Math. Ann. 213, 203—210 (1975)

On the Existence of Pathological Submeasures and the Construction of Exotic Topological Groups

Wojchiech Herer and Jens Peter Reus Christensen

The present paper originated from an attempt to solve a problem of Maharam (see [5]). She asked whether a sequentially point continuous outer measure defined on a ϱ-algebra $\mathscr{A}$ of subsets of some set X can be controlled by a countably additive probability measure (i.e. whether there exists a countably additive probability measure with the same null sets). Our results may be considered as a partial solution of this problem, since we give a "counterexample" which is ϱ-subadditive but not necessarily sequentially point continuous.

The result is applied to the construction of an abelian topological group with very strange properties.

First we fix some terminology. An algebra of sets is a Boolean algebra $\mathscr{A}$ of subsets of some set X. A real valued (finite) set function φ defined on $\mathscr{A}$ is called weakly finitely (countably) subadditive if

$$\varphi(\cup A_i) \leqq \sum_{i\in I}(A_i)$$

for any finite (countable) family A_i $(i\in I)$ of disjoint sets in $\mathscr{A}$. If this inequality holds for not necessarily disjoint families of sets we call the set function φ finitely or countably subadditive. Instead of "finitely subadditive" we shall usually write simply subadditive and instead of "countably subadditive" ϱ-subadditive.

A weakly subadditive (ϱ-subadditive) set function φ on $\mathscr{A}$ is called a submeasure (ϱ-submeasure) if its values are non negative, it is increasing (i.e. for $A \subsetneqq B$ we have always $\varphi(A) \leqq \varphi(B)$) and $\varphi(\emptyset)=0$. It is clear that a submeasure is subadditive. All ϱ-submeasures are automatically countably subadditive.

All measures considered in the present paper are finitely additive, real valued and non negative set functions. A submeasure φ on $\mathscr{A}$ is called pathological if it is not identically zero and there does not exist a non trivial measure u on $\mathscr{A}$ dominated by φ.

Let $\mathscr{F}$ be a paving of subsets of X such that there exist at least one finite covering of X with $\mathscr{F}$-sets. Let ξ be a non negative real valued setfunction defined on $\mathscr{F}$. Then the set function φ, defined for all subsets A of X as

$$\varphi(A) = \inf\{\Sigma\xi(B_k) \mid A \subsetneqq \cup B_k, B_k \in \mathscr{F}\}$$

where the infimum is taken over all finite coverings of A with $\mathscr{F}$-sets, is a submeasure defined on the Boolean algebra $\mathscr{P}(X)$ of all subsets of X.

It is easily seen that any submeasure can be obtained in this way. By this procedure any submeasure defined on $\mathscr{A}$ can be extended to the algebra of all subsets of X. Such an extension is of course maximal (in the pointwise ordering) between all extensions.

Our most important result in this section is the existence of pathological submeasures. We begin with a lemma.

Lemma 1. *Let X be a finite abelian group and $S \subseteqq X$ a non empty subset of X. Then for any measure u defined on all subsets of X we have*

$$u(X) \leqq (|X|/|S|) \sup \{u(x+S) \mid x \in X\},$$

where $|\cdot|$ denotes cardinality of a set.

Proof of Lemma. For every x X we have that

$$\sum_y \chi_S(y-x) u(\{y\}) \leqq \sup \{u(x+S) \mid x \in X\}.$$

Summing this inequality over all $x \in X$ and changing the order of summation we obtain the desired inequality.

Our first step in proving the existence of pathological submeasures is the following theorem.

