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Some Relations Between Curvature and Characteristic Classes

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1. Introduction

Let M be a Riemannian manifold and let M_m denote the tangent space to M at m. If P is a $2p$-dimensional subspace of M_m, Thorpe [7] has defined the $2p$th sectional curvature $\gamma_{2p}(m, P)$ and has proved the following theorems.

Theorem A. If M is compact and for some p with $2p \leqq \dim M$ we have $\gamma_{2p} \equiv 0$ on M, then the Euler characteristic $\chi(M) = 0$.

Theorem B. If M is compact and for some p we have γ_{2p} constant on M, then each Pontryagin class $P_k(M) = 0$ for $k \geqq p$.

We shall show that weaker assumptions about the curvature of M still imply the vanishing of the Euler characteristic and some of the higher Pontryagin classes. Let R^p denote the curvature operator corresponding to γ_{2p} (see § 2). Specifically we prove the following theorems in § 5.

Theorem (5.1). Assume M is compact. For each $m \in M$ denote by $\mathcal{N}_{R^p}(m)$ the maximal subspace of M_m on which R^p behaves like the pth curvature operator of a flat manifold. If $\dim \mathcal{N}_{R^p}(m)$ is constant and $2p \leqq \dim \mathcal{N}_{R^p}(m)$, then $\chi(M) = 0$.

Theorem (5.2). Assume M is compact. Let $\mathcal{N}_{B_p(K)}(m)$ denote the maximal subspace of M_m on which R^p behaves like the pth curvature operator of a space of constant curvature $\gamma_2 = K$. If $\dim \mathcal{N}_{B_p(K)}(m)$ is constant and $2p \leqq \dim \mathcal{N}_{B_p(K)}(m)$, then

$$P_k(M) = 0 \quad \text{for} \quad k \geqq p + \tfrac{1}{4}(\dim M - \dim \mathcal{N}_{B_p(K)}(m))$$

Actually we prove a slightly more general form of Theorem (5.2), which includes the results of Cheung and Hsiung [2].

The distributions $m \to \mathcal{N}_{R^p}(m)$ and $m \to \mathcal{N}_{B_p(K)}(m)$ have some additional features which are perhaps just as interesting as Theorems (5.1) and (5.2). In Theorems (4.3)–(4.5) we prove that $m \to \mathcal{N}_{R^p}(m)$ and $m \to \mathcal{N}_{B_p(K)}(m)$ are integrable, their integral manifolds are totally geodesic, and the integral manifolds have zero or constant γ_{2p}. Then under some additional assumptions we prove in Theorem (4.7) that the integral manifolds are complete.

2. Riemannian Double Forms

Let M be a (C^∞) differentiable manifold, let $\mathscr{F}(M)$ be the algebra of differentiable functions on M, and let $\mathfrak{X}(M)$ be the Lie algebra of vector fields on M. Following de Rham [3] we define a *double form of type* (p, q) on M to be an $\mathscr{F}(M)$-linear map

$$\omega : \mathfrak{X}(M)^p \times \mathfrak{X}(M)^q \to \mathscr{F}(M)$$

which is skew-symmetric in the first p-variables and also in the last q-variables. We shall use the notation

$$\omega(X_1, \ldots, X_p)(Y_1, \ldots, Y_q) \tag{2.1}$$

to denote the value of ω on the vector fields $X_1, \ldots, X_p, Y_1, \ldots, Y_q$. Then

$$\omega(X_1, \ldots, X_p) : \mathfrak{X}(M)^q \to \mathscr{F}(M)$$

is the $\mathfrak{X}(M)$-linear map whose value on the vector fields $Y_1, \ldots, Y_q$ is given by (2.1). If $p = q$ and

$$\omega(X_1, \ldots, X_p)(Y_1, \ldots, Y_p) = \omega(Y_1, \ldots, Y_p)(X_1, \ldots, X_p)$$

we say that ω is *symmetric*.

