqq \delta(p,q) \leqq \pi$, *for all* $p, q \in \partial\Sigma$.

4. Two-dimensional spaces

Throughout this section, X is a normed plane ($\dim X = 2$). Now $\partial\Sigma$ is (the range of) a simple closed curve, unique up to orientation (rectifiability follows from convexity), the length of which we denote by $2L = 2L(X)$. If $p, q \in \partial\Sigma$, there are exactly two simple curves from p to q in $\partial\Sigma$, and the sum of their lengths is $2L$; $\delta(p,q)$ is of course the length of the shorter of these, so that

$$\delta(p,q) \leqq L, \tag{4.1}$$

(4.2) $\delta(p,q) = L$ (both curves have equal lengths L) if and only if $p+q=0$.

The following known result establishes the existence of one pair of "conjugate diameters".

4.1. Lemma. *There exist $u, v \in \partial\Sigma$ that are the midpoints of consecutive sides of a parallelogram containing Σ.*

Proof. See [5]. In the simpler of DAY's two proofs a euclidean metric is introduced in X, and any parallelogram of least euclidean area containing Σ rather obviously satisfies the conclusion. In order to use a compactness argument to show that the least area is attained, an a priori bound for the diameter of the parallelograms is required, but not mentioned there; now a parallelogram of euclidean heights h_1, h_2 and area A has a diameter (euclidean length of the longer diagonal) $A^{\frac{1}{2}}(A(h_1^{-2}+h_2^{-2})+2(A^2h_1^{-2}h_2^{-2}-1)^{\frac{1}{2}})^{\frac{1}{2}}$; if Σ has euclidean diameter d and width b, there is a rectangle of sides b, d containing Σ; we may therefore restrict the parallelograms containing Σ (which have $h_1, h_2 \geqq b$) to have $A \leqq bd$, so that their diameters are $\leqq (2d(d+(d^2-b^2)^{\frac{1}{2}}))^{\frac{1}{2}}$.

4.2. Theorem. $L \leqq 4$; $L=4$ *if and only if* Σ *is a parallelogram.*

Proof. Let $u, v \in \partial\Sigma$ be as given by Lemma 4.1. The broken line with consecutive vertices $-u, v-u, u+v, u$ is contained in the boundary of the parallelogram Π that contains Σ, hence is a curve from $-u$ to u in $X \backslash \Sigma_0$; its length is $\|v\|+2\|u\|+\|v\|=4$. By Theorem 3.3, $L=\delta(-u, u) \leqq 4$.

If Σ is a parallelogram with u, v as midpoints of consecutive sides, the vertices are $\pm u \pm v$, and the length of each side is 2; hence $L=4$.

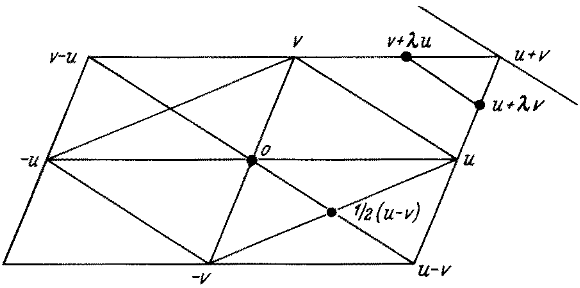

Fig. 1

Assume conversely that $L=4$, and let u, v, and the parallelogram Π be given as before (Fig. 1). If $\Sigma=\Pi$, Σ is a parallelogram, and the conclusion holds; otherwise, since Σ is convex and contained in Π, one of the vertices of Π does not belong to Σ; we assume that it is $u+v$ (otherwise interchange v and $-v$). Since the line through $u+v$ parallel to uv is a line of support of Π, there is a distinct parallel line of support of Σ that separates $u+v$ from the segment $uv \subset \Sigma$, and therefore contains points $v+\lambda u, u+\lambda v$ with $0 \leqq \lambda < 1$. Now the broken line with consecutive vertices $-u, v-u, v+\lambda u, u+\lambda v, u$ is a curve from $-u$ to u in $X \backslash \Sigma_0$, and therefore, by Theorem 3.3,

$$4=L=\delta(-u,u) \leqq \|v\|+(1+\lambda)\,\|u\|+(1-\lambda)\,\|u-v\|+\lambda\,\|v\|$$
$$=2(1+\lambda)+(1-\lambda)\,\|u-v\| \leqq 2(1+\lambda)+2(1-\lambda)=4\,;$$

equality must hold throughout; since $1-\lambda \neq 0$, this implies $\|u-v\|=2$, so that $\frac{1}{2}(u-v) \in \partial\Sigma$. Since this is the midpoint of the segment $u(-v)$, this whole segment lies in $\partial\Sigma$. Hence $u-v \notin \Sigma$, and interchanging v and $-v$ in the preceding argument we conclude that the segment uv is also contained in $\partial\Sigma$, and therefore Σ is the parallelogram with consecutive vertices $u, v, -u, -v$.

The following lemma belongs to the class of results on plane convex sets that are intuitively obvious, annoyingly awkward to prove, and probably published in some out-of-the way paper. Its ad-hoc proof is given for the sake of completeness.

4.3. Lemma. *Let* $p,q \in \partial\Sigma$ *be given*, $0 < \|q-p\| < 2$. *Set* $u = \|q-p\|^{-1}(q-p) \in \in \partial\Sigma$, *and let* $\mathfrak{s}$ *be the well-defined simple curve from* $-u$ *to* u *in* $\partial\Sigma$ *that contains* p, q. *Then* $\mathfrak{c} = \frac{1}{2}(p+q) + \frac{1}{2}\|q-p\|\,\mathfrak{s}$ *is a curve from* p *to* q *in* $X \backslash \Sigma_0$.

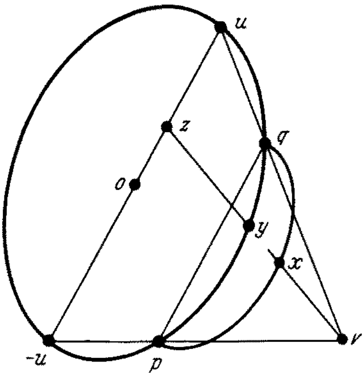

Fig. 2

Proof. (See Fig. 2). Since $\|q-p\| < 2$ — whence $p+q \neq 0$ — and $q-p = \|q-p\|\,u$, p and q are in the same open half-plane with edge $(-u)\,0\,u$; therefore $\mathfrak{s}$ is well defined, and $-u, p, q, u$ follow on it in that order. The lines $(-u)\,p$ and uq meet at $v = (2 - \|q-p\|)^{-1}(p+q)$; now $\|v\| = (2\|q\| - \|q-p\|)^{-1}\|p+q\| \geqq 1$, so $v \notin \Sigma_0$; and $2 - \|q-p\| > 0$, hence v lies in the same half-plane of edge $(-u)\,0\,u$ as p, q, and $\mathfrak{s}$. It follows from the convexity that Σ_0 is contained in the union of the open strip whose edges are the lines $(-u)\,0\,u$ and pq, and the open angle of vertex v and sides $vp(-u)$ and vqu. Now $\mathfrak{c}$ lies in the half-plane of edge pq that does not contain 0, hence contains v; $\mathfrak{c}$ lies a fortiori in the open half-plane of edge $(-u)\,0\,u$ that contains v.

Assume now by contradiction that $x \in \mathfrak{c} \cap \Sigma_0$; it follows from the preceding that x must be in the open triangle $v(-u)\,u$. The line vx meets the diameter $(-u)\,u$ in an interior point z, hence $z \in \Sigma_0$. Therefore the *segment* vz meets $\partial\Sigma$ in exactly one point, say y, and the segment vy does not meet Σ_0. Now y is the unique intersection of the line vx with $\partial\Sigma$ in the half-plane of edge $(-u)\,0\,u$ that contains v; it is hence the unique intersection of the line vx with $\mathfrak{s}$. On the other hand, $x \in \mathfrak{c}$ implies $x = \frac{1}{2}(p+q) + \frac{1}{2}\|q-p\|\,y'$ for some $y' \in \mathfrak{s}$, and $x - v = \frac{1}{2}\|q-p\|(y'-v)$; therefore $y = y'$, and x lies in the segment vy, which did not meet Σ_0; this contradicts the assumption on x.

Remark. It is not difficult to show that the restrictive assumptions on $\|q-p\|$ may be removed; the case $\|q-p\| = 2$, $p+q \neq 0$ requires a somewhat similar proof.

4.4. Theorem. *If* $p, q \in \partial\Sigma$, $\delta(p,q) \leqq \frac{1}{2}L\|q-p\| \leqq 2\|q-p\|$.

Proof. The second inequality follows from Theorem 4.2. The first inequality is trivial if $\|q-p\|=0$, and follows from (4.1) if $\|q-p\|=2$. We therefore assume that $0<\|q-p\|<2$; with $\mathfrak{s}$, $\mathfrak{c}$ as in Lemma 4.3, $\mathfrak{c}$ is a curve from p to q in $X\backslash\Sigma_0$, and $l(\mathfrak{c})=\frac{1}{2}\|q-p\|\, l(\mathfrak{s})=\frac{1}{2}L\|q-p\|$. The conclusion follows from Theorem 3.3 and the definition of δ.

