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The Langlands Quotient Theorem for p-adic Groups

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Langlands' paper [3] reduces the problem of determining and classifying all irreducible quasi-simple representations of a group $G = \mathbb{G}(\mathbb{R})$, where $\mathbb{G}$ is a connected and reductive algebraic group defined over $\mathbb{R}$, to the problem of determining and classifying all irreducible tempered representations of G. In this paper we shall state and prove the analogue of Langlands' theorem for p-adic groups.

The organization of this paper is as follows. The first section recalls notation. The second and third sections summarize known facts. The fourth section presents the statement of the Langlands quotient theorem. The final sections of the paper give the proof.

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1. Some Notation

Let $\mathbb{G}$ be a connected and reductive algebraic group defined over a non-archimedean local field Ω. Write G for the group of all Ω-points of $\mathbb{G}$.

In this paper we shall employ notations and terminology from [2] and [4]. We begin by recalling some of this terminology.

Throughout the paper we shall abuse notation by referring to Ω-groups and the corresponding groups of Ω-points by the same capital letter. The reader should have no difficulty in distinguishing, from the context, our intended usage.

We fix a minimal p-pair (P_0, A_0) $(P_0 = M_0N_0)$ of G. A split torus A of G is called a *standard torus* if $A \subset A_0$ and if A is a split component of some p-subgroup of G. A p-pair (P, A) is called a *semi-standard p-pair* if A is a standard torus. A semi-standard p-pair (P, A) is called a *standard p-pair* if $P \supset P_0$. Notation: $(P_1, A_1) \succ (P_2, A_2)$ means that $P_1 \supset P_2$ and $A_1 \subset A_2$. Every p-pair (P', A') of G is conjugate to a standard p-pair of G in the sense that: there exists $y \in G$ such that $(P'^y, A'^y) = (yP'y^{-1}, yA'y^{-1}) \succ (P_0, A_0)$.

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Let A_1 and A_2 be standard tori of G. We write $W(A_1|A_2)$ for the set of all homomorphisms $s: A_2 \to A_1$ which are induced by inner automorphisms of G. In particular, when $A_1 = A_2$, we write $W(A)$ or $W(G/A) = W(A|A)$. Note that $W(A)$ is the factor group $N_G(A)/Z_G(A)$, a finite group. Write $W_0 = W(A_0)$; W_0 is the relative Weyl group of G. For any A_1 and A_2 and $s \in W(A_1|A_2)$ there is at least one $s_0 \in W_0$ such that $s_0|A_2 = s$.

Let A be a standard torus of G. Let $X(A)$ denote the group of all rational characters of A. We write $\mathfrak{a} = \mathrm{Hom}(X(A), \mathbb{Z}) \bigotimes_{\mathbb{Z}} \mathbb{R}$ and call $\mathfrak{a}$ the *real Lie algebra of* A. We write $\mathfrak{a}^*$ (respectively, $\mathfrak{a}^*_{\mathbb{C}}$) for the dual space (respectively, complexified dual space) of $\mathfrak{a}$ ([2], § 7 or [4], § 0.5). Let M denote the centralizer of A. There is a natural mapping $H: M \to \mathfrak{a}$ and a pairing $\langle , \rangle : \mathfrak{a}^* \times \mathfrak{a} \to \mathbb{R}$ ([2], § 7). We assume a fixed W_0-invariant inner product on $\mathfrak{a}_0^*$, so that, when convenient, we may identify each $\mathfrak{a}^*$ with a subspace of $\mathfrak{a}_0^*$ and each $\mathfrak{a}$ with its dual $\mathfrak{a}^*$. We may assume that, when $\mathfrak{a}$ and $\mathfrak{a}^*$ are so identified, the pairing $\langle , \rangle$ on $\mathfrak{a}^* \times \mathfrak{a}$ and the scalar product $(,)$ on $\mathfrak{a}^* \times \mathfrak{a}^*$ are the same mapping.

Corresponding to any standard p-pair (P, A), we have a set of *simple roots* $\Sigma^0(P, A) \subset \mathfrak{a}^*$. The elements of $\Sigma^0(P, A)$ are the non-zero projections to $\mathfrak{a}^*$ of the simple roots $\Sigma^0(P_0, A_0) \subset \mathfrak{a}_0^*$. We write $+\mathfrak{a}^*$ for the subset of $\mathfrak{a}^*$ consisting of all linear combinations of elements of $\Sigma^0(P, A)$ having non-negative coefficients, $\mathfrak{a}^{*+}$ for the subset of $\mathfrak{a}^*$ consisting of all elements which, under the scalar product on $\mathfrak{a}^*$, map $+\mathfrak{a}^*$ to the non-negative reals. We write $+A$ and A^+ for the subsets of A which are mapped by H to $+\mathfrak{a}$ and $\mathfrak{a}^+$, respectively.

An element of $\mathfrak{a}^*$ is called *regular* if it is not orthogonal to any element of the set of reduced roots $\Sigma_r(P, A)$. Letting a vary in A, we write $a \underset{P}{\to} \infty$, if $\langle \alpha, H(a) \rangle \to \infty$ for all $\alpha \in \Sigma^0(P, A)$. If $a \in A^+$ and $H(a)$ is regular, then $a^n \underset{P}{\to} \infty$, $n \to \infty$.

For any standard torus A we write $\mathscr{P}(A)$ for the set of all p-subgroups P such that (P, A) is a semi-standard p-pair. For every $P \in \mathscr{P}(A)$ there is a unique $\bar{P} \in \mathscr{P}(A)$ such that $P \cap \bar{P} = M$ (i.e., $P = MN$, $\bar{P} = M\bar{N}$); $\bar{P}$ is called the *opposite* of P.

We write $\mathscr{E}_{\mathbb{C}}(G)$ [respectively, $\mathscr{E}(G)$] for the set of all classes of irreducible admissible representations of G [respectively, unitary admissible representations of G]. We write ${}^0\mathscr{E}_{\mathbb{C}}(G)$ [respectively, ${}^0\mathscr{E}(G)$] for the subset of $\mathscr{E}_{\mathbb{C}}(G)$ [respectively, $\mathscr{E}(G)$] consisting of all supercuspidal classes. We write $\mathscr{E}_2(G)$ for the set of all discrete series classes in $\mathscr{E}(G)$, ${}_w\mathscr{E}(G)$ for the subset of $\mathscr{E}(G)$ consisting off all classes of tempered representations.

