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The Infinitesimal Group of the Navier-Stokes Equations*

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Summary — Zusammenfassung

The Infinitesimal Group of the Navier-Stokes Equations. The Navier-Stokes equations for an incompressible viscous fluid admit time translation, time dependent change of the pressure origin, a scale change, rotation of axes, and time dependent spatial translation. No other transformations appear if dependence on derivatives is allowed.

Die infinitesimale Gruppe der Navier-Stokes Gleichungen. Die Navier-Stokes Gleichungen für ein inkompressibles, viskoses Fluid lassen eine Zeitverschiebung, eine zeitabhängige Verschiebung des Druckursprungs, eine Maßstabsänderung, eine Verdrehung der Achsen und eine zeitabhängige, räumliche Verschiebung zu. Andere Transformationen erscheinen nicht, wenn eine Abhängigkeit von den Ableitungen zugelassen wird.

0. Introduction

The Lie theory originated as the investigation of the infinitesimal group of a differential equation [2, §§ 25—27]. A differential equation is said to admit a change of variables if it takes the same form in the new variables as in the old. Such changes of variables evidently constitute a group, called the group of the equation. The infinitesimal group of the equation consists of those members of the group which correspond to small changes of the variables; we need not be precise at this point. Powerful techniques are available when the infinitesimal group is known and nontrivial, e.g., the order of an ordinary differential equation reduces by one for each independent generator of the infinitesimal group. In this paper we determine the infinitesimal group of the Navier-Stokes equations of fluid dynamics [1, p. 147].

1. The Navier-Stokes Equations

The velocity field (u, v, w) and pressure p of an incompressible viscous fluid satisfy the Navier-Stokes equations

$$\begin{aligned} u_t + uu_x + vu_y + wu_z + p_x - \nu(u_{xx} + u_{yy} + u_{zz}) &= 0 \\ v_t + uv_x + vv_y + wv_z + p_y - \nu(v_{xx} + v_{yy} + v_{zz}) &= 0 \\ w_t + uw_x + vw_y + ww_z + p_z - \nu(w_{xx} + w_{yy} + w_{zz}) &= 0 \\ u_x + v_y + w_z &= 0. \end{aligned} \tag{1}$$

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The subscripts denote partial derivatives with respect to time t and the usual cartesian coordinates x, y, z. The modified pressure p has absorbed the density factor, assumed to be constant, and includes also the potential of the external force field, if such is present [1, p. 176]. The kinematic viscosity ν is assumed to be constant. It is a consequence of (1) that

$$p_{xx} + p_{yy} + p_{zz} + u_x{}^2 + v_y{}^2 + w_z{}^2 + 2u_y v_x + 2u_z w_x + 2v_z w_y = 0 \tag{2}$$

and we may adjoin this to (1) upon occasion.

In certain places we will use the following condensed notation for the variables.

The independent variables x, y, z, t are denoted by x^i, where the indexing set is $i \in \{x, y, z, t\}$. Repeated indices $i, j, \ldots$ are to be summed over $\{x, y, z, t\}$.

The dependent variables u, v, w, p are denoted by u^α, where the indexing set is $\alpha \in \{u, v, w, p\}$. Repeated indices $\alpha, \beta, \ldots$ are to be summed over $\{u, v, w, p\}$. Subscripts $i, j, \ldots$ on the $u^\cdot$ denote the corresponding partial derivatives.

2. Infinitesimal Transformations

We consider infinitesimal transformations of the form

$$\begin{aligned} (x^i)' &= x^i + \varepsilon\xi^i, \qquad & i \in \{x, y, z, t\}, \\ (u^\alpha)' &= u^\alpha + \varepsilon\xi^\alpha, \qquad & \alpha \in \{u, v, w, p\}, \end{aligned} \tag{3}$$

where ε^2 is negligible. Both the coordinates $x^\cdot$ and the field $u^\cdot$ vary, and each $\xi^\cdot$ may be a function of x, y, z, t, u, v, w, p [2, § 28]. The corresponding infinitesimal operator is

$$X = \xi^i \frac{\partial}{\partial x^i} + \xi^\alpha \frac{\partial}{\partial u^\alpha}; \tag{4}$$

a smooth function f of all of the variables changes to $f' = f + \varepsilon(Xf) + 0(\varepsilon^2)$ under (3). (Throughout, smooth may as well mean infinitely differentiable, although something weaker will often suffice.)

Since (1) is a second order system, it is necessary to obtain the second extension of (4); this is an operator

$$X_{(2)} = X + \xi_i{}^\alpha \frac{\partial}{\partial u_i{}^\alpha} + \xi_{ij}^\alpha \frac{\partial}{\partial u_{ij}^\alpha}. \tag{5}$$

The subscripts on the ξ's are labels, and any partial differentiations of the ξ's will be made explicit. If L denotes the column consisting of the four expressions on the left hand side in (1), the Navier-Stokes equations $L = 0$ admit the infinitesimal transformation (3) provided $X_{(2)}L = 0$ whenever $L = 0$ [2, § 28]. That is, we must have

$$\begin{gathered} \xi_t{}^u + u\xi_x{}^u + v\xi_y{}^u + w\xi_z{}^u + \xi^u u_x + \xi^v u_y + \xi^w u_z + \xi_x{}^p - \nu(\xi_{xx}^u + \xi_{yy}^u + \xi_{zz}^u) = 0, \\ (+\text{ two others}), \\ \xi_x{}^u + \xi_y{}^v + \xi_z{}^w = 0, \end{gathered} \tag{6}$$

holding as a consequence of (1).