Remark. The only result of Section 3 we have used in this section is Theorem 3.3; the proof of Theorem 3.5 by means of Theorem 4.4 therefore involves no circularity.

Theorem 4.2 gives an upper bound for L; we now seek a lower bound.

4.5. Theorem. $L\geqq 3$; $L=3$ *if and only if* Σ *is an affinely regular hexagon.*

Proof. 1. Let $u\in\partial\Sigma$ be given. The function $x\to\|x-u\|$ is continuous on the connected set $\partial\Sigma$, and has the values 0, 2 at $x=u$, $x=-u$, respectively. There exists therefore $v\in\partial\Sigma$ with $\|v-u\|=1$, i.e., $v-u\in\partial\Sigma$. The points $v-u$, v are in the same half-plane of edge $(-u)\,0\,u$, and $-u$, $v-u$, v, u follow in that order on $\partial\Sigma$, since $v-(v-u)=u$. Therefore

$$L=\delta(-u,u)\geqq\|(v-u)-(-u)\|+\|v-(v-u)\|+\|u-v\|=\|v\|+\|u\|+\|v-u\|=3.$$

2. If Σ is the affinely regular hexagon with consecutive vertices u, v, w, $-u$, $-v$, $-w$, we have $u-v+w=0$, and hence $L=\|w-(-u)\|+\|v-w\|+\|u-v\|=\|v\|+\|u\|+\|w\|=3$.

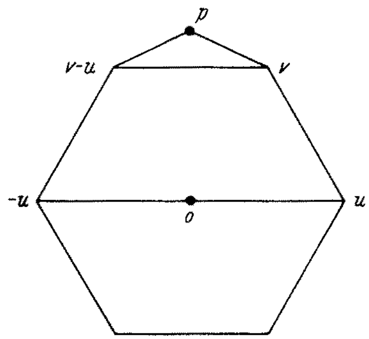

Fig. 3

3. Assume that $L=3$; choose u to be an extreme point of Σ, and let v be as in part 1 of the proof. We claim that the whole segment $(v-u)\,v$ lies in $\partial\Sigma$. Indeed, let p be the midpoint (in arc-length) of the shorter simple curve from $v-u$ to v in $\partial\Sigma$ (Fig. 3). We have

$$\begin{aligned}3=L=\delta(-u,u)&\geqq\|(v-u)-(-u)\|+\delta(v-u,p)+\delta(p,v)+\|u-v\|\\&=2+\delta(v-u,p)+\delta(p,v)\geqq 2+\|p-(v-u)\|+\|v-p\|\geqq 2+\|v-(v-u)\|=3.\end{aligned}$$

Therefore equality holds at each step, and we must have $\|p-(v-u)\|=\delta(v-u,p)=\delta(p,v)=\|v-p\|$ and $\|p-(v-u)\|+\|v-p\|=\|v-(v-u)\|=1$, whence $\|p-(v-u)\|=\|v-p\|=\frac{1}{2}$. Thus $2u+2(p-v)$, $2(v-p)\in\partial\Sigma$; but u is the midpoint of the segment having these endpoints, and was assumed to be an extreme point of Σ. Therefore both points must coincide with u, whence

$p = v - \frac{1}{2}u$, the midpoint of the segment $(v-u)\,v$; since $p \in \partial\Sigma$, the whole segment lies in $\partial\Sigma$.

If v is not an extreme point of Σ, the preceding implies that $v + \lambda u \in \partial\Sigma$ for some sufficiently small $\lambda > 0$; we assume $\lambda \leqq 1$. But then $\|(v+\lambda u) - u\| = \|\lambda v + (1-\lambda)(v-u)\| = 1$, and

$$3 = L = \delta(-u, u) \geqq \|(v-u) - (-u)\| + \|(v+\lambda u) - (v-u)\| + \|(v+\lambda u) - u\| = 1 + (1+\lambda) + 1 = 3 + \lambda > 3\,,$$

which is absurd.

Therefore v is also an extreme point, and by the same argument as above, applied to $v, v-u$ instead of u, v, it follows that the segment $(-u)(v-u)$ lies in $\partial\Sigma$; then $v-u$ is also an extreme point, and the segment uv lies in $\partial\Sigma$. Thus Σ is the affinely regular hexagon with consecutive vertices $u, v, v-u, -u, -v, u-v$.

Remark. The fact that $3 \leqq L \leqq 4$ and that the extreme values are attained by the planes with an affinely regular hexagon and a parallelogram, respectively, as unit disks, was shown in [13], and is proved here for completeness; it was not shown there that the extreme values are attained by these planes only.

5. Inner diameter, perimeter, girth

We now define the parameters associated with a normed space that are the main objects of study in the sequel. X again denotes a normed space with $\dim X \geqq 2$. The parameters are:

$$\begin{aligned} D(X) &= \sup\{\delta(p,q) : p, q \in \partial\Sigma\} \\ M(X) &= \sup\{\delta(-p,p) : p \in \partial\Sigma\} \\ m(X) &= \inf\{\delta(-p,p) : p \in \partial\Sigma\}\,. \end{aligned} \tag{5.1}$$

We may call $D(X)$ the *inner diameter* of $\partial\Sigma$, $2M(X)$ the *perimeter* of Σ, and $2m(X)$ the *girth* of Σ (cf. Lemma 5.1, (a)).

5.1. Lemma. (a):

$$\begin{aligned} m(X) &= \inf\{l(\mathfrak{c}) : \mathfrak{c} \textit{ a [simple] curve in } \partial\Sigma \textit{ with antipodal initial and final points}\} \\ &= \tfrac{1}{2}\inf\{l(\mathfrak{s}) : \mathfrak{s} \textit{ a simple closed curve in } \partial\Sigma,\ -\mathfrak{s} = \mathfrak{s} \textit{ (set-theoretically)}\}\,. \end{aligned} \tag{5.2}$$

(b): *If* $\dim X < \infty$, *the suprema and infima in* (5.1), (5.2) *are attained.*

Proof. Proof of (a). The first equality in (5.2) follows from the definitions of m and δ (cf. Theorem 3.3). If $\mathfrak{s}$ is a simple closed curve in $\partial\Sigma$, $-\mathfrak{s} = \mathfrak{s}$, and p is a point on $\mathfrak{s}$, then $\mathfrak{s}$ is obtained by joining end-to-end a simple curve $\mathfrak{c}$ from $-p$ to p in $\partial\Sigma$ and its reflection $-\mathfrak{c}$; and $l(\mathfrak{c}) = \frac{1}{2}l(\mathfrak{s})$, so that, in (5.2), $m(X) \leqq \frac{1}{2}\inf l(\mathfrak{s})$. Conversely, let $\mathfrak{c}$ be a simple curve from $-p$ to p in $\partial\Sigma$; $\mathfrak{c}$ joined end-to-end with $-\mathfrak{c}$ need not be a *simple* closed curve, since there may exist on $\mathfrak{c}$ a pair of antipodal points distinct from $-p, p$. Consider therefore the set $\{s_2 - s_1 : 0 \leqq s_1 \leqq s_2 \leqq l(\mathfrak{c}), g_{\mathfrak{c}}(s_1) + g_{\mathfrak{c}}(s_2) = 0\}$, which is non-empty (it contains $l(\mathfrak{c})$), compact, and bounded away from $0 (s_2 - s_1 \geqq \|g_c(s_2) - g_c(s_1)\| = 2)$. This set has a positive minimum, attained, say, for $s_1 = \sigma_1$, $s_2 = \sigma_2$. Then the arc of $\mathfrak{c}$ between arc-lengths σ_1, σ_2 contains no pair of antipodes except the endpoints; this arc, together with the corresponding arc of $-\mathfrak{c}$, constitute a simple closed curve $\mathfrak{s}$ in $\partial\Sigma$ with $-\mathfrak{s} = \mathfrak{s}$ and $l(\mathfrak{s}) = 2(\sigma_2 - \sigma_1) \leqq 2l(\mathfrak{c})$. Therefore, in (5.2), $m(X) \geqq \geqq \frac{1}{2}\inf l(\mathfrak{s})$, and equality holds.

Proof of (b). δ is continuous and, if $\dim X < \infty$, $\partial\Sigma$ is compact; therefore the suprema and infima in (5.1) are attained. The first infimum in (5.2) is attained in consequence of this and Theorem 3.3; the second infimum in (5.2) is attained, in view of the proof of (a), because the first one is.