Theorem 1. *Let $\varepsilon > 0$ be an arbitrary positive number. There exist a finite set X and a normalized submeasure φ* (i.e. $\varphi(X) = 1$) *defined on the algebra $\mathscr{P}(X)$ such that any measure u defined on this algebra and dominated by φ satisfies $u(X) \leqq \varepsilon$.*

Proof. Let h and n be natural numbers and let us consider a finite abelian group $X = G^h$, where G is some finite abelian group of cardinality n. Let us denote by k_i, for $i = 1, \ldots, h$, the coordinate projections of X onto G and let us define a set $S \subseteqq X$ as

$$S = \{x \in X \mid k_i(x) \neq 0 \quad \text{for all} \quad i = 1, \ldots, h\}.$$

Let us consider the set function ψ defined for every set $A \subseteqq X$ as

$$\psi(A) = \min \{|B| \mid B \subseteqq X, A \subseteqq B + S\}.$$

The set function ψ is a submeasure defined on the algebra $\mathscr{P}(X)$. Let us consider h elements $x_1, \ldots, x_h \in X$, then the element $x \in X$ defined by $k_i(x) = k_i(x_i)$ does not belong to the set $\{x_1, \ldots, x_h\} + S$. From this we conclude $\psi(X) > h$.

Now let us consider the normalized submeasure φ defined for $A \subseteqq X$ by $\varphi(A) = \psi(A)/\psi(X)$. Let u be an arbitrary measure on the algebra of subsets dominated by φ. We obtain by Lemma 1 that

$$u(X) \leqq (n^h/(n-1)^h)(1/h).$$

By choosing n and h suitably we conclude the proof of Theorem 1.

If $\mathscr{A}$ is an algebra of subsets of the set X we denote by $\Phi(\mathscr{A})$ the set of all normalized submeasures on $\mathscr{A}$ equipped with the topology of setwise convergence.

Theorem 2. *For every atomless algebra of sets $\mathscr{A}$, the set Π of all normalized pathological submeasures on $\mathscr{A}$ is a dense G_δ subset of $\Phi(\mathscr{A})$.*

Proof. The space $\Phi(\mathscr{A})$ is a compact Hausdorff space. An easy argument shows that the sets Φ_n defined for $n = 1, 2, \ldots,$ by

$$\Phi_n = \{\varphi \in \Phi(\mathscr{A}) \mid u(X) \geqq (1/n) \quad \text{for some measure} \quad u \leqq \varphi\}$$

are closed in $\Phi(\mathscr{A})$. Hence Π is a G_δ subset of $\Phi(\mathscr{A})$ and by the Baire property of compact Hausdorff spaces it suffices to show that the complements of the Φ_n's are dense in $\Phi(\mathscr{A})$ for $n = 1, 2, \ldots$.

Let φ be an arbitrary element of $\Phi(\mathscr{A})$ and let U be a basisneighbourhood of φ defined by

$$U = \{\psi \in \Phi(\mathscr{A}) \| \psi(A_i) - \varphi(A_i)| < \varepsilon, i = 1, \ldots, p\}$$

where $\varepsilon > 0$ and $A_i \in \mathscr{A}$. Let $C_1, \ldots, C_q$ be the atoms of the algebra of sets $\mathscr{B}$ generated by $A_1, \ldots, A_p$ and let us consider the algebras of sets $\mathscr{C}_i = \{A \in \mathscr{A} \mid A \subseteqq C_i\}$ for $i = 1, \ldots, q$. Since $\mathscr{A}$ is an atomless algebra of sets each of the algebras $\mathscr{C}_i$ contains an isomorphic copy of any finite algebra of sets. And since a submeasure defined on a subalgebra can always be extended on the whole algebra we obtain by Theorem 1 that for an arbitrary natural number n there exists submeasures φ_i on the algebras $\mathscr{C}_i$ with $\varphi_i(C_i) = \varphi(C_i)$ and such that any measure u on $\mathscr{C}_i$ dominated by φ_i satisfies $u(C_i) < (1/(nq))$. Let us define a set function ξ on the family $\mathscr{B} \cup \bigcup_{l=1}^{q} \mathscr{C}_i$ as follows: $\xi(A) = \varphi(A)$ if $A \in \mathscr{B}$ and $\xi(A) = \varphi_i(A)$ if $A \in \mathscr{C}_i$. The set function ψ is the submeasure defined by

$$\psi(A) = \inf\left\{\Sigma\xi(B_k) \mid A \subseteqq \cup B_k, B_k \in \mathscr{B} \cup \bigcup_{l=1}^{q} \mathscr{C}_l\right\}.$$