Let (P, A) $(P = MN)$ be a semi-standard p-pair of G. Let $\delta = \delta_P$ denote the modular function of P. Let $\sigma \in \omega \in \mathscr{E}_{\mathbb{C}}(M)$ and lift σ to a representation of P, trivial on N. We write $I(P, G, \sigma) = I(P, \sigma) = I(P, G, \omega) = I(P, \omega)$ for the induced representation $\mathrm{Ind}_P^G(\delta_P^{1/2}\sigma)$. The representation $I(P, \sigma)$ is admissible. If, in addition, σ is unitary, then so is $I(P, \sigma)$. For any $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ we define the quasi-character $\chi_\nu(m) = q^{\sqrt{-1}\langle \nu, H(m) \rangle}$ $(m \in M)$. The constant $q > 1$ is conventionally the module of Ω. Given σ as above, we set $\sigma_\nu(m) = \sigma(m)\chi_\nu(m)$ $(m \in M)$ and define $\sigma_\nu \in \omega_\nu \in \mathscr{E}_{\mathbb{C}}(M)$.

Let A_1 and A_2 be conjugate standard tori with M_1 and M_2 the respective centralizers. Let $\sigma \in \omega \in \mathscr{E}_{\mathbb{C}}(M_1)$ and let $s \in W(A_2|A_1)$ have representative $y = y(s) \in G$. Set $\sigma^y(m) = \sigma(y^{-1}my)$ $(m \in M_2)$. Write $\sigma^y \in \omega^s \in \mathscr{E}_{\mathbb{C}}(M_2)$. As the notation indicates, the class ω^s depends only upon s. We write $W(G/A_1, \omega)$ or $W(\omega)$ for the subgroup of $W(G/A_1)$ consisting of all elements which fix the class ω.

Let π_1 and π_2 be admissible representations of G. We write $\pi_1 \sim \pi_2$ if π_1 and π_2 are equivalent representations. It is easy to see that $I(P^y, \sigma^y) \sim I(P, \sigma)$ for all (P, σ) as above and $y \in G$.

Let X be a finite set. We write $[X]$ to denote the cardinality of X. Let π_1 be an admissible representation of G and let $\pi_2 \in C_2 \in \mathscr{E}_{\mathbb{C}}(G)$. We write $[\pi_2, \pi_1]$ or $[C_2, \pi_1]$ for the number of factors in a composition series for π_1 which are equivalent to π_2.

2. A Preliminary Classification of Irreducible Admissible Representations — as Components of $I(P, \omega)$, ω Supercuspidal

The purpose of this section and the next is to summarize some of the ideas which are developed in [4] in order to orient the reader relative to Langlands' classification. We shall also use some of the facts presented in these sections in proving Langlands' quotient theorem.

Let π be any admissible representation of G in a vector space $\mathscr{H}$. Let (P, A) $(P = MN)$ be a semi-standard p-pair of G. The Jacquet module $\bar{\mathscr{H}}(P) = \mathscr{H}/\mathscr{H}(P)$ may be defined as the universal object for all P-module morphisms from $\mathscr{H}$ to P-modules V on which N acts trivially. Any such V – in particular, $\bar{\mathscr{H}}(P)$ – may be regarded as an M-module. A basic theorem, due essentially to Jacquet (cf. [4], Corollary 2.3.6), asserts that $\bar{\mathscr{H}}(P)$ is an admissible M-module. We write $\pi(P)$ for the representation of M in $\bar{\mathscr{H}}(P)$. The functor $\mathscr{H} \mapsto \bar{\mathscr{H}}(P)$ from the category of admissible G-modules to the category of admissible M-modules is exact. We also remind the reader that the same is true of the functor $\mathrm{Ind}_P^G(\delta_P^{1/2} \cdot)$ from admissible M-modules to admissible G-modules for any $P \in \mathscr{P}(A)$. Recall also that both functors are transitive.

Now let π be an irreducible admissible representation of G and σ an irreducible admissible representation of M. The Frobenius reciprocity theorem ([4], Corollary 1.7.11) implies that π occurs as a subrepresentation of $I(P, \sigma)$ if and only if $\delta_P^{1/2}\sigma$ occurs as a quotient of $\pi(P)$. By [4], Theorem 3.3.1 and an easy induction, π occurs as a subrepresentation of $I(P, \sigma)$ if and only if π occurs as a quotient representation of $I(\bar{P}, \sigma)$. As we shall use this fact repeatedly, we emphasize that π occurs as a quotient of $I(P, \sigma)$ if and only if σ occurs as a quotient representation of $\delta_P^{1/2}\pi(\bar{P})$. We write $\mathfrak{Y}_\pi(P, A)$ for the set of all $\omega \in \mathscr{E}_{\mathbb{C}}(M)$ such that $\delta_P^{-1/2}\omega$ is the class of a subquotient of $\bar{\mathscr{H}}(\bar{P})$. The elements of $\mathfrak{Y}_\pi(P, A)$ are called *class exponents of* π *with respect to* (P, A). It is not always true that, if $\omega \in \mathfrak{Y}_\pi(P, A)$, then π is a quotient of $I(P, \omega)$; however, if there is one supercuspidal $\omega \in \mathfrak{Y}_\pi(P, A)$, then $\mathfrak{Y}_\pi(P, A) \subset {}^0\mathscr{E}_{\mathbb{C}}(M)$ and $\bar{\mathscr{H}}(\bar{P})$ is a direct sum of isotypic subspaces, each with a finite composition series. We shall describe this decomposition further in a moment.

Let σ and π be as in the preceding paragraph. Then $\sigma|A$ is, by Schur's lemma, a quasi-character χ_σ times $\sigma(1)$. We call χ_σ the *central exponent of* σ. We write $\mathfrak{X}_\pi(P, A)$ for the *set of exponents of* π *with respect to* (P, A), i.e., for the set of all central exponents of the class exponents of π.