The coefficients ξ_i^{α}, ξ_{ij}^{α} are worked out in Section 5. Suffice it to say for now that ξ_i^{α} involves first derivatives of the $\xi^{\cdot}$ and that ξ_{ij}^{α} involves second derivatives, and (6) modulo (1) is a system of homogeneous linear second order partial differential equations that the $\xi^{\cdot}$ must satisfy.

3. The Results

The details of the derivation are given in Section 5. System (1) admits the following linearly independent infinitesimal transformations, and no others.

(I) The operator

$$\frac{\partial}{\partial t}$$

generates the one parameter group of time translations $t' = t + h$, where $-\infty < h < \infty$ is a constant.

(II) With $G(t)$ an arbitrary smooth function, the operator

$$G(t)\,\frac{\partial}{\partial p}$$

corresponds to the transformation group $p' = p + g(t)$, where $g(t)$ is an arbitrary smooth function. The pressure change at each instant is uniform over the fluid and does not affect its motion. The group is not a Lie group.

(III) The integral form for the operator

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w} - 2p\frac{\partial}{\partial p}$$

is a one parameter group of scale changes:

$$\begin{gathered}(x', y', z') = k(x, y, z),\\ t' = k^2 t\\ (u', v', w') = (1/k)\,(u, v, w),\\ p' = (1/k^2)\,p, \qquad 0 < k < \infty.\end{gathered}$$

(IV) The infinitesimal rotations

$$y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} + v\frac{\partial}{\partial w} - w\frac{\partial}{\partial v}$$

$$z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + w\frac{\partial}{\partial u} - u\frac{\partial}{\partial w}$$

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}$$

generate a three parameter rotation group; the integral form need not be displayed. We note only that the velocities rotate with the coordinates.

(V) Let $E_x(t)$, $E_y(t)$, $E_z(t)$ be any smooth functions. Then (1) admits the time dependent displacement operators

$$E_x(t)\frac{\partial}{\partial x} + \dot{E}_x(t)\frac{\partial}{\partial u} - x\ddot{E}_x(t)\frac{\partial}{\partial p}$$

$$E_y(t)\frac{\partial}{\partial y} + \dot{E}_y(t)\frac{\partial}{\partial v} - y\ddot{E}_y(t)\frac{\partial}{\partial p}$$

$$E_z(t)\frac{\partial}{\partial z} + \dot{E}_z(t)\frac{\partial}{\partial w} - z\ddot{E}_z(t)\frac{\partial}{\partial p}.$$

The corresponding infinite dimensional group is not a Lie group:

$$\begin{aligned}
x' &= x + a(t)\\
y' &= y + b(t)\\
z' &= z + c(t)\\
u' &= u + \dot{a}(t)\\
v' &= v + \dot{b}(t)\\
w' &= w + \dot{c}(t)\\
p' &= p - x\ddot{a}(t) - y\ddot{b}(t) - z\ddot{c}(t),
\end{aligned}$$

where $a(t)$, $b(t)$, $c(t)$ are arbitrary smooth functions. The moving axes remain parallel to the fixed axes but the origin traces an arbitrary smooth path. The inertial reaction produced by the acceleration of the frame is balanced at each instant by a spatially constant pressure gradient.

It is to be emphasized that our considerations are entirely local in character. If a solution of (1) is given in a region and one of the above transformations is applied then the transformed quantities will satisfy (1) in the transformed region, with nothing said about boundary conditions.

Note that a Coriolis acceleration $2\boldsymbol{\Omega}\times\boldsymbol{u}$ cannot be balanced by a pressure gradient, in general, and the form of (1) is not preserved in moving axes if there is a non-vanishing angular velocity. In other words, the rotations (IV) cannot be time dependent.

4. Local Motion

The transformation group (V) may be useful in classifying the local behavior of solutions of (1), as follows. Let there be given a solution of (1) in a neighborhood of a point $P_0 = (x_0, y_0, z_0, t_0)$. The coordinates $(\alpha(t), \beta(t), \gamma(t))$ of a particle moving from P_0 with the fluid satisfy

$$\begin{aligned}
\dot{\alpha}(t) &= u(\alpha(t), \beta(t), \gamma(t), t)\\
\dot{\beta}(t) &= v(\alpha(t), \beta(t), \gamma(t), t)\\
\dot{\gamma}(t) &= w(\alpha(t), \beta(t), \gamma(t), t), \qquad t_0 \leqq t \leqq t_1,\\
\alpha(t_0) &= x_0, \qquad \beta(t_0) = y_0, \qquad \gamma(t_0) = z_0.
\end{aligned}$$

We assume that this system has a twice differentiable solution on a nonzero time interval. (It is sufficient to assume that u, v, w have continuous partial derivatives in a neighborhood of P_0.)

We introduce local coordinates, x', y', z' and local velocities u', v', w' according to

$$x' = x - \alpha(t), \qquad y' = y - \beta(t), \qquad z' = z - \gamma(t),$$

$$u'(x', y', z', t) = u\big(x' + \alpha(t), y' + \beta(t), z' + \gamma(t), t\big) - \dot{\alpha}(t), \text{ (similarly for } v', w').$$

The pressure in the local frame can be taken to be (we use also transformation (II)):

$$\begin{aligned} p'(x', y', z', t) &= p\big(x' + \alpha(t), y' + \beta(t), z' + \gamma(t), t\big) \\ &\quad - p\big(\alpha(t), \beta(t), \gamma(t), t\big) \\ &\quad + x'\ddot{\alpha}(t) + y'\ddot{\beta}(t) + z'\ddot{\gamma}(t). \end{aligned}$$

In the primed coordinates the relative field u', v', w', p' satisfies the Navier-Stokes equations for a fluid which remains at rest at the origin:

$$u'(0, 0, 0, t) = v'(0, 0, 0, t) = w'(0, 0, 0, t) = 0,$$
$$p'(0, 0, 0, t) = 0, \qquad t_0 \leqq t \leqq t_1.$$

Local stability against small disturbances which start within a small region around the moving origin may depend on a local Reynolds number appropriate to the region (cf. [5]). We do not pursue the matter here.