It is easy to see that ψ coincides with φ on $\mathscr{B}$ and hence $\psi \in U$. And every measure u on $\mathscr{A}$ dominated by ψ satisfies $u(X) < (1/n)$ so ψ belongs to the complement of Φ_n. This concludes the proof.

Corollary. *Let $\mathscr{A}$ be an algebra of sets having an atomless subalgebra. Then there exists a pathological submeasure on $\mathscr{A}$.*

The proof of the corollary is immediate. The next lemma may be considered as a kind of Hahn decomposition for weakly countably subadditive set functions.

Lemma 2. *Let $\mathscr{A}$ be a ϱ-algebra of subsets of X and ξ a weakly countably subadditive set function defined on $\mathscr{A}$ such that $\xi(X) > 0$ and with the property that every family A_t $(t \in T)$ of disjoint sets with $\xi(A_t) < 0$ for all $t \in T$ is at most countable. Then there exist a set $A_0 \in \mathscr{A}$ with $\xi(A_0) > 0$, $\xi(X \setminus A_0) \leqq 0$ and such that $\xi(A) \geqq 0$ for all $A \subseteqq A_0$.*

Proof of Lemma. By Zorns lemma we may choose a family $B_t (t \in T)$ of pairwise disjoint sets in $\mathscr{A}$ with $\xi(B_t) < 0$ which is maximal (by inclusion) among such families. The set $A_0 = X \setminus (\cup B_t)$ now satisfies the conditions of the lemma.

Now we shall state a very important property of pathological countably subadditive submeasures.

Theorem 2. *Let φ be a pathological countably subadditive submeasure defined on a ϱ-algebra $\mathscr{A}$ of subsets of some set X. There is neither a countably additive measure u on $\mathscr{A}$ with the property that $u(A) = 0$ implies $\varphi(A) = 0$ for all $A \in \mathscr{A}$ nor is there a non zero countably additive measure v such that $\varphi(A) = 0$ implies $v(A) = 0$ for all $A \in \mathscr{A}$.*

Proof. Let us assume that u is countably additive measure on $\mathscr{A}$ with the property that for all $A \in \mathscr{A}$ $u(A)=0$ implies $\varphi(A)=0$. We may assume that $u(X)<(X)$. Let us consider the set function ξ defined for $A \in \mathscr{A}$ as $\xi(A) = \varphi(A)-u(A)$. It is easy to see that ξ satisfies the assumptions of Lemma 2. Let A_0 be the set from Lemma 2. The restriction of u to A_0 is then a non trivial measure dominated by φ and this contradiction finishes the first part of the proof.

Let us assume now that v is a non zero countably additive measure on $\mathscr{A}$ with the property that for all $A \in \mathscr{A}$ $\varphi(A)=0$ implies $v(A)=0$. In that case one can prove (exactly in the same way as if φ would be a measure, see [3], Theorem *B*, § 20) that v is continuous with respect to φ, e.g. for any $\varepsilon>0$ there is a number $\delta>0$ such that for all $A \in \mathscr{A}$ $\varphi(A)<\delta$ implies $v(A)<\varepsilon$.

Let $r_0>0$ be such a real number that for every $A \in \mathscr{A}$ $\varphi(A) \leqq r_0$ implies $v(A)<(1/2)$ and let us consider the set function $\xi=\varphi-r_0 v$ on $\mathscr{A}$. We may assume (by choosing r_0 sufficiently small) that $\xi(X)>0$. It is again easy to see that the condition of Lemma 2 is fulfilled and we choose the set A_0. If v is normalized (which of course is no serious restriction) it is easily seen that the restriction of v to A_0 is non zero and dominated by φ. This contradiction concludes the proof. Note that the set A_0 has strictly positive measure in both parts of the proof. This, however, requires different arguments to verify.