Our main concern in what follows will be obtaining relations between D, M, m for the same space and for different spaces, and bounds for their values. The results in this section are somewhat scattered, and of varying degrees of obviousness. Much more is known about m than is about D and M.

5.2. Lemma. $2 \leqq m(X) \leqq M(X) \leqq D(X) \leqq 4$.

Proof. Obvious from the definitions and from Theorem 3.5.

5.3. Theorem. (a): *If* Y *is a subspace with* $\dim Y \geqq 2$, *then* $m(Y) \geqq m(X)$.

(b): *For any integer* $n > 1 + \frac{1}{2}m(X)$ *and any* $\sigma > 0$ *there exists a subspace* Y *with* $2 \leqq \dim Y \leqq n$ *such that*

$$m(Y) \leqq m(X)\left(1 - n^{-1}\left(1 + \tfrac{1}{2}m(X)\right)\right)^{-1} + \sigma\,; \tag{5.3}$$

therefore

$$m(X) = \inf\{m(Y) : Y \quad \text{a subspace}, \quad 2 \leqq \dim Y < \infty\}\,. \tag{5.4}$$

(c): *There exists a countable-dimensional subspace* Y *such that* $m(Y) = m(X)$.

Proof. (a) is trivial by Theorem 3.4, (a), since $\partial\Sigma(Y) \subset \partial\Sigma$. To prove (b) we assume n, σ given; by the definition of $m(X)$ there exists $p \in \partial\Sigma$ such that $n > 1 + \frac{1}{2}\delta_X(-p, p)$ and

$$\delta_X(-p, p)\left(1 - n^{-1}\left(1 + \tfrac{1}{2}\delta_X(-p, p)\right)\right)^{-1} \leqq m(X)\left(1 - n^{-1}\left(1 + \tfrac{1}{2}m(X)\right)\right)^{-1} + \tfrac{1}{2}\sigma\,.$$

We apply Theorem 3.4, (b) with $-p, p$ instead of p, q and $\frac{1}{2}\sigma$ instead of σ, and find (5.3), since $m(Y) \leqq \delta_Y(-p, p)$. (5.4) is an immediate consequence. The proof of (c) now follows from (a), (b) precisely as part (c) of Theorem 3.4 followed from parts (a), (b) of that theorem.

Remark. If $\dim X < \infty$, Lemma 5.1, (b) and the Remark to Theorem 3.4 show that (b) holds with $\sigma = 0$.

5.4. Theorem. *If* $\dim X = 2$, $3 \leqq m(X) = M(X) = D(X) = L(X) \leqq 4$; *the lower [upper] bound is attained if and only if* Σ *is an affinely regular hexagon [a parallelogram]*.

Proof. (4.1), (4.2), (5.1), and Theorems 4.2 and 4.5.

5.5. Theorem. *If* $\dim X < \infty$, $m(X) > 2$.

Proof. Assume, by contradiction, that $m(X) = 2$ (cf. Lemma 5.2). By Lemma 5.1, (b), there exists a point $p \in \partial\Sigma$ and a curve $\mathfrak{c}$ from $-p$ to p in $\partial\Sigma$ with $l(\mathfrak{c}) = 2$. If Y is the subspace spanned by $\mathfrak{c}$, we have $2 \leqq m(Y) \leqq l(\mathfrak{c}) = 2$, so we may assume without loss that $\mathfrak{c}$ spans X; set $\dim X = 1+n$, $n \geqq 1$. There exist points q_i on $\mathfrak{c}$, $i = 1, \ldots, n$, which, together with p, form a basis of X.

Now $2 = l(\mathfrak{c}) \geqq \|p+q_i\| + \|p-q_i\| \geqq \|p-(-p)\| = 2$, whence $\|p+q_i\| + \|p-q_i\| = 2$, $i=1, \ldots, n$. Since

$$p = \tfrac{1}{2}\|p+q_i\| \frac{p+q_i}{\|p+q_i\|} + \tfrac{1}{2}\|p-q_i\| \frac{p-q_i}{\|p-q_i\|}, \quad i=1, \ldots, n,$$

this makes p an interior point of each segment with endpoints

$$\|p \pm q_i\|^{-1}(p \pm q_i);$$

since these endpoints, and p, all lie in $\partial\Sigma$, so do the whole segments. The i-th segment has the direction of $2q_i + (\|p-q_i\| - \|p+q_i\|)\,p$; and these vectors, together with p, form a basis of X. Therefore p is an *interior* point of the (convex) intersection of $\partial\Sigma$ and a supporting hyperplane at p; and therefore there exists a point $q \neq p$ on $\mathfrak{c}$ that belongs to that intersection. Since the segment pq then lies in $\partial\Sigma$, so does its midpoint $\frac{1}{2}(p+q)$. But then $2 = l(\mathfrak{c}) \geqq \|p+q\| + \|q-p\| = 2 + \|q-p\| > 2$, a contradiction.

5.6. Theorem. *If* $\dim X \geqq 3$, $m(X) < 4$.

Proof. On account of Theorem 5.3, (a), there is no loss in assuming $\dim X = 3$. Assume, by contradiction, $m(X) = 4$ (cf. Lemma 5.2). If Y is any 2-dimensional subspace, $m(Y) = 4$ (Lemma 5.2, Theorem 5.3, (a)), and $\Sigma(Y)$ is a parallelogram (Theorem 5.4).

If p, q are distinct extreme points of Σ, and Y is a 2-dimensional subspace containing p, q, these points are extreme points of $\Sigma(Y)$, hence vertices of this parallelogram; therefore $\|q-p\| = 2$. The set of extreme points is thus finite, i.e., Σ is a polyhedron.

Let p, q, r be consecutive vertices of a face of Σ. By the preceding,

$$\|\tfrac{1}{2}(q+r) - \tfrac{1}{2}(p+q)\| = \tfrac{1}{2}\|r-p\| = 1.$$

But let Y be the 2-dimensional subspace containing $\frac{1}{2}(p+q), \frac{1}{2}(q+r)$; now these points are midpoints of edges of Σ, and either of these edges together with the midpoint of the other determine the plane of the face, which does not contain 0; therefore Y contains neither of these edges, and both points are vertices of the parallelogram $\Sigma(Y)$; hence $\|\frac{1}{2}(q+r) - \frac{1}{2}(p+q)\| = 2$, a contradiction.

The following two results concern 3-dimensional X; their proofs use this assumption strongly.

5.7. Lemma. *If* $\dim X = 3$ *and* $p, q \in \partial\Sigma$, *then* $2\delta(p,q) \leqq \delta(-p,p) + \delta(-q,q)$.

Proof. 1. By Theorem 3.3, (b), there exists a simple curve $\mathfrak{c}$ from $-p$ to p in $\partial\Sigma$, and a simple curve $\mathfrak{d}$ from $-q$ to q in $\partial\Sigma$, such that $l(\mathfrak{c}) = \delta(-p,p)$, $l(\mathfrak{d}) = \delta(-q,q)$. As in the proof of Lemma 5.1, (a), we see that there exist σ_1, σ_2, $0 \leqq \sigma_1 < \sigma_2 \leqq l(\mathfrak{c})$, such that the arc of $\mathfrak{c}$ between arc-lengths σ_1 and σ_2, joined end-to-end with the corresponding arc of $-\mathfrak{c}$, gives a simple closed curve $\mathfrak{s}$ in $\partial\Sigma$ with $-\mathfrak{s} = \mathfrak{s}$. Then $\partial\Sigma \setminus \mathfrak{s}$ consists of two components (Jordan Curve Theorem), and the mapping $u \to -u : \partial\Sigma \to \partial\Sigma$ maps each of these onto the other (rather than each onto itself) since it is orientation-reversing on $\partial\Sigma$, but orientation-preserving on $\mathfrak{s}$. Thus either $q\in\mathfrak{s}$, or $-q, q$ belong to different components of $\partial\Sigma\backslash\mathfrak{s}$; in either case, $\mathfrak{s}\cap\mathfrak{d}\neq\emptyset$, whence either $\mathfrak{c}\cap\mathfrak{d}\neq\emptyset$ or $-\mathfrak{c}\cap\mathfrak{d}\neq\emptyset$; we may assume the former without loss, since we might otherwise replace $\mathfrak{c}$ by $-\mathfrak{c}$ traversed in the opposite sense.

2. There exists, then, $u\in\mathfrak{c}\cap\mathfrak{d}$. Since reflection in 0 preserves δ, we have, using Theorem 3.5,

$$2(p,q)=\delta(p,q)+\delta(-p,-q)\leqq\delta(p,u)+\delta(u,q)+\delta(-p,u)+\delta(u,-q)\leqq$$
$$\leqq l(\mathfrak{c})+l(\mathfrak{d})=\delta(-p,p)+\delta(-q,q).$$

5.8. Theorem. *If* $\dim X=3$, *then* $M(X)=D(X)$.

Proof. For any $p,q\in\partial\Sigma$, Lemma 5.7 yields $\delta(p,q)\leqq\frac{1}{2}(\delta(-p,p)+\delta(-q,q))\leqq\leqq M(X)$. By (5.1), $D(X)\leqq M(X)$. The reverse inequality is trivial (Lemma 5.2).