Now let $\sigma \in \omega \in {}^0\mathscr{E}_{\mathbb{C}}(M)$. Then, for any $P_1 \in \mathscr{P}(A)$, $I(P, \omega)(P_1)$ has a finite composition series with the composition factors $\{\delta_{P_1}^{1/2}\omega^s | s \in W(G/A)\}$, each factor counted with the indicated multiplicity (Casselman first proved this for the principal series; cf. [4], Theorem 5.4.1.1 for the general case). In addition, each component of $I(P, \omega)$ accounts for some component of $I(P, \omega)(P_1)$ ([4], Corollary 5.4.4.6). This implies that $I(P, \omega)$ has a composition series whose length is bounded by $[W(G/A)]$.

We may now give a "preliminary" classification of all irreducible admissible representations of G. Let π be an irreducible admissible representation of G. According to Jacquet (cf. [4], Corollaries 2.4.2 and 2.8.3), if π is not supercuspidal, then $\pi(\bar{P}) \neq (0)$ for some proper standard p-pair (P, A) $(P=MN)$ of G. Taking (P, A) minimal, one obtains $\sigma \in \omega \in {}^0\mathscr{E}_{\mathbb{C}}(M)$ such that π is a quotient representation of $I(P, \omega)$.

Now let A_1 and A_2 be standard tori of G, M_1 and M_2 the respective centralizers. Let $\sigma_i \in \omega_i \in {}^0\mathscr{E}_{\mathbb{C}}(M_i)$, $i=1,2$. Then, for any $P_i \in \mathscr{P}(A_i)$ $(i=1,2)$, the induced representations $I(P_1, \omega_1)$ and $I(P_2, \omega_2)$ have either isomorphic or disjoint composition series. The composition series are isomorphic if and only if there exists an isomorphism $s \in W(A_1|A_2)$ such that $\omega_2^s = \omega_1$. In particular, the composition series of $I(P_1, \omega_1)$ (although, of course, not generally the representations which occur as irreducible quotients or subrepresentations) does not depend on the choice of $P_1 \in \mathscr{P}(A_1)$.

In summary, it suffices, in order to classify all elements of $\mathscr{E}_{\mathbb{C}}(G)$, to list a set of representatives $\{A\}$ for all conjugacy classes of standard tori, to list representatives for ${}^0\mathscr{E}_{\mathbb{C}}(M_A)/W(A)$ for all $A \in \{A\}$ $(M_A = Z_G(A))$, and, finally, choosing $P_A \in \mathscr{P}(A)$ for each $A \in \{A\}$, to list the components of $I(P_A, \omega)$ for all $\omega \in {}^0\mathscr{E}_{\mathbb{C}}(M_A)/W(A)$.

3. Tempered Representations and their Classification as Components of $I(P, \omega)$ $(\omega \in \mathscr{E}_2(M))$

Let π be an irreducible admissible representation of G. Then π belongs to the *discrete series* of G if and only if:

(1) the central exponent χ_π of π is unitary; and

(2) for every standard p-pair (P, A) $(P=MN)$ and every $\chi \in \mathfrak{X}_\pi(P, A)$, $\chi(a) \to 0$, $a \underset{P}{\to} \infty$ (cf. [4], Theorem 4.4.4). The representation π is *tempered* if and only if the following condition is fulfilled ([4], Lemma 4.5.3): For every standard p-pair (P, A) and every $\chi \in \mathfrak{X}_\pi(P, A)$, $|\chi(a)| \leqq 1$ for all $a \in A^+$. This condition implies that χ_π is unitary. If π is tempered and $\omega \in \mathfrak{Y}_\pi(P, A)$ has a unitary central exponent, then ω is itself a tempered class of M. Thus, a "minimal" tempered class exponent for π belongs to the discrete series of M. One sees easily from this that, for every tempered π, there is a standard (P, A) and an $\omega \in \mathscr{E}_2(M)$ such that $\pi \subset I(P, \omega)$. An irreducible admissible representation π is *not* tempered if and only if there is a standard p-pair (P, A) and $a \in A^+$ such that $|\chi(a^n)| \to \infty$, $n \to \infty$, for some $\chi \in \mathfrak{X}_\pi(P, A)$.

An irreducible admissible representation π of G is called *essentially tempered* if there is a quasi-character χ of G such that $\chi\pi$ is tempered.

Let us briefly mention the rather precise analogy which exists between the classification of irreducible admissible representations as components of $I(P, \omega)$ $[(P, A)$ $(P=MN)$ a standard p-pair; $\omega \in {}^0\mathscr{E}_{\mathbb{C}}(M)]$ and the classification of irreducible tempered representations as components of $I(P, \omega)$ $[(P, A)$ as before; $\omega \in \mathscr{E}_2(M)]$. We have already observed that every irreducible tempered representation of G is a component (direct summand!) of $I(P, \omega)$ for some standard p-pair (P, A) $(P=MN)$, some $\omega \in \mathscr{E}_2(M)$.

Let (P, A) $(P=MN)$ be a standard p-pair of G, $\sigma \in \omega \in \mathscr{E}_2(M)$. For any $P' \in \mathscr{P}(A)$ the representation $\delta_{P'}^{-1/2} I(P, \omega)(P')$ has a tempered direct summand whose composition factors are $\{\omega^s | s \in W(G/A)\}$, again with the indicated multiplicities. As before, this leads to the following result: Let A_1 and A_2 be standard tori, M_1 and M_2 the respective centralizers. The (unitary) induced class $C^G_{M_i}(\omega_i)$ depends only on $\omega_i \in \mathscr{E}_2(M_i)/W(G/A_i)$, not on $P_i \in \mathscr{P}(A_i)$, $i=1,2$; $C^G_{M_1}(\omega_1)=C^G_{M_2}(\omega_2)$ if and only if A_1 and A_2 are conjugate and $\omega_1^s=\omega_2$ for some $s\in W(A_2|A_1)$; otherwise the representations are disjoint. The procedure for listing and classifying tempered representations goes exactly as at the end of the preceding section.

The theorem of the next section shows that the disjoint union

$$\coprod_{\substack{(P,A)\succ(P_0,A_0)\\(P=M_AN)}} \{(\omega,\nu)|\omega\in {}_w\mathscr{E}(M_A),\ \nu\in\mathfrak{a}^{*+} \text{ and regular}\}$$

is in natural one-one correspondence with $\mathscr{E}_{\mathbb{C}}(G)$.