Let us consider a normalized pathological submeasure φ defined on the algebra of clopen sets in the Cantor space $K=\{0, 1\}^{N}$. Then the formula

$$\varphi(A)=\inf\left\{\sum_i \varphi(A_i) \mid A \subseteq \cup A_i\right\},$$

where the infimum is taken over all countable coverings of A with clopen sets, defines an extension of φ to a countably subadditive submeasure defined on all subsets of the Cantor space K. That this infimum is indeed an extension of φ can be seen by a trivial compactness argument. In the sequel a submeasure defined on the clopen sets shall always be extended with this procedure. Of course the extended submeasure is also pathological. The following result applies the continuum hypothesis.

Theorem 3. *There exists a σ-algebra $\mathscr{A}$ of sets with the property that every non zero countably subadditive submeasure defined on it is non pathological.*

Proof. It is known that, assuming the continuum hypothesis, there exists a σ-algebra of sets $\mathscr{A}$ with a non zero countably additive measure u such that $u(A)>0$ for all uncountable sets $A \in \mathscr{A}$ (see [6], Proposition 20.1*). It is easily seen using Theorem 2 that this algebra has the property stated in the theorem.

We shall now apply the preceding material in the construction of abelian topological groups with very strange properties. First we make some introductory definitions.

Let $(G, +, \mathscr{O})$ be an abelian topological group.

The topological group G is called exotic if there does not exist any non trivial strongly continous unitary representations of G into $L(H)$ over some Hilbert space H. It is very easy to show that G is exotic if and only if there does not exist any non trivial continuous positive definite functions on G. A curious property of exotic groups is given by the following theorem.

Theorem 4. *Let $(G, +, \mathcal{O})$ be a metrizable exotic group. Let θ be a group homomorphism of G into the group of homeomorphisms of some compact Hausdorff space $(\Omega, \mathscr{P})$. Suppose that for each $\omega \in \Omega$ the mapping from G into Ω defined by $g \dashrightarrow (\theta g)(\omega)$ is continuous.*

Then each minimal closed $\theta(G)$ invariant subset of Ω consists of a single point which is a fixpoint for all homeomorphisms in. $\theta(G)$.

Proof. Let $A \subseteqq \Omega$ be a minimal closed $\theta(G)$ invariant subset. Suppose that A is not a single point. Then of course no point of A is a fixpoint for all elements of $\theta(G)$. In our definition of exotic group we required the group to be abelian. The usual fixed point theorem implies that there exists an $\theta(G)$ invariant Radon probability measure u supported by A (there may of course exist several). The Lebesgue theorem of dominated convergence is easily seen to imply the representation of G constructed by considering $\theta(g)$ as a unitary operator in $L^2(u)$ is strongly continuous. We need the metrizability of G since the Lebesgue theorem is only valid for sequences. The continuity of the functions $\langle \theta(g)f, f \rangle$ is then first established for continuous functions f on A and then by approximation for $f \in L^2(A, u)$. By choosing a suitable continuous function f we see that our representation is non trivial. This concludes the proof.

We do not know whether the above property characterizes exotic groups.

A similar argument as the above proof yields that if G is a metrizable exotic group then every extreme invariant mean on the space of bounded uniformly continuous functions is multiplicative. We also do not know whether or not this property is characteristic for exotic groups.

The above property of course implies in particular that there exists at least one invariant mean m on the space $UCB(G)$ of uniformly continuous bounded functions on G, which is multiplicative. Hence there exists an ultrafilter $\mathscr{F}$ on G such that

$$m(f) = \lim_{\mathscr{F}} f(g)$$

for $f \in UCB(G)$.