We consider some special spaces.

5.9. Theorem. *If* X *is an inner-product space*, $m(X)=M(X)=D(X)=\pi$.

Proof. Lemma 3.6.

5.10. Lemma. *For every normed space* Z *with* $\dim Z\geqq 1$, $M(R\oplus Z)=D(R\oplus Z)=4$.

Proof. On account of Lemma 5.2 it is sufficient to show that if $\mathfrak{c}$ is a curve from $(-1)\oplus 0$ to $1\oplus 0$ in $\partial\Sigma(R\oplus Z)$, then $l(\mathfrak{c})\geqq 4$. Intuitively, it takes arcs of lengths at least 1, 2, 1 to get from the "centre" to the "rim" of the "bottom" of $\Sigma(R\oplus Z)$, from there to the "rim" of the "lid", and from there to the "centre" of the "lid", respectively.

More rigorously, define $\Phi: R\oplus Z\to R\oplus R$ by $\Phi(\lambda\oplus z)=\lambda\oplus\|z\|_Z$. Φ is obviously norm-preserving and distance-decreasing, hence continuous and curve-shortening. Thus $\Phi(\mathfrak{c})$ is a curve from $(-1)\oplus 0$ to $1\oplus 0$ in $\Phi(\partial\Sigma(R\oplus Z))\subset\subset\partial\Sigma(R\oplus R)$, and $\Sigma(R\oplus R)=[-1,1]\oplus[-1,1]$ is a parallelogram. By (4.2) and Theorem 4.2, $l(\mathfrak{c})\geqq l(\Phi(\mathfrak{c}))\geqq\delta_{R\oplus R}((-1)\oplus 0, 1\oplus 0)=L(R\oplus R)=4$.

For a given normed space Z, $\dim Z\geqq 1$, we define $R\oplus' Z$ as algebraically identical with $R\oplus Z$, but with the norm $\|\lambda\oplus z\|_{R\oplus' Z}=\max\{|\lambda|,\frac{1}{2}|\lambda|+\|z\|_Z\}$. Obviously, $\|\lambda\oplus z\|_{R\oplus Z}\leqq\|\lambda\oplus z\|_{R\oplus' Z}\leqq\frac{3}{2}\|\lambda\oplus z\|_{R\oplus Z}$, so that $R\oplus Z$ and $R\oplus' Z$ are isomorphic under the identity mapping.

5.11. Lemma. *For every normed space* Z *with* $\dim Z\geqq 1$, $D(R\oplus' Z)\leqq 3$.

Proof. Let $p=\lambda\oplus z\in\partial\Sigma(R\oplus' Z)$ be given. Then p lies in 2-dimensional subspace Y spanned by $1\oplus 0$ and some $0\oplus z_0$, $z_0\in\partial\Sigma(Z)$ (if $z\neq 0$, this subspace is unique). But $\Sigma(Y)$ is the affinely regular hexagon with consecutive vertices $0\oplus z_0, 1\oplus\frac{1}{2}z_0, 1\oplus -\frac{1}{2}z_0$, etc., as follows from the definition of the norm in $R\oplus' Z$. By Theorems 3.4, (a) and 4.5, and (4.2), we have $\delta(-p,1\oplus 0)+\delta(1\oplus 0,p)\leqq\leqq\delta_Y(-p,1\oplus 0)+\delta_Y(1\oplus 0,p)=3$. If $q\in\partial\Sigma(R\oplus' Z)$ is another point we thus have (cf. proof of Lemma 5.7):

$$2\delta(p,q)=\delta(p,q)+\delta(-p,-q)\leqq\delta(p,1\oplus 0)+\delta(1\oplus 0,q)+$$
$$+\delta(-p,1\oplus 0)+\delta(1\oplus 0,-q)\leqq 3+3=6,$$

whence $\delta(p,q)\leqq 3$. Since $p,q\in\partial\Sigma(R\oplus' Z)$ were arbitrary, the conclusion follows.

6. Isomorphism classes

This section constitutes a digression, of some independent interest, on the structure of classes of normed spaces. Its development goes somewhat beyond the immediate application to the study in hand.

In the sequel we shall be dealing with an *isomorphism class* $\mathbf{X}$, to be understood as the class of all normed spaces isomorphic to a given one; inasmuch as we are concerned with properties that are congruence-invariant, the set-theoretical difficulties in such a formulation can be overcome by assuming all the spaces in the class to have one and the same underlying linear space of the appropriate dimension; but we shall make no explicit use of this assumption. We denote by $\dim \mathbf{X}$ the common dimension of all spaces in $\mathbf{X}$. In particular, $\mathbf{X}_n$ will be the unique isomorphism class containing spaces of dimension n, $1 \leqq n < \infty$.

In the isomorphism class $\mathbf{X}$, congruence is an equivalence relation. If $X \in \mathbf{X}$, its *congruence class* – equivalence class with respect to congruence – is denoted by $\tilde{X}$; the collection of all congruence classes of $\mathbf{X}$ is denoted by $\tilde{\mathbf{X}}$. We intend to introduce in $\tilde{\mathbf{X}}$ a natural pseudo-metric.

If X, Y are isomorphic spaces, we set

$$\Delta(X, Y) = \inf\{\log \|T\| \, \|T^{-1}\| : T \text{ an isomorphism from } X \text{ to } Y\}.$$

The following lemma summarizes some obvious properties of Δ, the proof of which is left to the reader.

6.1. Lemma. *If X, Y, Z, W are isomorphic normed spaces, then*

$$\begin{aligned}
&\Delta(X, Y) = \Delta(Y, X) \geqq 0, \\
&\Delta(X, Z) \leqq \Delta(X, Y) + \Delta(Y, Z), \\
&\Delta(X, Y) = 0 \quad \text{if} \quad X, Y \quad \text{are congruent}, \\
&\Delta(X, Z) = \Delta(Y, W) \quad \text{if} \quad X, Y \quad \text{are congruent and} \quad Z, W \\
&\qquad\qquad\qquad\qquad \text{are congruent.}
\end{aligned}$$

On each isomorphism class $\mathbf{X}$, Δ is thus a congruence-invariant pseudo-metric. Every continuous function on $(\mathbf{X}, \Delta)$ is congruence-invariant.

If $\mathbf{X}$ is an isomorphism class, Lemma 6.1 shows that Δ induces a function $\tilde{\Delta}$ on $\tilde{\mathbf{X}} \times \tilde{\mathbf{X}}$, as follows: if $X, Y \in \mathbf{X}$, $\tilde{\Delta}(\tilde{X}, \tilde{Y}) = \Delta(X, Y)$.

6.2. Theorem. *For each isomorphism class $\mathbf{X}$, $\tilde{\Delta}$ is a pseudo-metric on $\tilde{\mathbf{X}}$. If φ is a continuous function on $(\mathbf{X}, \Delta)$, there exists a unique function $\tilde{\varphi}$ on $\tilde{\mathbf{X}}$ such that $\varphi(X) = \tilde{\varphi}(\tilde{X})$ for all $X \in \mathbf{X}$; and $\tilde{\varphi}$ is continuous on $(\tilde{\mathbf{X}}, \tilde{\Delta})$.*

Proof. Lemma 6.1 and the definition of $\tilde{\Delta}$.

Remark. A pseudo-metric very similar to $\tilde{\Delta}$ is described by DVORETZKY [6; p. 156] for finite-dimensional spaces (where it is a metric; see below); it is obviously related to the metric introduced by SHEPHARD [19] for convex sets; see also [20]. The following theorem is sketched by DVORETZKY [6; p. 156], and is similar to the results of MACBEATH [14] and SHEPHARD [19] for affine-equivalence classes of convex sets; we give a proof for completeness, and also because our definitions are slightly different from DVORETZKY's.

Added in proof. The author became aware, after submitting the manuscript of this paper, of the fact that the pseudo-metrics Δ and $\tilde{\Delta}$ had been introduced, for Banach spaces and congruence classes of such, respectively, by BANACH and MAZUR: see [2; pp. 242—243], where BANACH queries whether $\tilde{\Delta}$ is indeed non-zero for distinct congruence classes (i.e., a metric).