4. The Statement of Langlands' Quotient Theorem

The main purpose of this paper is to prove the following theorem (cf. [3], Lemmas 3.13, 3.14, and 4.2):

Theorem 4.1. (1) *Let (P,A) $(P=MN)$ be a standard p-pair of G and let $\sigma\in\omega\in {}_w\mathscr{E}(M)$. Let $\nu\in\mathfrak{a}^{*+}$ and assume that ν is regular. Then $I(P,\omega_{-\sqrt{-1}\nu})$ has exactly one irreducible quotient representation.*

We write $J(\sigma,\nu)$ for the irreducible quotient representation of (1).

(2) *Let (P_i,A_i) $(P_i=M_iN_i)$ be a standard p-pair of G, let $\sigma_i\in\omega_i\in {}_w\mathscr{E}(M'')$, and let $\nu_i\in\mathfrak{a}_i^{*+}$ be regular, $i=1,2$. If $J(\sigma_1,\nu_1)\sim J(\sigma_2,\nu_2)$, then $P_1=P_2$, $\omega_1=\omega_2$, and $\nu_1=\nu_2$.*

(3) *Let π be an irreducible admissible representation of G. Then there exists a standard p-pair (P,A) $(P=MN)$, an irreducible tempered representation σ of M, and a regular element $\nu\in\mathfrak{a}^{*+}$ such that $\pi\sim J(\sigma,\nu)$.*

5. Defining the Langlands Quotient Representation J(σ'', ν'')—the Proof of (1)

Proposition 5.1. *Let (P'',A'') $(P''=M''N'')$ be a standard p-pair of G. Let $\sigma''\in\omega''\in {}_w\mathscr{E}(M''$. Let $\nu''\in\mathfrak{a}''^{*+}$ and assume that ν'' is regular. Then $I(P'',\omega''_{-\sqrt{-1}\nu''})$ has exactly one irreducible quotient representation.*

Proof. Let (P'',A'') $(P''=M''N'')$, (P',A') $(P'=M'N')$, and (P,A) $(P=MN)$ be standard p-pairs of G such that $(P'',A'')\succ(P',A')\succ(P,A)$. By [4], Corollary 4.5.11 we may assume that there is a representation $\sigma'\in\omega'\in\mathscr{E}_2(M')$ such that σ'' occurs as a quotient of $I(P'\cap M'',\sigma')$ and, by [4], Corollary 2.4.2 and Theorem 4.4.4, we may further assume that σ' occurs as a quotient of $I(P\cap M',\sigma_{\sqrt{-1}\nu_0})$ where $\sigma\in\omega\in {}^0\mathscr{E}(M)$ and ν_0 is a linear combination of the elements of $\Sigma^0(P\cap M',A)$ with all positive coefficients. Clearly, any quotient representation of $I(P'',\sigma''_{-\sqrt{-1}\nu''})$ is also a quotient representation of $I(P,\omega_{\sqrt{-1}(\nu_0-\nu'')})$. Since $I(P',\omega')$ is tempered, every element $\omega^s_{\sqrt{-1}s\nu_0}\in\mathfrak{Y}_{I(P',\omega')}(P,A)\,(s\in W(G/A))$ satisfies $s\nu_0\in+\mathfrak{a}^*$.

The proof of Proposition 5.1 continues via a series of lemmas.

We call a regular element $\nu''\in\mathfrak{a}''^*$ *generic* if $\nu_0-\nu''$ is fixed by no element of $W(G/A)-W(M''/A)$.

Lemma 5.2. *The generic elements of $\mathfrak{a}''^*$ comprise an open dense subset of $\mathfrak{a}''^*$.*

Proof. Fix $s\in W(G/A)-W(M''/A)$. It is enough to show that the set of all regular $\nu''\in\mathfrak{a}''^*$ such that $s(\nu_0-\nu'')=\nu_0-\nu''$ lie in (at most) a hyperplane of $\mathfrak{a}''^*$. If ν'' is regular and both $s(\nu_0-\nu'')=\nu_0-\nu''$ and $s\nu_0=\nu_0$, then $s\nu''=\nu''$, which is impossible, since $s\notin W(M''/A)$. We may assume $s\nu_0\neq\nu_0$. If $s\nu_0-s\nu''=\nu_0-\nu''$, then $(\nu'', -s^{-1}\nu_0) = (\nu_0-s\nu_0, \nu_0)\neq 0$. Since $\nu''=0$ is *not* a solution of this linear equation, the set of solutions constitutes a hyperplane of $\mathfrak{a}''^*$.

Lemma 5.3. *Let μ be an irreducible admissible representation of M''. If ν'' is generic, then*

$$[\omega_{\sqrt{-1}(\nu_0-\nu'')}, \delta^{1/2}_{P\cap M''}\mu(\bar{P}\cap M'')] = [\omega_{\sqrt{-1}(\nu_0-\nu'')}, \delta^{1/2}_{P}I(P'',\mu)(\bar{P})] ,$$

the two sides being non-zero if and only if μ is equivalent to a quotient of $I(P\cap M'', \omega_{\sqrt{-1}(\nu_0-\nu'')})$.

Proof. It follows from Frobenius' reciprocity theorem that μ occurs as a quotient of $\delta^{1/2}_{P''}I(P'',\mu)(\bar{P}'')$. Therefore, by the transitivity of the Jacquet functor, $[\omega_{\sqrt{-1}(\nu_0-\nu'')}, \delta^{1/2}_{P\cap M''}\mu(\bar{P}\cap M'')] \leqq [\omega_{\sqrt{-1}(\nu_0-\nu'')}, \delta^{1/2}_{P}I(P'',\mu)(\bar{P})]$. By Frobenius' reciprocity theorem the left side is non-zero if and only if μ is equivalent to a quotient of $I(P\cap M'', \omega_{\sqrt{-1}(\nu_0-\nu'')})$.