The invariance of m is seen to imply that

$$F - F = \{g_1 - g_2 \mid g_1, g_2 \in F\}$$

is dense for all $F \in \mathscr{F}$. This solves negatively the hypothesis stated in [1] Chapter 5.

In the sequel M shall denote the real topological vector space of equivalence classes (modulo Lebesgue nul sets) of Borel measurable functions on the unit interval with the topology of convergence in measure. The space M is a complete separable and metrizable real topological vector space.

Theorem 5. *Let E be a real topological vector space which is complete, separable and metrizable. Then E is exotic as a topological group if and only if there does not exist any non trivial continuous linear operators from E into M.*

Proof. Let θ be a non trivial continuous linear operator from E into M. Let $r \in \boldsymbol{R}$ be an arbitrary real number. The (equivalence class of a) function $\exp(\mathrm{ir}\, \theta(x)(t))$ on the unit interval can be identified with a unitary (multiplication) operator on $L^2([0, 1])$ (of course the unit interval is considered with Lebesgue measure). For fixed r we obtain a strongly continuous unitary representation of the topological group E and by choosing r suitably we can obtain that this representation becomes non trivial.

Conversely let φ be a non trivial strongly continuous unitary representation of the topological group E into $L(H)$ over some Hilbert space H. Of course it is no serious restriction to assume that H is separable. Using a well known structure theorem of abelian Von Neumann algebras (see [2]) we can assume that $\varphi(x)$ for $x \in E$ is an equivalence class of a measurable function on [0, 1] such that $|\varphi(x)(t)| = 1$ for all $t \in [0, 1]$. It follows easily from well known results that for each $x \in E$ there exists a unique (equivalence class of a) real measurable function $\theta(x)$ on the unit interval such that for all reals $r \in \boldsymbol{R}$ we have

$$\varphi(rx)(t) = \exp(\mathrm{ir}\, \theta(x)(t))$$

for almost every $t \in [0, 1]$.

$\theta(x)$ is a linear operator from E into M. Since we have for $x \in E$

$$\theta(x) = \lim_{n \to 0} n(\varphi(n^{-1}x) - 1)$$

where the limit holds in the topology for convergence in measure, we conclude that θ is a Borel measurable mapping from E into M being a pointwise limit of continuous mappings. From this it follows that θ is indeed continuous (see [1]). This concludes the proof of the theorem.

Theorem 6. *Let E be a real topological vector space which is complete, separable and metrizable. Let θ be a continuous linear operator from E into M. Then there exists a dense G_δ-set $A \subsetneqq E$ such that for all $x \in A$ the (equivalence class of) set $\{t \in [0, 1] \mid \theta(x)(t) \neq 0\}$ is the same and such that this (equivalence class of) set contains the (equivalence class of) set $\{t \in [0, 1\} \mid \theta(y)(t) \neq 0\}$ for all $y \in E$.*

Proof. Let P be the space of (equivalence classes of) measurable subset of the unit interval equipped with the topology of convergence in measure. P may be identified naturally with a closed subset of M. We define the mapping $S: M \dashrightarrow P$ by letting $S(f)$ be the equivalence class of the set $\{t \in [0, 1] \mid f(t) \neq 0\}$.

Lemma 3. *The mapping $S: M \dashrightarrow P$ defined above is Borel measurable from the Polish space M into the Polish space P.*

Proof of Lemma. A Borel measurable lifting is a mapping L which chooses to each equivalence class $f \in M$ a Borel measurable function $L(f)$ on the unit interval such that the induced mapping from $M \times [0, 1]$ into the reals defined by

$$(f, t) \dashrightarrow L(f)(t)$$

is measurable with respect to the product Borel structure and such that the equivalence class of $L(f)$ is f. A Borel measurable lifting cannot be linear (see [1]). The usual martingale argument by means of which one can show the existence of linear liftings of L^∞ (see [1]) can easily be modified to yield the existence of Borel measurable liftings of L^∞. Using such a lifting a Borel measurable lifting of M is easily constructed. Let k be the function of two real variables defined by

$$k(a, b) = \begin{cases} 0 & \text{if} \quad (a \neq 0 \quad \text{and} \quad b = 1) \quad \text{or} \quad (a = 0 \quad \text{and} \quad b = 0) \\ 1 & \text{else.} \end{cases}$$