6.3. Theorem. *For each integer* $n \geqq 1$, $(\tilde{\mathbf{X}}_n, \tilde{\Delta})$ *is a compact metric space.*

Proof. 1. If $X, Y \in \mathbf{X}_n$, with $\tilde{\Delta}(\tilde{X}, \tilde{Y}) = \Delta(X, Y) = 0$, there exists a sequence (T_j) of isomorphisms from X to Y with $\lim_{j\to\infty} \|T_j\| \, \|T_j^{-1}\| = 1$. Since T_j may be replaced by $\|T_j^{-1}\| \, T_j$ without altering the value of the limit, we may assume without loss that $\lim_{j\to\infty} \|T_j\| = \lim_{j\to\infty} \|T_j^{-1}\| = 1$. In the n^2-dimensional space of (bounded) linear mappings from X to Y, the bounded sequence (T_j) has a subsequence converging to, say, T; and since the sequence of inverses is bounded, T is invertible, i.e., an isomorphism; and $\|T\| = \lim_{j\to\infty} \|T_j\| = 1$, $\|T^{-1}\| = \lim_{j\to\infty} \|T_j^{-1}\| = 1$, so that T is a congruence. Thus $\tilde{X} = \tilde{Y}$. We conclude that $\tilde{\Delta}$ is a metric on $\tilde{\mathbf{X}}_n$.

2. We consider an n-dimensional linear space E, and in it an n-dimensional ellipsoid S with its centre at 0; thus S is the unit ball of a euclidean space E_S. Let $\mathbf{K}$ be the class of all closed convex sets $K \subset E$ such that $-K = K$, $S \subset K \subset \subset n^{\frac{1}{2}} S$ ("closed" means radially closed or, equivalently, closed in the natural Hausdorff topology of E, i.e., the topology of E_S). On $\mathbf{K}$ we define the *Hausdorff metric* with respect to S:

$$\Delta_H(K, K') = \inf\{\lambda : K \subset K' + \lambda S, K' \subset K + \lambda S\}.$$

It is well known that the infimum is attained and that Δ_H is a metric; and, by the Blaschke Selection Theorem (cf., e.g., [7; pp. 64—67]), $(\mathbf{K}, \Delta_H)$ is compact.

3. In view of the assumption on $\mathbf{K}$, there is associated with each $K \in \mathbf{K}$ the unique normed space $E_K \in \mathbf{X}_n$ that has E as underlying linear space and $\Sigma(E_K) = K$: its norm is the Minkowski functional of K.

We consider the mapping $\Phi : K \to \tilde{E}_K : (\mathbf{K}, \Delta_H) \to (\tilde{\mathbf{X}}_n, \tilde{\Delta})$. This mapping is uniformly continuous, in fact Lipschitzian: indeed, let $K, K' \in \mathbf{K}$ be given, set $\lambda = \Delta_H(K, K')$, and let T be the identity mapping from E_K to $E_{K'}$. We have $K \subset K' + \lambda S \subset (1+\lambda) K'$, $K' \subset K + \lambda S \subset (1+\lambda) K$, so that $\|T\|, \|T^{-1}\| \leqq 1 + \lambda = 1 + \Delta_H(K, K')$; and

$$\tilde{\Delta}(\Phi(K), \Phi(K')) = \Delta(E_K, E_{K'}) \leqq \log \|T\| \, \|T^{-1}\| \leqq$$
$$\leqq 2 \log(1 + \Delta_H(K, K')) \leqq 2 \Delta_H(K, K').$$

4. The mapping Φ from the compact space $(\mathbf{K}, \Delta_H)$ into $(\tilde{\mathbf{X}}_n, \tilde{\Delta})$ is thus continuous; to complete the proof, it remains to show that Φ is surjective. For any $X \in \mathbf{X}_n$, let V be some bijective linear mapping from X to E; then $V\Sigma(X)$ is a symmetric bounded closed convex set in E. By a result of JOHN [11], there exists a bijective linear mapping $W : E \to E$ such that $S \subset W V \Sigma(X) \subset n^{\frac{1}{2}} S$, i.e., $W V \Sigma(X) \in \mathbf{K}$. Now obviously $W V$ is a congruence from X to $E_{W V \Sigma(X)}$, so that $\Phi(W V \Sigma(X)) = \tilde{X}$. Since $\tilde{X}$ was an arbitrary element of $\tilde{\mathbf{X}}_n$, Φ is surjective.

6.4. Corollary. *If φ is a real-valued continuous function on $(\mathbf{X}_n, \Delta)$, φ attains its infimum and supremum on X_n.*

Proof. Theorems 6.2 and 6.3.

7. Continuity of the parameters

We use the concepts introduced in the preceding section in order to compare the metrical properties of unit spheres in different normed spaces.

Let X, Y be isomorphic normed spaces, $\dim X = \dim Y \geqq 2$, and let T be an isomorphism from X to Y. T induces the homeomorphism $\tau : \partial\Sigma(X) \to \partial\Sigma(Y)$ defined by $\tau(p) = \|Tp\|_Y^{-1} Tp$—its inverse is given by $\tau^{-1}(p') = \|T^{-1}p'\|_X^{-1} T^{-1}p'$, as is easily verified.

7.1. Lemma. *Let X, Y, T, τ be as described. If $p, q \in \partial\Sigma(X)$, then*

$$|\delta_Y(\tau(p), \tau(q)) - \delta_X(p, q)| \leqq 6(\|T\| \, \|T^{-1}\| - 1).$$

Proof. 1. Let $\mathfrak{c}$ be a curve from p to q in $\partial\Sigma(X)$; the curve $\|T^{-1}\| \, T\mathfrak{c}$ is in $Y \backslash \Sigma_0(Y)$, since $x \in \mathfrak{c}$ implies $\|T^{-1}\| \, \|Tx\|_Y \geqq \|x\|_X = 1$; and $l(\|T^{-1}\| \, T\mathfrak{c}) \leqq \|T\| \, \|T^{-1}\| \, l(\mathfrak{c})$ (the former length measured in Y, the latter in X).

Let $\mathfrak{d}$ be the curve from $\tau(p)$ to $\tau(q)$ in $Y \backslash \Sigma_0(Y)$ consisting, consecutively, of the radial segment from $\tau(p)$ to $\|T^{-1}\| \, Tp$, the curve $\|T^{-1}\| \, T\mathfrak{c}$, and the radial segment from $\|T^{-1}\| \, Tq$ to $\tau(q)$. We have, using Theorem 3.3, (a),

$$\delta_Y(\tau(p), \tau(q)) \leqq l(\mathfrak{d}) = (\|T^{-1}\| \, \|Tp\|_Y - 1) + l(\|T^{-1}\| \, T\mathfrak{c}) + \\ + (\|T^{-1}\| \, \|Tq\|_Y - 1) \leqq 2(\|T\| \, \|T^{-1}\| - 1) + \|T\| \, \|T^{-1}\| \, l(\mathfrak{c}).$$

Since $\mathfrak{c}$ was an arbitrary curve from p to q in $\partial\Sigma(X)$, we have, using Theorem 3.5 (or Lemma 5.2)

$$\delta_Y(\tau(p), \tau(q)) \leqq 2(\|T\| \, \|T^{-1}\| - 1) + \|T\| \, \|T^{-1}\| \, \delta_X(p, q) \leqq \\ \leqq \delta_X(p, q) + 6(\|T\| \, \|T^{-1}\| - 1). \tag{7.1}$$

2. Repeating the same argument, with X, Y, T, τ, p, q replaced by $Y, X, T^{-1}, \tau^{-1}, \tau(p), \tau(q)$, respectively, we have

$$\delta_X(p, q) \leqq \delta_Y(\tau(p), \tau(q)) + 6(\|T^{-1}\| \, \|T\| - 1). \tag{7.2}$$

Combination of (7.1) and (7.2) yields the conclusion.

7.2. Theorem. *If X, Y are isomorphic normed spaces,* $\dim X = \dim Y \geqq 2$, *then*

$$|D(Y) - D(X)|, |M(Y) - M(X)|, |m(Y) - m(X)| \leqq 6(e^{\Delta(X,Y)} - 1). \tag{7.3}$$

For every isomorphism class $\mathbf{X}$ (except $\mathbf{X}_1$), D, M, m are continuous — indeed locally Lipschitzian — (congruence-invariant) functions on $(\mathbf{X}, \Delta)$.

Proof. For every isomorphism T from X to Y, the corresponding homeomorphism τ is antipode-preserving. The definitions and Lemma 7.1 then yield $|D(Y) - D(X)|$, $|M(Y) - M(X)|$, $|m(Y) - m(X)| \leqq 6(\|T\| \, \|T^{-1}\| - 1)$. Since T is an arbitrary isomorphism, the conclusion follows by the definition of Δ.

Before proceeding to discuss the extrema of the functions D, M, m on the isomorphism classes, we interpolate a striking application to the estimation of m of DVORETZKY's Sphericity Theorem [6], concerning "quasi-spherical" sections of convex sets of sufficiently high dimension. We first state, in our terminology, the special case of this theorem that we require.