We remark at this point that the Bernoulli integral is essentially invariant for the transformations (V). That is, suppose the fluid velocity field is irrotational: a velocity potential ψ exists such that $\boldsymbol{u} = \nabla\psi$, in vector notation. The Bernoulli integral is the familiar $h = \dot{\psi} + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + p$, satisfying $\nabla h = 0$. It is clear that the primed field for transformations (V) above is also irrotational, with a velocity potential $\psi' = \psi + \dot{a}x' + \dot{b}y' + \dot{c}z'$, say. If h' denotes the Bernoulli integral formed with the quantities of the primed field, it is a straightforward matter to verify that $h' - h$ is a function only of t.

5. The Derivation

The explicit form of the coefficients in the second extension (5) is to be obtained from the recipe of [2, § 28], namely, for our tensor notation,

$$\xi_i^{\alpha} = \frac{d\xi^{\alpha}}{dx^i} - u_k^{\alpha}\frac{d\xi^k}{dx^i},$$

$$\xi_{ij}^{\alpha} = \frac{d\xi_i^{\alpha}}{dx^j} - u_{ik}^{\alpha}\frac{d\xi^k}{dx^j}.$$

Taking the total derivatives indicated, we find

$$\xi_i^{\alpha} = \frac{\partial\xi^{\alpha}}{\partial x^i} + \frac{\partial\xi^{\alpha}}{\partial u^{\beta}}u_i^{\beta} - \frac{\partial\xi^k}{\partial x^i}u_k^{\alpha} - \frac{\partial\xi^k}{\partial u^{\beta}}u_i^{\beta}u_k^{\alpha} \tag{7}$$

for the coefficients of the first extension, and for the second extension,

$$\begin{aligned}
\xi^{\alpha}_{ij} &= \frac{d}{dx^j}\left\{\frac{\partial\xi^\alpha}{\partial x^i} + \frac{\partial\xi^\alpha}{\partial u^\beta}u_i{}^\beta - \frac{\partial\xi^k}{\partial x^i}u_k{}^\alpha - \frac{\partial\xi^k}{\partial u^\beta}u_i{}^\beta u_k{}^\alpha\right\} - \left\{\frac{\partial\xi^k}{\partial x^j} + \frac{\partial\xi^k}{\partial u^\beta}u_j{}^\beta\right\}u^\alpha_{ik} \\
&= \frac{\partial^2\xi^\alpha}{\partial x^j\,\partial x^i} + \frac{\partial^2\xi^\alpha}{\partial u^\beta\,\partial x^i}u_j{}^\beta + \frac{\partial^2\xi^\alpha}{\partial x^j\,\partial u^\beta}u_i{}^\beta + \frac{\partial^2\xi^\alpha}{\partial u^\gamma\,\partial u^\beta}u_j{}^\gamma u_i{}^\beta \\
&\quad - \frac{\partial^2\xi^k}{\partial x^j\,\partial x^i}u_k{}^\alpha - \frac{\partial^2\xi^k}{\partial u^\beta\,\partial x^i}u_j{}^\beta u_k{}^\alpha - \frac{\partial^2\xi^k}{\partial x^j\,\partial u^\beta}u_i{}^\beta u_k{}^\alpha \\
&\quad - \frac{\partial^2\xi^k}{\partial u^\gamma\,\partial u^\beta}u_j{}^\gamma u_i{}^\beta u_k{}^\alpha + \frac{\partial\xi^\alpha}{\partial u^\beta}u^\beta_{ji} - \frac{\partial\xi^k}{\partial x^i}u^\alpha_{jk} - \frac{\partial\xi^k}{\partial x^j}u^\alpha_{ik} \\
&\quad - \frac{\partial\xi^k}{\partial u^\beta}\left(u^\beta_{ji}u_k{}^\alpha + u_i{}^\beta u^\alpha_{jk} + u_j{}^\beta u^\alpha_{ik}\right).
\end{aligned} \tag{8}$$

Equations (7) and (8) are general formulas for a second extension; nothing of (1) is involved.

We consider first the divergence condition in (6). That is, it is required that $\xi_x{}^u + \xi_y{}^v + \xi_z{}^w = 0$ hold by virtue of (1). The quantity in question is