Now let ν'' be generic. Since

$$[\omega_{\sqrt{-1}(\nu_0-\nu'')}, \delta^{1/2}_{P}I(P, \omega_{\sqrt{-1}(\nu_0-\nu'')})(\bar{P})] = [W(G/A, \omega_{\sqrt{-1}(\nu_0-\nu'')})] = [W(M''/A, \omega_{\sqrt{-1}(\nu_0-\nu'')})] ,$$

we see that

$$\begin{aligned}&[\omega_{\sqrt{-1}(\nu_0-\nu'')}, \delta^{1/2}_{P}I(P, \omega_{\sqrt{-1}(\nu_0-\nu'')})(\bar{P})]\\ &= [\omega_{\sqrt{-1}(\nu_0-\nu'')}, \delta^{1/2}_{P\cap M''}I(P\cap M'', \omega_{\sqrt{-1}(\nu_0-\nu'')})(\bar{P}\cap M'')]\\ &= [\omega_{\sqrt{-1}\nu_0}, \delta^{1/2}_{P\cap M''}I(P\cap M'', \omega_{\sqrt{-1}\nu_0})(\bar{P}\cap M'')] .\end{aligned}$$

This implies that, for every component μ of $I(P\cap M'', \omega_{\sqrt{-1}(\nu_0-\nu'')})$,

$$[\omega_{\sqrt{-1}(\nu_0-\nu'')}, \delta^{1/2}_{P\cap M''}\mu(\bar{P}\cap M'')] = [\omega_{\sqrt{-1}(\nu_0-\nu'')}, \delta^{1/2}_{P}I(P'',\mu)(\bar{P})] .$$

The lemma is proved.

Corollary 5.4. *Let μ be an irreducible admissible representation of M''. Let $\nu''\in\mathfrak{a}''^*$ be generic. Then there exist equivalent quotient representations of $I(P'',\mu)$ and $I(P, \omega_{\sqrt{-1}(\nu_0-\nu'')})$ if and only if μ occurs as a quotient representation of $I(P\cap M'', \omega_{\sqrt{-1}(\nu_0-\nu'')})$.*

Proof. By Frobenius' reciprocity theorem $I(P'',\mu)$ and $I(P, \omega_{\sqrt{-1}(\nu_0-\nu'')})$ have equivalent quotient representations if and only if $\omega_{\sqrt{-1}(\nu_0-\nu'')}$ occurs as a quotient of $\delta^{1/2}_{P}I(P'',\mu)(\bar{P})$. By Casselman's lemma affirming that supercuspidals are projectives in the category of admissible representations (with a fixed central exponent), $\omega_{\sqrt{-1}(\nu_0-\nu'')}$ occurs as a quotient if and only if

$$[\omega_{\sqrt{-1}(\nu_0-\nu'')}, \delta^{1/2}_{P}I(P'',\mu)(\bar{P})] > 0 .$$

Thus, Lemma 5.3 implies this corollary.

Lemma 5.5. *If ν'' is generic and μ is equivalent to any quotient of*

$$I(P\cap M'', \omega_{\sqrt{-1}(\nu_0-\nu'')}) ,$$

then $I(P'',\mu)$ has only one irreducible quotient representation.

Proof. Let $I(P'',\mu)$ have more than one irreducible quotient representation. Then there is a morphism of G-modules $I(P'',\mu)\to\pi_1\oplus\pi_2$, where π_1 and π_2 are admissible representations of G satisfying $[\mu,\delta_{P''}^{1/2}\pi_i(\bar{P}'')]>0$, $i=1,2$. If μ is a quotient of $I(P\cap M'',\omega_{\sqrt{-1}(\nu_0-\nu'')})$, then $[\omega_{\sqrt{-1}(\nu_0-\nu'')},\ \delta_{P\cap M''}^{1/2}\mu(\bar{P}\cap M'')]>0$. Therefore, by the exactness of the Jacquet functor,

$$[\omega_{\sqrt{-1}(\nu_0-\nu'')},\delta_{P\cap M''}^{1/2}\mu(\bar{P}\cap M'')]<[\omega_{\sqrt{-1}(\nu_0-\nu'')},\delta_P^{1/2}I(P'',\mu)(\bar{P})]\ .$$

If ν'' is generic, this contradicts Lemma 5.3.

Lemma 5.6. *Let σ'' be a tempered quotient of $I(P\cap M'',\omega_{\sqrt{-1}\nu_0})$ and let ν'' be any regular element of $\mathfrak{a}''^{*+}$. Then $I(P'',\sigma''_{-\sqrt{-1}\nu''})$ has exactly one irreducible quotient representation.*

Proof. It is enough to show that $[\omega_{\sqrt{-1}(\nu_0-\nu'')},\delta_P^{1/2}I(P'',\sigma''_{-\sqrt{-1}\nu''})(\bar{P})]=[\omega_{\sqrt{-1}(\nu_0-\nu'')},\delta_{P\cap M''}^{1/2}\sigma''(\bar{P}\cap M'')]$; otherwise we arrive at a contradiction as in Lemma 5.5. By the first part of the proof of Lemma 5.3 we know $\geqq$ and, by Lemma 5.3, the equality for ν'' generic. The composition factors for $\delta_P^{1/2}I(P'',\sigma''_{-\sqrt{-1}\nu''})(\bar{P})$ consist of all the classes $\omega^s_{\sqrt{-1}s(\nu_0-\nu'')}$, counting multiplicities, where s lies in a certain fixed subset X of $W(G/A)$. Let $Y=X\cap W(\omega_{\sqrt{-1}(\nu_0-\nu'')})$. For ν'' generic, $Y\subset W(M''/A)$ and

$$[Y]=[\omega_{\sqrt{-1}(\nu_0-\nu'')},\delta_P^{1/2}I(P'',\sigma'')(\bar{P})]=[\omega_{\sqrt{-1}(\nu_0-\nu'')},\delta_{P\cap M''}^{1/2}\sigma''_{-\sqrt{-1}\nu''}(\bar{P}\cap M'')]\ .$$

We shall show that the same is true for all regular $\nu''\in\mathfrak{a}''^{*+}$.

Let $s\in X-Y$ and suppose that $s(\nu_0-\nu'')=\nu_0-\nu''$. Then, by analytic continuation, $\omega_{s\nu'+\sqrt{-1}s\nu_0}\in\mathfrak{D}_{C^G_{M'}(\omega'_{\nu'})}(P,A)$ for almost all $\nu'\in\mathfrak{a}'^*$ and thus, since $C^G_{M'}(\omega'_{\nu'})$ is a tempered class, $s\nu_0\in+\mathfrak{a}^*$. Since $\nu''-s\nu''=\nu_0-s\nu_0$ and, as is well known, $\nu''-s\nu''\in+\mathfrak{a}^*$, it follows that $s\nu_0$ lies in the subspace of $\mathfrak{a}^*$ spanned by $\Sigma^0(P\cap M',A)$. Thus, $s\nu''=\nu''-\nu_1$, where $(\nu_1,\nu'')=0$. This implies that $(s\nu'',s\nu'')>(\nu'',\nu'')$, a contradiction, unless $s\nu''=\nu''$, in which case, $s\nu_0=\nu_0$ and, by the regularity of ν'', $s\in W(M''/A)$. Thus, $Y\subset W(M''/A)$ for all regular ν''. The proofs of Lemma 5.6 and of Proposition 5.1 are complete.