Let L be a Borel measurable lifting of M and let us identify an element of P with its characteristic function. We have then

$$G(S)=\{(f,A)\in M\times P \mid A=S(f)\}=\{(f,A)\mid \int_0^1 k(L(f)(t), L(A)(t))dt=0\}.$$

This shows that the graph of the mapping S is Borel measurable and S is therefore Borel measurable (see [1]). This finishes the proof of the lemma.

Since S is Borel measurable we may choose a dense G_δ subset $A\subsetneqq E$ such that the restriction of $S\circ\theta$ to A is continuous from A to P (see [1]). Let D be an arbitrary countable dense subset of E which is a vector space over the rationals. We may assume that A is invariant under translations with elements in D. It is now very easy to see that A has the properties stated in the theorem and the proof is finished.

Let now u be a normalized pathological submeasure defined on the clopen subsets of the Cantor space $K=\{0,1\}^N$. The submeasure u is extended in the natural way to a countably subadditive submeasure defined on all subsets (also denoted u). We may assume that $u(A)>0$ for each clopen set $A\neq\emptyset$. Let us consider the space $C(K)$ of continuous real valued functions on K equipped with the topology of convergence in u "measure"; with this topology $C(K)$ is a separable, metrizable and separated linear topological space. Let G be the completion of $C(K)$ with the above mentioned topology. The space G is a complete, separable and metrizable linear topological space. Each element of G may be identified with an equivalence class (modulo u null sets) of Borel measurable functions on K.

Theorem 7. *The space G constructed above is exotic as a topological group.*

Proof. Suppose G is not exotic. Let θ be a non trivial continuous linear operator from G into M. Let $A\subsetneqq K$ be a clopen subset; we consider G_A the closed linear subspace of G consisting of all elements whose support (u essentially) is contained in A. Let $T\subsetneqq G_A$ be a dense G_δ subset of G_A with the properties of Theorem 6. For any $f\in T$ we define $\alpha(A)$ to be the Lebesgue measure of the set

$$\{t\in[0,1] \mid \theta(f)(t)\neq 0\};$$

the definition is valid according to the preceding results.

From Theorem 6 we easily conclude that α is a non trivial submeasure defined on the algebra of clopen sets; moreover α has the (very much) stronger form of subadditivity given by

$$\alpha(A\cup B)+\alpha(A\cap B)\leqq\alpha(A)+\alpha(B)$$

for all clopen sets $A, B\in K$. It is well known that this strong form of subadditivity implies the existence of a measure v on the clopen sets dominated by α and such that $v(K)=\alpha(K)$ (see [4]). The continuity of θ implies that α is continuous with respect to u in the sense that to $e>0$ exists $d>0$ such that $u(A)\leqq d\Rightarrow\alpha(A)\leqq e$ for all clopen sets $A\subsetneqq K$.

The measure v has a unique extension to a countably additive Borel measure on K. The continuity of α and hence of v with respect to u now gives a contradiction with Theorem 2. This concludes the proof.

Some problems with relation to the preceding results remain open. The most interesting of those problems is whether or not there exist a pathological submeasure u defined on the Borel field of K such that u satisfies $u(A_n) \dashrightarrow u(A)$ for all sequences A_n of Borel sets whose characteristic functions tends pointwise to the characteristic function of the Borel set A.

Acknowledgements. The authors are thankful to Flemming Topsøe for stimulating and encouraging discussions.

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Dr. W. Herer
Dr. J. P. R. Christensen
Department of Mathematics
University of Kopenhagen
Universitetsparken 5
DK-21 Kopenhagen Ø, Denmark

(Received May 20, 1974)