$$\begin{aligned}
&\frac{\partial\xi^u}{\partial x} + \frac{\partial\xi^u}{\partial u}u_x + \frac{\partial\xi^u}{\partial v}v_x + \frac{\partial\xi^u}{\partial w}w_x + \frac{\partial\xi^u}{\partial p}p_x - \left[\frac{\partial\xi^x}{\partial x}u_x + \frac{\partial\xi^y}{\partial x}u_y + \frac{\partial\xi^z}{\partial x}u_z + \frac{\partial\xi^t}{\partial x}u_t\right] \\
&\quad - \left[\frac{\partial\xi^x}{\partial u}u_x{}^2 + \frac{\partial\xi^y}{\partial u}u_xu_y + \frac{\partial\xi^z}{\partial u}u_xu_z + \frac{\partial\xi^t}{\partial u}u_xu_t + \frac{\partial\xi^x}{\partial v}v_xu_x + \frac{\partial\xi^y}{\partial v}v_xu_y\right. \\
&\qquad + \frac{\partial\xi^z}{\partial v}v_xu_z + \frac{\partial\xi^t}{\partial v}v_xu_t + \frac{\partial\xi^x}{\partial w}w_xu_x + \frac{\partial\xi^y}{\partial w}w_xu_y + \frac{\partial\xi^z}{\partial w}w_xu_z + \frac{\partial\xi^t}{\partial w}w_xu_t \\
&\qquad \left. + \frac{\partial\xi^x}{\partial p}p_xu_x + \frac{\partial\xi^y}{\partial p}p_xu_y + \frac{\partial\xi^z}{\partial p}p_xu_z + \frac{\partial\xi^t}{\partial p}p_xu_t\right] + \frac{\partial\xi^v}{\partial y} \\
&\quad + \frac{\partial\xi^v}{\partial u}u_y + \frac{\partial\xi^v}{\partial v}v_y + \frac{\partial\xi^v}{\partial w}w_y + \frac{\partial\xi^v}{\partial p}p_y - \left[\frac{\partial\xi^x}{\partial y}v_x + \frac{\partial\xi^y}{\partial y}v_y + \frac{\partial\xi^z}{\partial y}v_z + \frac{\partial\xi^t}{\partial y}v_t\right] \\
&\quad - \left[\frac{\partial\xi^x}{\partial u}u_yv_x + \frac{\partial\xi^y}{\partial u}u_yv_y + \cdots\right] + \frac{\partial\xi^w}{\partial z} + \frac{\partial\xi^w}{\partial u}u_z + \frac{\partial\xi^w}{\partial v}v_z + \frac{\partial\xi^w}{\partial w}w_z + \frac{\partial\xi^w}{\partial p}p_z \\
&\quad - \left[\frac{\partial\xi^x}{\partial z}w_x + \frac{\partial\xi^y}{\partial z}w_y + \frac{\partial\xi^z}{\partial z}w_z + \frac{\partial\xi^t}{\partial z}w_t\right] - \left[\frac{\partial\xi^x}{\partial u}u_zw_x + \frac{\partial\xi^y}{\partial u}u_zw_y + \cdots\right].
\end{aligned}$$

Since no second derivatives of the $u^{\cdot}$ appear, the only substitution available from (1) is $u_x + v_y + w_z = 0$. The quadratic terms in the derivatives contribute

$$-\frac{\partial\xi^k}{\partial u^\beta}u_\sigma{}^\beta u_k{}^\sigma$$

where σ (summed) denotes u or x, v or y, w or z. This is independent of (1) unless it is proportional to $u_x + v_y + w_z = u_\sigma{}^\sigma$. The only proportionality possible is the vanishing one, i.e.,

$$\frac{\partial\xi^k}{\partial u^\beta} = 0.$$

This is to say, the coordinate displacements cannot involve the dependent variables.

The remaining conditions are more or less obvious:

$$\frac{\partial \xi^u}{\partial p} = \frac{\partial \xi^v}{\partial p} = \frac{\partial \xi^w}{\partial p} = 0$$

$$\frac{\partial \xi^t}{\partial x} = \frac{\partial \xi^t}{\partial y} = \frac{\partial \xi^t}{\partial z} = 0$$

because p_x, p_y, p_z and u_t, v_t, w_t do not appear in $u_x + v_y + w_z$, and

$$\frac{\partial \xi^u}{\partial x} + \frac{\partial \xi^v}{\partial y} + \frac{\partial \xi^w}{\partial z} = 0, \tag{9}$$

$$\frac{\partial \xi^u}{\partial u} - \frac{\partial \xi^x}{\partial x} = \frac{\partial \xi^v}{\partial v} - \frac{\partial \xi^y}{\partial y} = \frac{\partial \xi^w}{\partial w} - \frac{\partial \xi^z}{\partial z} = A, \qquad \text{say,}$$

$$\frac{\partial \xi^u}{\partial v} = \frac{\partial \xi^x}{\partial y}, \quad \frac{\partial \xi^u}{\partial w} = \frac{\partial \xi^x}{\partial z}, \quad \frac{\partial \xi^v}{\partial u} = \frac{\partial \xi^y}{\partial x},$$

$$\frac{\partial \xi^u}{\partial v} = \frac{\partial \xi^y}{\partial z}, \quad \frac{\partial \xi^w}{\partial u} = \frac{\partial \xi^z}{\partial x}, \quad \frac{\partial \xi^w}{\partial v} = \frac{\partial \xi^z}{\partial y},$$

where A may be a function of all of the variables. Integrability of this system requires

$$\frac{\partial A}{\partial u} = \frac{\partial A}{\partial v} = \frac{\partial A}{\partial w} = 0.$$

Since we already have $\partial A/\partial p = 0$, the solution is evidently

$$\begin{aligned}
\xi^u &= \left(\frac{\partial \xi^x}{\partial x} + A\right) u + \frac{\partial \xi^x}{\partial y} v + \frac{\partial \xi^x}{\partial z} w + B_x \\
\xi^v &= \frac{\partial \xi^y}{\partial x} u + \left(\frac{\partial \xi^y}{\partial y} + A\right) v + \frac{\partial \xi^y}{\partial z} w + B_y \\
\xi^w &= \frac{\partial \xi^z}{\partial x} u + \frac{\partial \xi^z}{\partial y} v + \left(\frac{\partial \xi^z}{\partial z} + A\right) w + B_z
\end{aligned} \tag{10}$$

where now A and B_x, B_y, B_z are functions of x, y, z, t. The first equations of (9) becomes

$$\begin{aligned}
\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\} \left[\frac{\partial \xi^x}{\partial x} + \frac{\partial \xi^y}{\partial y} + \frac{\partial \xi^z}{\partial z} + A\right] &= 0 \\
\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} &= 0.
\end{aligned} \tag{11}$$