7.3. Lemma. *There exists a number $\kappa > 0$ with the following property: for any $\varepsilon > 0$, any integer $n \geqq e^{\kappa^2 \varepsilon^{-2}}$, and any n-dimensional normed space X there exists a 2-dimensional subspace Y such that $\Delta(Y, E^2) \leqq \varepsilon$, where E^2 is a 2-dimensional euclidean space.*

Remark. This special case is stated, qualitatively, in [6; p. 156]. Observe that here the "asphericity" of $\Sigma(Y)$ — in DVORETZKY's terminology — is $1 - e^{-\varepsilon}$, rather than ε, but this does not affect the result, except possibly for the value of κ.

We have not computed an estimate of κ, since DVORETZKY's numerical estimates are likely to be very much too large, especially for the particular case of plane sections.

7.4. Theorem. *There exists a number $\kappa_0 > 0$ such that, if X is a normed space with $2 \leqq \dim X < \infty$, then $m(X) \leqq \pi + \kappa_0 \log^{-\frac{1}{2}}(\dim X)$.*

Proof. Set $n = \dim X$. By Lemma 7.3 there exists a 2-dimensional subspace Y of X such that $\Delta(Y, E^2) \leqq \kappa \log^{-\frac{1}{2}} n$. Now Theorem 7.2, with Theorems 5.3, (a) and 5.9, yields

$$m(X) \leqq m(Y) \leqq m(E^2) + 6(\exp(\kappa \log^{-\frac{1}{2}} n) - 1) \leqq \pi + 6\kappa \log^{-\frac{1}{2}} n \exp(\kappa \log^{-\frac{1}{2}} n),$$

and the conclusion follows, with $\kappa_0 = 6\kappa \exp(\kappa \log^{-\frac{1}{2}} 2)$.

7.5. Corollary. *If X is an infinite-dimensional normed space, then $m(X) \leqq \pi$.*

Proof. Theorems 7.4 and 5.3, (b).

8. Extreme values

We investigate the extreme values of the functions D, M, m, as the space varies in an isomorphism class. For obvious reasons we exclude once and for all the isomorphism class $\mathbf{X}_1$. If $\mathbf{X}$ is an isomorphism class, we define:

$$\begin{aligned} D^*(\mathbf{X}) &= \sup\{D(X) : X \in \mathbf{X}\} & D_*(\mathbf{X}) &= \inf\{D(X) : X \in \mathbf{X}\} \\ M^*(\mathbf{X}) &= \sup\{M(X) : X \in \mathbf{X}\} & M_*(\mathbf{X}) &= \inf\{M(X) : X \in \mathbf{X}\} \\ m^*(\mathbf{X}) &= \sup\{m(X) : X \in \mathbf{X}\} & m_*(\mathbf{X}) &= \inf\{m(X) : X \in \mathbf{X}\}. \end{aligned} \tag{8.1}$$

In particular, for each integer $n \geqq 2$, we set $D^*(n) = D^*(\mathbf{X}_n), \ldots, m_*(n) = m_*(\mathbf{X}_n)$. We say that $D^*(\mathbf{X})$, etc., *is attained* if the corresponding supremum or infimum in (8.1) is attained. Trivial bounds for these extrema are provided by Lemma 5.2.

Two of the extrema are easily determined:

8.1. Theorem. *For every isomorphism class $\mathbf{X}, D^*(\mathbf{X}) = M^*(\mathbf{X}) = 4$; both extrema are attained. In particular, $D^*(2) = M^*(2) = 4$, and each is attained exactly when $\Sigma(X)$ is a parallelogram.*

Proof. By Lemma 2.1 there exists a normed space Z with $\dim Z \geqq 1$ such that $R \oplus Z \in \mathbf{X}$. Therefore, by Lemma 5.10, $D^*(\mathbf{X}) \geqq D(R \oplus Z) = 4$, $M^*(\mathbf{X}) \geqq$ $\geqq M(R \oplus Z) = 4$. Equality follows by Lemma 5.2. The two-dimensional result follows from Theorem 5.4.

For the other extrema, we have the following basic application of the theory in Sections 6 and 7:

8.2. Theorem. *$D_*(n), M_*(n), m^*(n), m_*(n)$ are all attained for every n, $2 \leqq n < \infty$.*

Proof. Theorem 7.2 and Corollary 6.4.

Less is known abouth the values of D_*, M_* than is about those of m_*, m^*; we discuss the latter two first.

8.3. Theorem. (a): *$m_*(2) = 3$, and is attained exactly when $\Sigma(X)$ is an affinely regular hexagon.*

(b): *$2 < m_*(n) \leqq 3$ for all n, $2 \leqq n < \infty$; $(m_*(n))$ is a non-increasing sequence; if $m_*(\infty) = \lim_{n\to\infty} m_*(n)$, then $2 \leqq m_*(\infty) \leqq 3$, and*

$$(8.2) \qquad m_*(n) - m_*(\infty) \leqq n^{-1}((1+\tfrac{1}{2}m_*(\infty))^{-1} - n^{-1})^{-1} \leqq 15n^{-1} \quad \textit{for all } n \geqq 3\,.$$

(c): *For every infinite-dimensional isomorphism class $\mathbf{X}$, $m_*(\mathbf{X}) = m_*(\infty)$. For every infinite cardinal $\aleph$ there exists an isomorphism class $\mathbf{X}$ with $\dim \mathbf{X} = \aleph$, such that $m_*(\mathbf{X}) = m_*(\infty)$ is attained.*

Proof. Proof of (a). Theorem 5.4.

Proof of (b). By Theorems 5.5 and 8.2, $m_*(n) > 2$. If $n \geqq 2$, Theorem 5.3, (a) yields

$$m_*(n+1) \leqq \inf\{m(R \oplus X) : X \in \mathbf{X}_n\} \leqq m_*(n)$$

since $m(R \oplus X) \leqq m(\{0\} \oplus X) = m(X)$. Thus $(m_*(n))$ is non-increasing, and $m_*(n) \leqq m_*(2) = 3$. The limit $m_*(\infty)$ exists, and $2 \leqq m_*(\infty) \leqq 3$.

Let n, n' be integers, $n' \geqq n \geqq 3$. Let $X \in \mathbf{X}_{n'}$ be such that $m(X) = m_*(n')$ (Theorem 8.2). Since $1 + \frac{1}{2}m(X) \leqq 1 + \frac{1}{2} \cdot 3 < n$, we may apply Theorem 5.3, (b), with its appended Remark, and find a subspace Y of X, $\dim Y \leqq n$, such that

$$m_*(n) \leqq m_*(\dim Y) \leqq m(Y) \leqq m(X)\left(1 - n^{-1}(1 + \tfrac{1}{2}m(X))\right)^{-1}$$
$$= m_*(n')\left(1 - n^{-1}(1 + \tfrac{1}{2}m_*(n'))\right)^{-1}.$$

Letting n' tend to ∞ and observing that $m_*(\infty) \leqq 3 \leqq n$, we obtain (8.2).

Proof of (c). Let $\mathbf{X}$ be any infinite-dimensional isomorphism class. For every $X \in \mathbf{X}$, Theorem 5.3, (b) implies

$$m(X) = \inf\{m(Y) : Y \text{ a subspace, } 2 \leqq \dim Y < \infty\} \geqq$$
$$\geqq \inf\{m_*(n) : 2 \leqq n < \infty\} = m_*(\infty)\,;$$

thus $m_*(\mathbf{X}) \geqq m_*(\infty)$. Conversely, for each integer $n \geqq 2$ let $Y_n \in \mathbf{X}_n$ be such that $m(Y_n) = m_*(n)$ (Theorem 8.2). By Lemma 2.1, there exists a normed space Z_n such that $Y_n \oplus Z_n \in \mathbf{X}$. But then, using Theorem 5.3, (a),

$$m_*(\mathbf{X}) \leqq m(Y_n \oplus Z_n) \leqq m(Y_n \oplus \{0\}) = m(Y_n) = m_*(n)\,.$$

Since this holds for all n, $m_*(\mathbf{X}) \leqq \lim_{n\to\infty} m_*(n) = m_*(\infty)$, and equality holds.

With Y_n as in the preceding paragraph, we form $Y_\infty = \bigoplus_2^\infty Y_n$, the space uf all sequences $\bigoplus_2^\infty y_n$, $y_n \in Y_n$, with only finitely many non-zero terms, and with the norm $\left\|\bigoplus_2^\infty y_n\right\|_{Y_\infty} = \max_n \|y_n\|_{Y_n}$. Obviously $\dim Y_\infty = \aleph_0$, and since Y_∞ has, for each $n \geqq 2$, a subspace canonically congruent to Y_n, we have $m_*(\infty) \leqq \leqq m(Y_\infty) \leqq m(Y_n) = m_*(n)$. Since this holds for all n, we must have $m(Y_\infty) = m_*(\infty)$.

For any infinite cardinal $\aleph$, let Z be a normed space with $\dim Z = \aleph$; then $\dim(Y_\infty \oplus Z) = \aleph_0 + \aleph = \aleph$, and $m_*(\infty) \leqq m(Y_\infty \oplus Z) \leqq m(Y_\infty \oplus \{0\}) = m(Y_\infty) = m_*(\infty)$. If $\mathbf{X}$ is the isomorphism class of $Y_\infty \oplus Z$, then $\dim \mathbf{X} = \aleph$, and $m_*(\mathbf{X}) = m_*(\infty)$ is attained at $Y_\infty \oplus Z$.