Proposition 5.7. *Let $\sigma'_i\in\omega'_i\in\mathscr{E}_2(M')$, let σ'_i occur as a quotient representation of $I(P\cap M',M',\sigma_{\sqrt{-1}\nu_0})$, and let σ''_i be an irreducible component of $I(P'\cap M'',M'',\sigma'_i)$, $i=1,2$. If σ''_1 is not equivalent to σ''_2, then $J(\sigma''_1,\nu'')$ is not equivalent to $J(\sigma''_2,\nu'')$. If π is any irreducible quotient representation of $I(P\cap M'',M'',\sigma_{\sqrt{-1}\nu_0})$ and $I(P'',G,\pi_{-\sqrt{-1}\nu''})$ has a quotient equivalent to $J(\sigma''_1,\nu'')$, then $\pi\sim\sigma''_1$.*

Proof. The first assertion follows from the second; we prove the second. Suppose π is an irreducible quotient of $I(P\cap M'',\omega_{\sqrt{-1}\nu_0})$ such that $\pi\nsim\sigma''$ and such that $I(P'',\sigma''_{-\sqrt{-1}\nu''})$ and $I(P'',\pi_{-\sqrt{-1}\nu''})$ have equivalent quotient representations. Then there are surjective morphisms $I(P'',\pi_{-\sqrt{-1}\nu''})\to\lambda$ and $I(P'',\sigma''_{-\sqrt{-1}\nu''})\to\lambda$, where λ is irreducible. By Frobenius reciprocity $[\pi_{-\sqrt{-1}\nu''},\delta_{P''}^{1/2}\lambda(\bar{P}'')]>0$ and $[\omega_{\sqrt{-1}(\nu_0-\nu'')},\delta_{P\cap M''}^{1/2}\pi_{-\sqrt{-1}\nu''}(\bar{P}\cap M'')]>0$. It follows that

$$\begin{aligned}&[\omega_{\sqrt{-1}(\nu_0-\nu'')},\delta_P^{1/2}I(P'',\sigma''_{-\sqrt{-1}\nu''})(\bar{P})]\\&\quad\geqq[\omega_{\sqrt{-1}(\nu_0-\nu'')},\delta_P^{1/2}\lambda(\bar{P})]>[\omega_{\sqrt{-1}(\nu_0-\nu'')},\delta_{P\cap M''}^{1/2}\sigma''_{-\sqrt{-1}\nu''}(\bar{P}\cap M'')]\ .\end{aligned}$$

This contradicts the fact that these multiplicities are equal (see the proof of Lemma 5.6).

Proposition 5.8. *Let ν'' be a regular element of $\mathfrak{a}''^{*+}$. Let σ'' be any irreducible tempered representation of M''. If $J(\sigma'', \nu'')$ is equivalent to a quotient representation of $I(P, \omega_{\sqrt{-1}(\nu_0-\nu'')})$, then σ'' is equivalent to a quotient representation of $I(P\cap M'', \omega_{\sqrt{-1}\nu_0})$.*

Proof. If ν'' is generic, Corollary 5.4 implies the present Proposition. If $J=J(\sigma'', \nu'')$ is equivalent to a quotient of $I(P, \omega_{\sqrt{-1}(\nu_0-\nu'')})$ and σ'' is not equivalent to a quotient of $I(P, \omega_{\sqrt{-1}\nu_0})$, then $[\omega_{\sqrt{-1}(\nu_0-\nu'')}, \delta_P^{1/2} J(\bar{P})]>0$ and

$$[\omega_{\sqrt{-1}(\nu_0-\nu'')}, \delta_{P\cap M''}^{1/2}\sigma''(\bar{P}\cap M'')]=0 .$$

Define the set $Y\subset W(G/A, \omega_{\sqrt{-1}(\nu_0-\nu'')})$ relative to $I(P'', \sigma''_{-\sqrt{-1}\nu''})$ just as in the proof of Lemma 5.6. Then $Y\cap W(M''/A)=\emptyset$. On the other hand, $I(P'', \sigma'')$ is tempered. If $s\in Y$, then $\omega_{\sqrt{-1}s\nu_0+\nu''}\in\mathfrak{Y}_{I(P'', \sigma''_{\nu''})}(P, A)$ for all $\nu''\in\mathfrak{a}''^*$. Therefore, $s\nu_0\in+\mathfrak{a}^*$. Since $s(\nu_0-\nu'')=\nu_0-\nu''$ with $\nu''\in\mathfrak{a}''^{*+}$, we have $\nu_0-s\nu_0=\nu''-s\nu''\in+\mathfrak{a}^*$. Therefore, both ν_0 and $s\nu_0$ are linear combinations of elements of $\Sigma^0(P\cap M', A)$. Thus, $(\nu'', \nu''-s\nu'')=0$, which implies that $\nu''=s\nu''$ and contradicts $Y\cap W(M''/A)=\emptyset$. Proposition 5.8 is proved.

Proposition 5.9. *Let σ'' be any irreducible tempered representation of M''. Let $\pi=J(\sigma'', \nu'')$. Let $J(P, \omega_{\sqrt{-1}(\nu_0-\nu'')})$ and $J(P\cap M'', \omega_{\sqrt{-1}\nu_0})$ denote the maximum completely reducible quotient representations of, respectively, $I(P, \omega_{\sqrt{-1}(\nu_0-\nu'')})$ and $I(P\cap M'', \omega_{\sqrt{-1}\nu_0})$. Then $[\pi, J(P, \omega_{\sqrt{-1}(\nu_0-\nu'')})]=[\sigma'', J(P\cap M'', \omega_{\sqrt{-1}\nu_0})]$.*

Proof. Frobenius' reciprocity theorem implies that

$$[\pi, J(P, \omega_{\sqrt{-1}(\nu_0-\nu'')}]=[\omega_{\sqrt{-1}(\nu_0-\nu'')}, \delta_P^{1/2}\pi(\bar{P})]$$

and

$$[\sigma'', J(P\cap M'', \omega_{\sqrt{-1}\nu_0})]=[\omega_{\sqrt{-1}\nu_0}, \delta_{P\cap M''}^{1/2}\sigma''(\bar{P}\cap M'')] .$$

Lemma 5.3 implies that the right side multiplicities are equal when ν'' is generic. The proof of Lemma 5.6 implies that they are equal for all regular $\nu''\in\mathfrak{a}''^{*+}$.