The first equation of (6) has 683 terms, formally, when we use (7) and (8) for the explicit form of the coefficients. The conditions obtained so far produce substantial reductions, however. For one thing, all terms involving $u_. u_. u_.$ and $u_. u_{..}$ disappear. The remaining second derivative terms come from the combination $\xi^u_{xx} + \xi^u_{yy} + \xi^u_{zz}$ and are, omitting coefficient $-\nu$,

$$\left(\frac{\partial \xi^x}{\partial x} + A\right) \Delta u + \frac{\partial \xi^x}{\partial y} \Delta v + \frac{\partial \xi^x}{\partial z} \Delta w - 2 \left[\frac{\partial \xi^x}{\partial x} u_{xx} + \frac{\partial \xi^x}{\partial y} u_{xy} + \frac{\partial \xi^x}{\partial z} u_{xz} \right.$$
$$\left. + \frac{\partial \xi^y}{\partial x} u_{yx} + \frac{\partial \xi^y}{\partial y} u_{yy} + \frac{\partial \xi^y}{\partial z} u_{yz} + \frac{\partial \xi^z}{\partial x} u_{zx} + \frac{\partial \xi^z}{\partial y} u_{zy} + \frac{\partial \xi^z}{\partial z} u_{zz}\right]$$

where $\Delta u = u_{xx} + u_{yy} + u_{zz}$ denotes the usual Laplacian. This is independent of (1) unless the coefficients satisfy

$$\begin{gathered}\frac{\partial \xi^x}{\partial x} = \frac{\partial \xi^y}{\partial y} = \frac{\partial \xi^z}{\partial z} = C, \qquad \text{say,} \\ \frac{\partial \xi^x}{\partial y} + \frac{\partial \xi^y}{\partial x} = 0, \qquad \frac{\partial \xi^x}{\partial z} + \frac{\partial \xi^z}{\partial x} = 0, \qquad \frac{\partial \xi^y}{\partial z} + \frac{\partial \xi^z}{\partial y} = 0,\end{gathered} \tag{12}$$

where C may be a function of x, y, z, t. The conditions (12) are to the effect that the symmetric part of the tensor $\partial \xi^i / \partial x^j$ is scalar, i.e.,

$$\frac{1}{2}\left(\frac{\partial \xi^i}{\partial x^j} + \frac{\partial \xi^j}{\partial x^i}\right) = C\delta_{ij}, \qquad i, j = x, y, z.$$

The antisymmetric part is a curl, and we introduce functions D_x, D_y, D_z of x, y, z, t by

$$\begin{aligned} D_x &= -\frac{\partial \xi^y}{\partial z} = \frac{\partial \xi^z}{\partial y} \\ D_y &= -\frac{\partial \xi^z}{\partial x} = \frac{\partial \xi^x}{\partial z} \\ D_z &= -\frac{\partial \xi^x}{\partial y} = \frac{\partial \xi^y}{\partial x}. \end{aligned} \tag{13}$$

We now give (6) explicity. We use the simplifications found thus far, we substitute from (1) for the second derivatives, and we collect terms. The first Eq. of the set assumes the form

$$\begin{aligned}
&\left(2C - \frac{\partial \xi^t}{\partial t}\right) u_t + \left[-\frac{\partial \xi^x}{\partial t} + \nu \Delta \xi^x + (2C + A)\, u + \frac{\partial \xi^p}{\partial u} - 2\nu\left(\frac{\partial C}{\partial x} + \frac{\partial A}{\partial x}\right) + B_x\right] u_x \\
&\quad + \left[-\frac{\partial \xi^y}{\partial t} + \nu \Delta \xi^y + (2C + A)\, v - 2\nu\left(\frac{\partial C}{\partial y} + \frac{\partial A}{\partial y}\right) + B_y\right] u_y \\
&\quad + \left[-\frac{\partial \xi^z}{\partial t} + \nu \Delta \xi^z + (2C + A)\, w - 2\nu\left(\frac{\partial C}{\partial z} + \frac{\partial A}{\partial z}\right) + B_z\right] u_z \\
&\quad + \left[\frac{\partial \xi^p}{\partial v} - 2\nu \frac{\partial D_z}{\partial x}\right] v_x - 2\nu \frac{\partial D_z}{\partial y} v_y - 2\nu \frac{\partial D_z}{\partial z} v_z \\
&\quad + \left[\frac{\partial \xi^p}{\partial w} - 2\nu \frac{\partial D_y}{\partial x}\right] w_x - 2\nu \frac{\partial D_y}{\partial y} w_y - 2\nu \frac{\partial D_y}{\partial z} w_z \\
&\quad + \left(\frac{\partial \xi^p}{\partial p} - A\right) p_x + \left(\frac{\partial B_x}{\partial t} + \frac{\partial \xi^p}{\partial x}\right) \\
&\quad + \left(\frac{\partial C}{\partial x} + \frac{\partial A}{\partial x}\right) u^2 + \left(\frac{\partial C}{\partial y} + \frac{\partial A}{\partial y} - \frac{\partial D_z}{\partial x}\right) uv + \left(\frac{\partial C}{\partial z} + \frac{\partial A}{\partial z} + \frac{\partial D_y}{\partial x}\right) uw \\
&\quad - \frac{\partial D_z}{\partial y} v^2 + \left(\frac{\partial D_y}{\partial y} - \frac{\partial D_z}{\partial z}\right) vw + \frac{\partial D_y}{\partial z} w^2 \\
&\quad + \left[\frac{\partial C}{\partial t} + \frac{\partial A}{\partial t} - \nu \Delta (C + A) + \frac{\partial B_x}{\partial x}\right] u \\
&\quad + \left[-\frac{\partial D_z}{\partial t} + \nu \Delta D_z + \frac{\partial B_x}{\partial y}\right] v + \left[\frac{\partial D_y}{\partial t} - \nu \Delta D_y + \frac{\partial B_x}{\partial z}\right] w = 0.
\end{aligned} \tag{14}$$