8.4. Theorem. (a): $m^*(2) = 4$, *and is attained exactly when* $\Sigma(X)$ *is a parallelogram.*

(b): $\pi \leqq m^*(n) < 4$ *for all* n, $3 \leqq n < \infty$; $(m^*(n))$ *is a non-increasing sequence, with* $\lim_{n\to\infty} m^*(n) = \pi$, *and there exists a number* $\kappa_0 > 0$ *such that*

$$m^*(n) - \pi \leqq \kappa_0 \log^{-\frac{1}{2}} n, \quad \textit{for all} \quad n \geqq 2. \tag{8.3}$$

(c): *For every infinite-dimensional isomorphism class* $\mathbf{X}$, $m^*(\mathbf{X}) \leqq \pi$. *If* $\mathbf{X}$ *contains an inner-product space (and such* $\mathbf{X}$ *exists with* $\dim \mathbf{X} = \aleph$ *for each infinite cardinal* $\aleph$*), then* $m^*(\mathbf{X}) = \pi$ *is attained.*

Proof. Proof of (a). Theorem 5.4.

Proof of (b) *and* (c). 1. If $\mathbf{X}$ is an isomorphism class containing an inner-product space X, then $m^*(\mathbf{X}) \geqq m(X) = \pi$ (Theorem 5.9); in particular, $m^*(n) \geqq \geqq \pi$, $2 \leqq n < \infty$.

If $2 \leqq n < \infty$ and X is a normed space with $\dim X \geqq n$, there exists an n-dimensional subspace Y of X and hence, by Theorem 5.3, (a), $m(X) \leqq m(Y) \leqq \leqq m^*(n)$. Therefore $(m^*(n))$ is non-increasing, and $m^*(X) \leqq \lim_{n\to\infty} m^*(n)$ for every infinite-dimensional isomorphism class X.

2. By Theorems 5.6 and 8.2, $m^*(n) < 4$ for $3 \leqq n < \infty$. By part 1 of this proof and Theorem 7.4, $\pi \leqq m^*(n) \leqq \pi + \kappa_0 \log^{-\frac{1}{2}} n$, whence $\lim_{n\to\infty} m^*(n) = \pi$; and (8.3) and the conclusions in (c) follow.

8.5. Theorem. (a): $M_*(2) = D_*(2) = 3$, *and each is attained exactly when* $\Sigma(X)$ *is an affinely regular hexagon.*

(b): $M_*(3) = D_*(3) \leqq 3$.

(c): $M_*(\mathbf{X}) \leqq D_*(\mathbf{X}) \leqq 3$ *for every isomorphism class* $\mathbf{X}$.

Proof. Proof of (a). Theorem 5.4.

Proof of (b) *and* (c). By Lemma 2.1 there exists a normed space Z with $\dim Z \geqq 1$ such that $R \oplus Z \in \mathbf{X}$; but since $R \oplus Z$ is isomorphic to $R \oplus' Z$ (p. 71), Lemmas 5.2 and 5.11 yield $M_*(\mathbf{X}) \leqq D_*(\mathbf{X}) \leqq D(R \oplus' Z) \leqq 3$.

From this and from Theorem 5.8 we obtain $M_*(3) = D_*(3) \leqq 3$.

9. Conjectures and remarks

The reader will have observed that the results we have obtained do not penetrate very deeply beneath the surface. It is therefore appropriate that, in closing, we state some rather rash conjectures; if nothing else, disproving their strongest forms may produce enough insight and encouragement to obtain more substantial information. Although there are many points in the paper that raise questions, we restrict ourselves to the explicit statement of three points, leaving the others to the reader's curiosity. Conjecture 9.3 is the problem that originally motivated the research leading to this paper (see [17]).

9.1. *Conjecture. For every normed space* X *with* $\dim X \geqq 4$, $D(X) = M(X)$.

9.2. *Subsidiary conjecture. For every isomorphism class* $\mathbf{X}$, $D_*(\mathbf{X}) = M_*(\mathbf{X})$; *or, at least,* $D_*(n) = M_*(n)$, $4 \leqq n < \infty$.

9.3. *Conjecture.* $m_*(\infty) = 3$. *Equivalently,* $m_*(n) = 3$, $3 \leqq n < \infty$.

9.4. *Subsidiary conjectures.* (a): $M_*(\mathbf{X}) = D_*(\mathbf{X}) = 3$ *for every isomorphism class* X (Lemma 5.2, Theorems 8.3 and 8.5).

(b): $m_*(3) = 3$.

9.5. *Conjecture.* $m^*(3) = \pi$. *Equivalently,* $m^*(n) = \pi$, $3 \leqq n < \infty$.

Well-known results on the approximation of finite-dimensional convex sets (cf. [7; pp. 67—71]) may be restated in our terminology as follows, using the argument in the proof of Theorem 6.3: for each integer n, $2 \leqq n < \infty$, the class of spaces $X \in \mathbf{X}_n$ such that $\Sigma(X)$ is a polytope — i.e., has only a finite set of extreme points — is dense in $(\mathbf{X}_n, \Delta)$; and the same is true of the class of the $X \in \mathbf{X}_n$ with smooth $\Sigma(X)$, indeed with $\partial\Sigma(X)$ of any fixed degree of continuous differentiability. Since the parameters D, M, m are continuous in $(\mathbf{X}_n, \Delta)$, the attemps at proving or disproving the conjectures may thus be restricted to either class of spaces. The use of spaces with polytopic unit balls yields a kind of "combinatorial" approach, and some important steps for a reduction of Conjecture 9.3 by this method have been carried out by E.G. STRAUS (private communication). The use of spaces with smooth unit balls, on the other hand, suggests an approach by differential-geometric methods: here $\partial\Sigma(X)$ is a Finsler space, and the problems involve geodesics. The author does not conceal his surprise at the apparent intractability of even the three-dimensional case of Conjectures 9.3 and 9.5.

Added in proof. KLEE has shown that every finite-dimensional space with a polytopic unit ball is congruent to a subspace of the normed space — call it $\oplus R$ — of all sequences of real numbers with only finitely many non-zero terms provided with the maximum norm [12; Prop. 4.5], and consequently to a subspace of the Banach space c_0 (see [12; Prop. 4.7] for this together with a strong converse). It follows from the remarks of the preceding paragraph and from Theorems 5.3, (a) and 8.3 that $m_*(\infty) = m(\oplus R) = m(c_0)$.

It may be remarked, finally, that some additional results may be obtained if attention is restricted to spaces that are symmetric with respect to a maximal subspace, i.e., that admit a non-trivial self-congruence leaving such a subspace pointwise invariant. This matter will be dealt with elsewhere.

Added in proof. We have settled Conjecture 9.3 in the negative: indeed, $m_*(\infty) = 2$. This will be shown in an Addendum to the present paper.

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Addendum: Inner Diameter, Perimeter, and Girth of Spheres

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This note supplements the author's paper [3], which is assumed to be known and shall be referred to in the sequel as S, its contents being quoted as, e.g., Theorem S. 5.2, formula S(8.2). Our purpose is to examine the girth of cube-shaped(strictly speaking, parallelotopic) spheres and use them to refute Conjecture S. 9.3; the situation for large dimensions is indeed as bad as it can be: it turns out that $m_*(\infty)=2$.

Unless otherwise noted, the normed space X will be finite-dimensional. If $\dim X = n \geqq 2$ and the unit ball Σ is a polytope, its maximal (i.e., $(n-1)$-dimensional) faces shall be simply called its *faces*; their union is $\partial\Sigma$.

A curve obtained by joining end-to-end straight-line segments, each traversed once from one endpoint to the other, is a *polygon*; there is exactly one way of thus building up a polygon in which no consecutive segments are collinear and traversed in the same sense; the segments in this representation, considered as arcs, are the *edges* of the polygon, and their endpoints its *vertices*.

If Σ is a polytope, it seems obvious that shortest distances in $\partial\Sigma$ are attained by polygons:

1. Lemma. *If Σ is a polytope, $p, q \in \partial\Sigma$, there exists a polygon $\mathfrak{p}$ from p to q in $\partial\Sigma$ such that $l(\mathfrak{p}) = \delta(p, q)$. This $\mathfrak{p}$ is simple.*

Proof. We order the faces of Σ in some arbitrary way. By Theorem S. 3.3 there exists a curve $\mathfrak{c}$ from p to q such that $l(\mathfrak{c}) = \delta(p, q)$. We modify $\mathfrak{c}$ successively by replacing, at the mth stage, the arc between the "first" and "last" points of the curve (obtained at the preceding stage) in the mth face (in the chosen order) by the straight-line segment, traversed once, from the former to the latter. We end up with a polygon $\mathfrak{p}$ from p to q in $\partial\Sigma$ with $\delta(p, q) \leqq l(\mathfrak{p}) \leqq l(\mathfrak{c}) = \delta(p, q)$. $\mathfrak{p}$ is simple, since otherwise a strictly shorter polygon from p to q in $\partial\Sigma$ could be constructed from it in the obvious way, which would be absurd.