6. Uniqueness—the Proof of (2)

Proposition 6.1. *Let (P_i'', A_i'') $(P_i''=M_i''N_i'')$ be a standard p-pair of G $(i=1,2)$. Let $\sigma_i''\in\omega_i''\in{}_w\mathscr{E}(M_i'')$ and let $\nu_i''\in\mathfrak{a}_i''^*$ be a regular element in the positive chamber of $\mathfrak{a}_i''^*$ corresponding to P_i''. If $J(\sigma_1'', \nu_1'')\sim J(\sigma_2'', \nu_2'')$, then $P_1''=P_2''$, $\omega_1''=\omega_2''$, and $\nu_1''=\nu_2''$.*

Proof. Let (P_i, A_i) $(P_i=M_iN_i)$ and (P_i', A_i') $(P_i'=M_i'N_i')$ be standard p-pairs of G such that $(P_i'', A_i'')\succ(P_i', A_i')\succ(P_i, A_i)$ $(i=1,2)$. Let $\sigma_i\in\omega_i\in{}^0\mathscr{E}(M_i)$ and $\nu_i\in\mathfrak{a}_i^*$ be such that $I(P_i\cap M_i', \omega_{i,\sqrt{-1}\nu_i})$ has an irreducible quotient representation $\sigma_i'\in\omega_i'\in\mathscr{E}_2(M_i')$ such that σ_i'' is a summand of $I(P_i'\cap M_i'', \omega_i')$. If $J=J(\sigma_1'', \nu_1'')\sim J(\sigma_2'', \nu_2'')$, then A_1 is conjugate to A_2 and there exists $s\in W(A_2|A_1)$ such that $\omega_1^s=\omega_2$ and $s(\nu_1-\nu_1'')=\nu_2-\nu_2''$ ([4], Corollary 5.3.2.2). If $\omega_{2,s(\sqrt{-1}\nu_1+\nu_1')}$ is not a class exponent with respect to (P_2, A_2) for $C_{M_1'}^G(\omega'_{1,\nu_1'})$ for all $\nu_1'\in\mathfrak{a}_1'^*$, then $\omega_{2,s\sqrt{-1}(\nu_1-\nu_1')}$ cannot be a class exponent with respect to (P_2, A_2) for J, since all the class exponents of J are analytic continuations of class exponents for the classes $C_{M_1'}^G(\omega'_{1,\nu_1'})$. Since

$C^G_{M'_2}(\omega'_{2,\nu'_2})$ is tempered, $s^{-1}\nu_2$ is a linear combination of elements of $\Sigma^0(P_1, A_1)$ with non-negative coefficients. On the other hand, since $(\nu_2, \nu''_2)=0$ and $\nu''_2=\nu_2+s\nu''_1-s\nu_1$, we see that $(\nu_2,\nu_2)=(s\nu_1,\nu_2)-(s\nu''_1,\nu_2)$. Thus, $(s\nu_1,\nu_2)\geqq(\nu_2,\nu_2)$; similarly $(s\nu_1,\nu_2)\geqq(\nu_1,\nu_1)$. By Schwarz's inequality $(s\nu_1,\nu_2)^2=(\nu_1,\nu_1)(\nu_2,\nu_2)$, or $s\nu_1=\pm\nu_2$. Since both $s\nu_1$ and ν_2 are linear combinations with positive coefficients of elements of $\Sigma^0(P_2, A_2)$, we have $s\nu_1=\nu_2$ and $s\nu''_1=\nu''_2$. Since ν''_1 and ν''_2 lie in $\mathfrak{a}_0^{*+}$ and are regular in $\mathfrak{a}_1''^{*+}$ and $\mathfrak{a}_2''^{*+}$, respectively, we may conclude that $s\nu''_1=\nu''_1=\nu''_2$, $\mathfrak{a}_1''^{*+}=\mathfrak{a}_2''^{*+}$, and $s\in W(M''_1/A_0)$.

Thus, we may assume $M''_1=M''_2=M''$, $P''_1=P''_2=P''$, and $\nu''_1=\nu''_2=\nu''$. By Proposition 5.8 σ''_1 is equivalent to a quotient representation of $I(P_2\cap M'', \omega_{\sqrt{-1}\nu_2})$. By Proposition 5.7 $\sigma''_1\sim\sigma''_2$.

Let J be a representation of the same class as $J(\sigma''_1, \nu''_1)$, where σ''_1 and ν''_1 are as in Proposition 6.1.

Corollary 6.2. *Let (P, A) $(P=MN)$ be a semi-standard p-pair of G. Let σ be a tempered representation of M and $\nu\in\mathfrak{a}^*$, ν regular and in the chamber which is positive relative to P. Let $J\sim J(\sigma,\nu)$. Then there exists $s_0\in W_0$ and a representative $y=y(s_0)\in G$ such that $(P^y, A^y)=(P''_1, A''_1)$, $M^y=M''_1$, $\sigma^y\sim\sigma''_1$, and $s_0\nu=\nu''_1$.*

Proof. Let y be chosen such that (P^y, A^y) is a standard p-pair. Then $I(P,\sigma_{-\sqrt{-1}\nu})$ $\sim I(P^y, \sigma^y_{-\sqrt{-1}s_0\nu})$. Furthermore, the above hypotheses imply that $s_0\nu$ is a regular element in a positive chamber of the dual real Lie algebra of A^y. The corollary follows from the proposition.

7. Existence—the Proof of (3)

For the "existence" theorem we shall need Langlands' Lemma 4.4, [3], p. 87. We shall restate this lemma in our own terminology.