The form of ξ^p is not yet determined, but it cannot involve the derivatives, so the coefficients of the derivatives must vanish. It is apparent that $A = -2C$ must hold, and the first set in (11) then gives

$$\frac{\partial C}{\partial x} = \frac{\partial^2 \xi^x}{\partial x^2} = \frac{\partial^2 \xi^y}{\partial x\, \partial y} = \frac{\partial^2 \xi^z}{\partial x\, \partial z} = 0$$

$$\frac{\partial C}{\partial y} = \frac{\partial^2 \xi^x}{\partial y\, \partial x} = \frac{\partial^2 \xi^y}{\partial y^2} = \frac{\partial^2 \xi^z}{\partial y\, \partial z} = 0$$

$$\frac{\partial C}{\partial z} = \frac{\partial^2 \xi^x}{\partial z\, \partial x} = \frac{\partial^2 \xi^y}{\partial z\, \partial y} = \frac{\partial^2 \xi^z}{\partial z^2} = 0,$$

implying that C depends at most on t. The v_y, v_z, w_y, w_z coefficient conditions in (14) are

$$\frac{\partial D_z}{\partial y} = -\frac{\partial^2 \xi^x}{\partial y^2} = \frac{\partial^2 \xi^y}{\partial y\, \partial x} = \Gamma, \qquad \text{say,}$$

$$\frac{\partial D_z}{\partial z} = -\frac{\partial^2 \xi^x}{\partial z\, \partial y} = \frac{\partial^2 \xi^y}{\partial z\, \partial x} = 0$$

$$\frac{\partial D_y}{\partial y} = -\frac{\partial^2 \xi^z}{\partial y\, \partial x} = \frac{\partial^2 \xi^x}{\partial y\, \partial z} = 0$$

$$\frac{\partial D_y}{\partial z} = -\frac{\partial^2 \xi^z}{\partial z\, \partial x} = \frac{\partial^2 \xi^x}{\partial z^2} = \Gamma.$$

We now argue that the second and third equations of (6) will give a similar set of conditions, to be obtained by cyclic permutation of the indices. The solution of the system is evidently $\Gamma = 0$ and

$$\xi^x = Cx - D_z y + D_y z + E_x$$

$$\xi^y = D_z x + Cy - D_x z + E_y$$

$$\xi^z = -D_y x + D_x y + Cz + E_z$$

where now the coefficients depend on t but not on x, y, z.

The remaining conditions from (14) are

$$\begin{aligned}
&\frac{\partial \xi^t}{\partial t} = 2C \\
&B_x = \frac{\partial \xi^x}{\partial t} \\
&B_y = \frac{\partial \xi^y}{\partial t} \\
&B_z = \frac{\partial \xi^z}{\partial t} \\
&\frac{\partial \xi^p}{\partial u} = \frac{\partial \xi^p}{\partial v} = \frac{\partial \xi^p}{\partial w} = 0 \\
&\frac{\partial \xi^p}{\partial p} = -2C
\end{aligned} \qquad (15)$$

$$\begin{aligned} \frac{\partial B_x}{\partial x} &= \frac{\partial C}{\partial t} \\ \frac{\partial B_x}{\partial y} &= \frac{\partial D_z}{\partial t} \\ \frac{\partial B_x}{\partial z} &= -\frac{\partial D_y}{\partial t} \\ \frac{\partial \xi^p}{\partial x} &= -\frac{\partial B_x}{\partial t}; \end{aligned} \tag{15}$$

to these we add the corresponding conditions obtained by permutation of the indices. The second condition in (11) yields $\partial C/\partial t = 0$, so C is a constant. We also find

$$\frac{\partial D_z}{\partial t} = \frac{\partial B_x}{\partial y} = \frac{\partial^2 \xi^x}{\partial y\, \partial t} = -\frac{\partial D_z}{\partial t}$$

$$\frac{\partial D_y}{\partial t} = -\frac{\partial B_x}{\partial z} = -\frac{\partial^2 \xi^x}{\partial z\, \partial t} = -\frac{\partial D_y}{\partial t},$$

so that D_y, D_z are constants, as is D_x by symmetry.

The form of ξ^p is determined as

$$\xi^p = -2Cp + F$$

where F as a function of x, y, z, t must satisfy

$$\frac{\partial F}{\partial x} = -\frac{\partial B_x}{\partial t} = -\frac{\partial^2 \xi^x}{\partial t^2} = -\ddot{E}_x.$$

By symmetry, F must be of the form

$$F = -x\ddot{E}_x - y\ddot{E}_y - z\ddot{E}_z + G$$

where G is a function of t. The last remaining condition of (15) gives

$$\xi^t = 2Ct + H$$

where H is a constant.

The independent parameters in the ξ system are constants C, D_x, D_y, D_z, H and the arbitrary smooth functions $E_x(t)$, $E_y(t)$, $E_z(t)$, $G(t)$. These are associated with the linearly independent infinitesimal operators listed in Section 2.