2. Lemma. *If Σ is a polytope, there exist $p \in \partial\Sigma$ and a simple polygon $\mathfrak{p}$ from $-p$ to p such that $m(X) = l(\mathfrak{p})$.*

Proof. By Lemma S. 5.1, (b), $m(X) = \min\{\delta(-p, p) : p \in \partial\Sigma\}$. The conclusion follows from Lemma 1.

We consider in particular, for $n = 2, 3, \ldots$, the space $R_n = \bigoplus_1^n R$, algebraically the outer direct sum of n copies of R (whence $\dim R_n = n$), and provided with the maximum norm $\left\| \bigoplus_1^n x^j \right\| = \max\{|x^j| : 1 \leqq j \leqq n\}$; this space is sometimes known as l_n^∞. Then $\Sigma_n = \Sigma(R_n) = \bigoplus_1^n [-1, 1]$ is a parallelotope, or "cube". The face of Σ_n defined by $x^j = 1\,[x^j = -1]$ is the *upper[lower] jth face* of Σ_n, $j = 1, \ldots, n$.

3. Lemma. $m(R_n) \geqq 2n(n-1)^{-1}$, $n = 2, 3, \ldots$

Proof. By Lemma 2 there exists $p \in \partial\Sigma_n$ and a polygon $\mathfrak{p}$ from p_0 to $-p_0$ with $l(\mathfrak{p}) = m(R_n)$; assume that its vertices are, in succession, $p_0, p_1, \ldots, p_{k-1}$, $p_k = -p_0$. Each edge of $\mathfrak{p}$ lies in some face of Σ_n; performing an appropriate congruence of R_n onto itself, if necessary, we may assume that the faces involved belong to exactly the 1st, 2nd, ..., rth pair of upper and lower faces, $1 < r \leqq n$. We consider the number $L = \sum_{j=1}^{r} \sum_{i=1}^{k} |p_i^j - p_{i-1}^j|$, where $p_i = \bigoplus_{j=1}^{n} p_i^j$. For each j, $1 \leqq j \leqq r$, some vertex, say $p_{h(j)}$, $1 \leqq h(j) \leqq k$, lies in the upper or lower jth face, so that $p_{h(j)}^j = \pm 1$ and $\sum_{i=1}^{k} |p_i^j - p_{i-1}^j| \geqq |p_{h(j)}^j - p_0^j| + |p_n^j - p_{h(j)}^j| = |\pm 1 - p_0^j| + |-p_0^j \mp 1| = 2$. Therefore

$$L \geqq \sum_{j=1}^{r} 2 = 2r. \tag{1}$$

On the other hand, for each $i, 1 \leqq i \leqq k$, we have $|p_i^j - p_{i-1}^j| \leqq \|p_i - p_{i-1}\|$, $j = 1, \ldots, r$; and the edge $p_{i-1}p_i$ lies in some face, say the (upper or lower) $q(i)$th, $1 \leqq q(i) \leqq r$, so that $p_i^{q(i)} = p_{i-1}^{q(i)} = \pm 1$. Therefore $\sum_{j=1}^{r} |p_i^j - p_{i-1}^j| \leqq \leqq (r-1)\|p_i - p_{i-1}\|$, and

$$L \leqq \sum_{i=1}^{k} (r-1)\|p_i - p_{i-1}\| = (r-1)\, l(\mathfrak{p}) = (r-1)\, m(R_n). \tag{2}$$

Comparison of (1), (2) yields $m(R_n) \geqq 2r(r-1)^{-1} \geqq 2n(n-1)^{-1}$.

4. Lemma. *Let $\mathfrak{p}_n$ be the polygon in $R_n (n \geqq 2)$ with successive vertices $p_0, p_1, \ldots, p_n$ given by $p_i = \bigoplus_{j=1}^{n} p_i^j$, where*

$$p_i^j = \begin{cases} (n-1)^{-1}(n-1-2(i-j)) & 1 \leqq j \leqq i \leqq n \\ (n-1)^{-1}(n+1-2(j-i)) & 0 \leqq i < j \leqq n \end{cases}$$

(whence $p_0 = -p_n$). Then $\mathfrak{p}_n$ is a simple polygon from p_0 to $p_n = -p_0$ in $\partial\Sigma_n$ and $l(\mathfrak{p}_n) = 2n(n-1)^{-1}$.

Proof. $|p_i^j| \leqq 1$ and $p_{i-1}^i = p_i^i = 1$, $i = 1, \ldots, n$, so that the edge $p_{i-1}p_i$ lies in the upper ith face of Σ_n, $i = 1, \ldots, n$; thus $\mathfrak{p}_n$ lies in $\partial\Sigma_n$; it is simple, since each edge lies in a different face of Σ_n. Further,

$$p_i^j - p_{i-1}^j = 2(n-1)^{-1}\operatorname{sgn}(j-i), \quad i, j = 1, \ldots, n, \tag{3}$$

so that $l(\mathfrak{p}_n) = \sum_{i=1}^{n} \|p_i - p_{i-1}\| = 2n(n-1)^{-1}$.

5. Theorem. $m(R_n) = 2n(n-1)^{-1}$, $n = 2, 3, \ldots$

Proof. Lemmas 3, 4.

6. Lemma. *With $\mathfrak{p}_n$ defined as in Lemma 4, the linear span of $\mathfrak{p}_n$ is all R_n if n is even, and is the $(n-1)$-dimensional subspace R'_{n-1} defined in R_n by $\sum_{1}^{n}(-1)^j x^j = 0$ if n is odd. In the latter case, $m(R'_{n-1}) = 2n(n-1)^{-1}$.*

Proof. The span of $\mathfrak{p}_n$ is the span of $p_1, \ldots, p_n$; it is also the span of $q_1, \ldots, q_n$, where $q_i = p_i - p_{i-1}$, $i = 1, \ldots, n$ (recall that $p_0 = -p_n$), for indeed $2p_i = \sum_{k=1}^{i} q_k - \sum_{k=i+1}^{n} q_k$, $i = 1, \ldots, n$. If $q_i = \bigoplus_{j=1}^{n} q_i^j$, the q_i^j are given by (3). The matrix $((q_i^j))$ is thus skew-symmetric, all elements with $j > i$ being equal; if n is even it is non-singular — its inverse is $((r_i^j))$ with $r_i^j = \frac{1}{2}(n-1)(-1)^{i+j}\operatorname{sgn}(j-i)$ — and if n is odd it is singular and its rank is $n-1$ (as a consequence of the even-order case). Also,
$$\sum_{j=1}^{n}(-1)^j q_i^j = 2(n-1)^{-1}\left(-\sum_{j=1}^{i-1}(-1)^j + \sum_{j=i+1}^{n}(-1)^j\right) = -2(n-1)^{-1}\sum_{j=1}^{n-1}(-1)^j,$$
$i = 1, \ldots, n$, and the sum vanishes when n is odd. This yields the conclusion. The last part of the statement follows, since $\mathfrak{p}_n$ is in $\partial\Sigma(R'_{n-1})$, whence $m(R'_{n-1}) \leqq \leqq l(\mathfrak{p}_n) = 2n(n-1)^{-1}$, but $m(R'_{n-1}) \geqq m(R_n) = 2n(n-1)^{-1}$ (Theorem S. 5.3, (a), Theorem 5).

7. Theorem. *$m_*(2n+1) \leqq m_*(2n) \leqq 2+n^{-1}$, $n=1,2,\ldots$, whence $m_*(\infty)=2$.*

Proof. The first inequality holds by Theorem S. 8.3. For given n and R'_{2n} as in Lemma 6, we have $\dim R'_{2n} = 2n$, whence $m_*(2n) \leqq m(R'_{2n}) = 2+n^{-1}$.

Remark 1. The bound $m_*(n) \leqq m(R_n) = 2+2(n-1)^{-1}$ is weaker.

Remark 2. Theorem 7 implies, for the infinite-dimensional spaces mentioned in S, p. 78, $m(\oplus R) = m(c_0) = 2$.

Beyond exploding Conjecture S. 9.3, and consequently rendering nugatory the remarks based on it in [2] and [1; Remark 2 to Theorem 111.D, p. 352], the results obtained here do not supersede any given in S, except that Theorem 7 is an improvement on the estimate S (8.2). In particular, the question still remains whether $m_*(3)=3$; we might replace the disproved conjecture, somewhat diffidently, by the query whether equality holds in Theorem 7 for all n.

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