Let (P, A) $(P=MN)$ be a standard p-pair of G. Using our fixed W_0-invariant inner product on $\mathfrak{a}_0^*$, we identify the real Lie algebra $\mathfrak{a}$ of A and its dual $\mathfrak{a}^*$ with one another and with a subspace of $\mathfrak{a}_0^*$. The set of simple roots $\Sigma^0(P, A)$ consists of the set of projections $\alpha_0\mapsto\alpha$ to $\mathfrak{a}^*$ of a subset $S(P)\subset\Sigma^0(P_0, A_0)$. The set of weights $\{\lambda_{\alpha_0}|\alpha_0\in S(P)\}\subset\mathfrak{a}^*$. Clearly, $(\lambda_{\alpha_0},\alpha'_0)=(\lambda_{\alpha_0},\alpha')=\delta_{\alpha,\alpha'}$. We also have $(\alpha,\alpha')\leqq 0$ for any two distinct elements of $\Sigma^0(P, A)$. Let $\mathfrak{z}$ denote the real Lie algebra of the split component Z of G.

Lemma 7.1 (*Langlands*). *Let $\nu\in\mathfrak{a}^*$ and assume that ν is orthogonal to $\mathfrak{z}$. Then there is a unique subset $F(\nu)\subset\Sigma^0(P, A)$ such that $\nu=\sum_{\alpha\in F} c_\alpha\lambda_\alpha-\sum_{\alpha\notin F} d_\alpha\alpha$, where $c_\alpha>0$ and $d_\alpha\geqq 0$.*

Proof. As Langlands formulated his version of this lemma only for the case $\Sigma^0(P_0, A_0)$ and $\mathfrak{a}_0^*$, and not for an arbitrary set of parabolic roots, we must remark: Langlands' proof depends only on the fact that a set of simple roots satisfies $(\alpha_0,\alpha'_0)\leqq 0$ $(\alpha_0\neq\alpha'_0)$ and the dual basis of weights (consequently) satisfies $(\lambda_{\alpha_0},\lambda_{\alpha'_0})\geqq 0$. Since these relations hold for elements of $\Sigma^0(P, A)$ and the corresponding dual basis of weights, Langlands' proof also applies to our case.

Proposition 7.2. *Let π be an irreducible admissible representation of G. Then there exists a standard p-pair (P'', A'') $(P''=M''N'')$, an irreducible tempered representation σ'' of M'', and a regular element $\nu''\in\mathfrak{a}''^{*+}$ such that $\pi\sim J(\sigma'',\nu'')$.*

Proof. If π is essentially tempered, take $(P, A)=(G, Z)$. Since all elements of $\mathfrak{z}^*$ are regular, this case is obvious. We shall consider only the case in which the central exponent χ_π of π is unitary. The general case follows trivially from this case. For every standard p-pair (P, A) $(P=MN)$ of G consider the set of exponents $\mathfrak{X}_\pi(P, A)$. If $\chi\in\mathfrak{X}_\pi(P, A)$, then, by Lemma 7.1, there exists $F=F(\chi)\subset\Sigma^0(P, A)$ such that $\log_q|\chi(a)|=\left(\sum_{\alpha\in F} c_\alpha\lambda_\alpha-\sum_{\alpha\notin F} d_\alpha\alpha, H(a)\right)$ for all $a\in A$. If $F(\chi)=\emptyset$ for all $\chi\in\mathfrak{X}_\pi(P, A)$ for all standard p-pairs (P, A), then π is tempered (cf. § 3). Otherwise, we may choose (P, A) and $\chi\in\mathfrak{X}_\pi(P, A)$ such that $F(\chi)$ has maximum nonzero cardinality. Given such a (P, A) and χ, we may (and do) define $(P'', A'')\succ(P, A)$, where A'' is the largest torus in the kernel of all the root characters ξ_α of A such that $\alpha\in\Sigma^0(P, A)-F(\chi)$. Setting $\chi''=\chi|A''$, we see that $F(\chi'')=\Sigma^0(P'', A'')$, since $\Sigma^0(P'', A'')$ is the set of projections of the elements of $F(\chi)$ and, for each $\alpha\in F(\chi)$, $\lambda_\alpha\in\mathfrak{a}''^*\subset\mathfrak{a}^*$ and $\log_q|\chi''(a)|=\left(\sum_{\alpha\in F(\chi)} c_\alpha\lambda_\alpha, H(a)\right)$ $(a\in A'')$. Clearly, $[F(\chi)]=[F(\chi'')]$. Let $\sigma''\in\omega''\in\mathfrak{Y}_\pi(P'', A'')$ be such that χ'' is the central exponent of the class ω''. Also assume, without loss of generality, that $\delta_{\bar{P}''}^{1/2}\sigma''$ occurs as a quotient of $\pi(\bar{P}'')$. It follows from the choice of (P'', A'') and χ'' that the weight $v''=v''(\chi'')$ defined such that $\log_q|\chi''(a'')|=\langle v'', H(a'')\rangle$ for all $a''\in A''$ is a regular element of $\mathfrak{a}''^*$ and that $(v'', \alpha)>0$ for all $\alpha\in\Sigma^0(P'', A'')$.

We claim that ω'' is essentially tempered. To see this note that, if (P_1, A_1) $(P_1=M_1N_1)$ is a standard p-pair of G such that $(P'', A'')\succ(P_1, A_1)$, then $({}^*P_1=P_1\cap M'', A_1)$ is a standard p-pair of M''. If $\chi_1\in\mathfrak{X}_{\omega''}({}^*P_1, A_1)$, then $\chi_1\in\mathfrak{X}_\pi(P_1, A_1)$ and $\chi_1|A''=\chi''$. Thus, if $a_1\in A_1^+({}^*P_1)\cap{}^0M''$ (${}^0M''=\text{kernel } H_{M''}$), then

$$\log_q|\chi_1(a_1)|=\left(\sum_{\alpha\in F_1} c_\alpha^{(1)}\lambda_\alpha-\sum_{\alpha\notin F_1} d_\alpha^{(1)}\alpha, H(a_1)\right),$$

where $F_1\subset\Sigma^0(P_1, A_1)$. However, by the maximality of $[F]$, we may identify F_1 and $F(\chi'')$, so

$$\log_q|\chi_1(a_1)|=\left(-\sum_{\alpha\notin F_1} d_\alpha^{(1)}\alpha, H(a_1)\right),$$

which implies that $|\chi_1(a_1)|\leqq 1$ for all a_1 as above. Thus, ω'' is essentially tempered and the proposition follows.

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