6. Disturbance Equations

Suppose a solution of (1) is given in a region. We wish to consider neighboring solutions of the form $u + \varepsilon\xi^u, \ldots, p + \varepsilon\xi^p$ where ε^2 is negligible. If we use vector notation $\boldsymbol{\xi} = (\xi^u, \xi^v, \xi^w)$, $\boldsymbol{u} = (u, v, w)$, the disturbance field satisfies [3, § 7]

$$\begin{aligned} \frac{\partial \boldsymbol{\xi}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{\xi} + (\boldsymbol{\xi} \cdot \nabla)\, \boldsymbol{u} + \nabla \xi^p - \nu \Delta \boldsymbol{\xi} &= 0 \\ \nabla \cdot \boldsymbol{\xi} &= 0. \end{aligned} \tag{16}$$

For given $\boldsymbol{u}$ this is a homogeneous linear second order system for $\boldsymbol{\xi}$, ξ^p. A solution gives $\boldsymbol{\xi}$, ξ^p as a functional of the values of $\boldsymbol{u}$ and its derivatives, but the dependence on these is not pointwise. That is, such a disturbance field is not the infinitesimal transformation of a one parameter group, in general, even if dependence on the derivatives of $\boldsymbol{u}$ is allowed. We show this as follows.

Let us reformulate the infinitesimal transformation considerations to allow for dependence on the derivatives. We introduce 16 new dependent variables u_i^α, these being functions of x, y, z, t; the subscript no longer denotes partial differentiation. We replace (1) by a system of 20 first order Eqs. in 24 variables:

$$u_t + uu_x + vu_y + wu_z + p_x - \nu\left[\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right] = 0,$$

$$(+\text{ two others}),$$

$$u_x + v_y + w_z = 0,$$

$$\frac{\partial u}{\partial x} - u_x = 0, \quad \frac{\partial u}{\partial y} - u_y = 0,$$

$$(+\ 14\text{ others}).$$

The derived Eq. (2) also becomes first order:

$$\frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y} + \frac{\partial p_z}{\partial z} + u_x^2 + v_y^2 + w_z^2 + 2u_y v_x + 2u_z w_x + 2v_z w_y = 0.$$

The divergence condition is of actual order zero, i.e., $u_x + v_y + w_z = 0$ is an algebraic constraint on the dependent variables.

We consider infinitesimal operators with symbol

$$X = \xi^i \frac{\partial}{\partial x^i} + \xi^\alpha \frac{\partial}{\partial u^\alpha} + \xi_i^\alpha \frac{\partial}{\partial u_i^\alpha};$$

each coefficient may be a function of all 24 variables. The first extension requires new notation:

$$X_{(1)} = X + \xi^\alpha_{(i)} \frac{\partial}{\partial\left(\dfrac{\partial u^\alpha}{\partial x^i}\right)} + \xi^\alpha_{i(j)} \frac{\partial}{\partial\left(\dfrac{\partial u_i^\alpha}{\partial x^j}\right)}.$$

Observe that all partial derivatives are made explicit. Formula (7) no longer applies, since we have a different set of variables. Instead,

$$\begin{aligned}
\xi^\alpha_{(i)} &= \frac{d\xi^\alpha}{dx^i} - \frac{\partial u^\alpha}{\partial x^k}\frac{d\xi^k}{dx^i} \\
&= \frac{\partial \xi^\alpha}{\partial x^i} + \frac{\partial \xi^\alpha}{\partial u^\beta}\frac{\partial u^\beta}{\partial x^i} + \frac{\partial \xi^\alpha}{\partial u_j^\beta}\frac{\partial u_j^\beta}{\partial x^i} \\
&\quad - \frac{\partial u^\alpha}{\partial x^k}\left\{\frac{\partial \xi^k}{\partial x^i} + \frac{\partial \xi^k}{\partial u^\beta}\frac{\partial u^\beta}{\partial x^i} + \frac{\partial \xi^k}{\partial u_j^\beta}\frac{\partial u_j^\beta}{\partial x^i}\right\}
\end{aligned}$$

and

$$\xi^{\alpha}_{i(j)} = \frac{d\xi_i{}^\alpha}{dx^j} - \frac{\partial u_i{}^\alpha}{\partial x^k}\frac{d\xi^k}{dx^j}$$

$$= \frac{\partial \xi_i{}^\alpha}{\partial x^j} + \frac{\partial \xi_i{}^\alpha}{\partial u^\beta}\frac{\partial u^\beta}{\partial x^j} + \frac{\partial \xi_i{}^\alpha}{\partial u_k{}^\beta}\frac{\partial u_k{}^\beta}{\partial x^j}$$

$$- \frac{\partial u_i{}^\alpha}{\partial x^k}\left\{\frac{\partial \xi^k}{\partial x^j} + \frac{\partial \xi^k}{\partial u^\beta}\frac{\partial u^\beta}{\partial x^j} + \frac{\partial \xi^k}{\partial u_m{}^\beta}\frac{\partial u_m{}^\beta}{\partial x^j}\right\}.$$

The given system $\mathsf{M} = 0$ admits an infinitesimal transformation if $X_{(1)}\mathsf{M} = 0$ whenever $\mathsf{M} = 0$, where M denotes the column of 21 left hand sides. That is, the ξ's must satisfy

$$\xi_t{}^u + u\xi_x{}^u + v\xi_y{}^u + w\xi_z{}^u + \xi^u u_x + \xi^v u_y + \xi^w u_z$$
$$+ \xi_x{}^p - \nu[\xi^u_{x(x)} + \xi^u_{y(y)} + \xi^u_{z(z)}] = 0,$$

$$(+ \text{ two others}),$$

$$\xi_x{}^u + \xi_y{}^v + \xi_z{}^w = 0,$$

$$\xi^p_{x(x)} + \xi^p_{y(y)} + \xi^p_{z(z)} + 2[\xi_x{}^u u_x + \xi_y{}^u u_y + \xi_z{}^u u_z$$
$$+ \xi_y{}^u v_x + \xi_x{}^v u_y + \xi_z{}^u w_x + \xi_x{}^w u_z + \xi_z{}^v w_y + \xi_y{}^w v_z] = 0,$$

and also

$$\xi^u_{(x)} - \xi_x{}^u = 0, \qquad \xi^u_{(y)} - \xi_y{}^u = 0, \qquad (+ \text{ 14 others}).$$

If we substitute $\partial u^\alpha/\partial x^i = u_i{}^\alpha$ in these last equations they take the form

$$\xi_i{}^\alpha = \frac{\partial \xi^\alpha}{\partial x^i} + \frac{\partial \xi^\alpha}{\partial u^\beta}u_i{}^\beta - \frac{\partial \xi^k}{\partial x^i}u_k{}^\alpha - \frac{\partial \xi^k}{\partial u^\beta}u_i{}^\beta u_k{}^\alpha + \left[\frac{\partial \xi^\alpha}{\partial u_j{}^\beta} - \frac{\partial \xi^k}{\partial u_j{}^\beta}u_k{}^\alpha\right]\frac{\partial u_j{}^\beta}{\partial x^i}. \tag{17}$$

Now, the following combinations of the derivatives are the only ones available from the given system:

$$\frac{\partial u_x{}^\alpha}{\partial x} + \frac{\partial u_y{}^\alpha}{\partial y} + \frac{\partial u_z{}^\alpha}{\partial z}, \qquad \frac{\partial}{\partial x^i}(u_x + v_y + w_z).$$

The derivatives $\partial u_j{}^\beta/\partial x^i$ can be eliminated from $\xi_i{}^\alpha$ in (17) only if

$$\frac{\partial \xi^\alpha}{\partial u_j{}^\beta} - \frac{\partial \xi^k}{\partial u_j{}^\beta} - u_k{}^\alpha = A^\alpha \Delta(\beta, \mathrm{j}), \qquad \text{say}, \tag{18}$$

where $\Delta(\beta, j)$ is defined by

$$\Delta(u, x) = \Delta(v, y) = \Delta(w, z) = 1,$$
$$\Delta(\beta, y) = 0 \qquad \text{otherwise}.$$

The integrability condition for system (18) works out to be

$$\frac{\partial \xi^m}{\partial u_j{}^\beta}\delta_{\alpha,\gamma} - \frac{\partial \xi^j}{\partial u_m{}^\gamma}\delta_{\alpha,\beta} = \frac{\partial A^\alpha}{\partial u_j{}^\beta}\Delta(\gamma, m) - \frac{\partial A^\alpha}{\partial u_m{}^\gamma}\Delta(\beta, j). \tag{19}$$

Suppose $\Delta(\gamma, m) = 0$, $\Delta(\beta, j) = 0$, and $\alpha = \gamma$ in (19). Then

$$\frac{\partial \xi^m}{\partial u_j^\beta} = \frac{\partial \xi^j}{\partial u_m^\gamma} \delta_{\gamma,\beta} \qquad (\gamma \text{ not summed}).$$

If m, β, j are given satisfying $\Delta(\beta, j) = 0$ there is a γ satisfying $\gamma \neq \beta$ and $\Delta(\gamma, m) = 0$, since each condition exludes at most one value of γ. There follows

$$\frac{\partial \xi^m}{\partial u_j^\beta} = 0 \qquad \text{if} \qquad \Delta(\beta, j) = 0;$$

in other words, ξ^m can depend only on u_x, v_y, w_z as far as the u_j^β are concerned. But then (18) gives

$$\frac{\partial \xi^\alpha}{\partial u_j^\beta} = 0 \qquad \text{if} \qquad \Delta(\beta, j) = 0,$$

so ξ^α also is a function only of u_x, v_y, w_z among the u_j^β.

In (19) assume that $\Delta(\beta, j) = 1$ and $\Delta(\gamma, m) = 0$, and set $\alpha = \gamma$:

$$\frac{\partial \xi^m}{\partial u_j^\beta} = -\frac{\partial A^\gamma}{\partial u_m^\gamma} \qquad \text{if} \qquad \Delta(\beta, j) = 1, \qquad \Delta(\gamma, m) = 0 \quad (\gamma \text{ not summed}).$$

The right hand side is independent of β, j, which is to say

$$\frac{\partial \xi^m}{\partial u_x} = \frac{\partial \xi^m}{\partial v_y} = \frac{\partial \xi^m}{\partial w_z}.$$

It is straightforward from this that ξ^m depends on the u_j^β only as a function of $u_x + v_y + w_z$. All solution manifolds are contained in the subspace $u_x + u_y + w_z = 0$, however, so ξ^m does not depend on the u_j^β at all. We now apply the same argument to

$$A^\alpha = \frac{\partial \xi^\alpha}{\partial u_j^\beta} \qquad \text{if} \qquad \Delta(\beta, j) = 1,$$

to find that ξ^α cannot depend on the u_j^β.

With ξ^i, ξ^α independent of the variables u_j^β, we are reduced to essentially the same system as the one treated previously. There are no other infinitesimal transformations of (1) than the ones listed in Section 2, even with dependence on derivatives admitted.

7. Conclusions

The infinitesimal group of the Navier-Stokes equations is spanned by the operators listed in Section 3. The infinitesimal group does not enlarge if dependence on derivatives is admitted. The usefulness of the group remains to be shown — the Navier-Stokes equations are non-linear, and the techniques of representation theory are not immediately applicable.

The present treatment was an outgrowth of conversations with B. P. Bogert, A. J. Claus, and F. M. Labianca in a seminar on the perturbation theory of the Navier-Stokes equations. In particular, the question came up as to whether the coordinate stretching of Lighthill [4] has to do with a transformation group; it does not, according to Section 6.

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