As de Rham has noted, it is possible to define the exterior product $\omega \wedge \theta$ of two double forms ω and θ of types (p, q) and (r, s), respectively, by the formula

$$\begin{aligned}(\omega \wedge \theta)&(X_1, \ldots, X_{p+r})(Y_1, \ldots, Y_{q+s}) \\ &= \sum_{\substack{\varrho \in Sh(p,r) \\ \sigma \in Sh(q,s)}} \varepsilon_\varrho \varepsilon_\sigma\, \omega(X_{\varrho_1}, \ldots, X_{\varrho_p})(Y_{\sigma_1}, \ldots, Y_{\sigma_q})\, \theta(X_{\varrho_{p+1}}, \ldots, X_{\varrho_{p+r}})(Y_{\sigma_{q+1}}, \ldots, Y_{\sigma_{q+r}})\end{aligned} \tag{2.2}$$

for $X_1, \ldots, X_{p+r}, Y_1, \ldots, Y_{q+s} \in \mathfrak{X}(M)$. Here $Sh(p, r)$ denotes the set of all (p, r)-shuffles; specifically

$$Sh(p, r) = \{\varrho \in \mathfrak{S}_{p+r} : \varrho_1 < \varrho_2 < \cdots < \varrho_p \quad \text{and} \quad \varrho_{p+1} < \cdots < \varrho_{p+r}\},$$

where $\mathfrak{S}_{p+r}$ is the symmetric group of degree $p + \mathrm{r}$. It is not difficult to show that $\wedge$ is an associative multiplication and that

$$\omega \wedge \theta = (-1)^{pq+rs}\theta \wedge \omega \tag{2.3}$$

where ω has type (p, q) and θ has type (r, s).

We shall find three further operations on double forms useful. Let $\nabla_X (X \in \mathfrak{X}(M))$ be an affine connection on M (see [4]).

Definition. Let ω be a double form of type (p, q) on M and let $X_1, \ldots, X_{p+1}$, $Y_2, \ldots, Y_q \in \mathfrak{X}(M)$. Then double forms $\nabla_Z(\omega)$ of type (p, q), $D\omega$ of type $(p + 1, q)$, and ω' of type $(p+1, q-1)$ are defined by the formulas:

$$\nabla_Z(\omega)(X_1, \ldots, X_p) = \nabla_Z(\omega(X_1, \ldots, X_p)) - \sum_{j=1}^{p} \omega(X_1, \ldots, \nabla_Z X_j, \ldots, X_p); \quad (2.4)$$

$$(D\omega)(X_1, \ldots, X_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1} \nabla_{X_j}(\omega)(X_1, \ldots, \hat{X}_j, \ldots, X_{p+1}). \quad (2.5)$$

$$\omega'(X_1, \ldots, X_{p+1})(Y_2, \ldots, Y_q) = \sum_{j=1}^{p+1} (-1)^{j+1} \omega(X_1, \ldots, \hat{X}_j, \ldots, X_{p+1})(X_j, Y_2, \ldots, Y_q). \quad (2.6)$$

(Note that D depends on ∇ for $q>0$.)

Proposition (2.1). *Let ω and θ be double forms on M of types (p, q) and (r, s) respectively. Then we have the following formulas*

$$\nabla_Z(\omega \wedge \theta) = \nabla_Z(\omega) \wedge \theta + \omega \wedge \nabla_Z(\theta); \quad (2.7)$$

$$D(\omega \wedge \theta) = D\omega \wedge \theta + (-1)^p \omega \wedge D\theta; \quad (2.8)$$

$$(\omega \wedge \theta)' = \omega' \wedge \theta + (-1)^{p+q} \omega \wedge \theta'. \quad (2.9)$$

Proof. (2.7) follows directly from (2.2). We prove (2.9); the proof of (2.8) is similar. We define operators ι_X and $\eta_X (X \in \mathfrak{X}(M))$ as follows. Let ω be a double form of type (p, q), and let $X_1, \ldots, X_p, Y_1, \ldots, Y_q \in \mathfrak{X}(M)$. Then

$$\iota_X(\omega)(X_1, \ldots, X_{p-1})(Y_1, \ldots, Y_q) = \omega(X, X_1, \ldots, X_{p-1})(Y_1, \ldots, Y_q),$$
$$\eta_X(\omega)(X_1, \ldots, X_p)(Y_1, \ldots, Y_{q-1}) = \omega(X_1, \ldots, X_p)(X, Y_1, \ldots, Y_{q-1}).$$

We have the formula

$$\iota_X(\omega') = \eta_X(\omega) - \iota_X(\omega)'.$$

Then (2.9) is proved by induction using this equation and the fact that ι_X and η_X are skew-derivations.

Next we assume that M has a Riemannian metric $\langle , \rangle$, and that ∇ is the Riemannian connection corresponding to $\langle \ , \ \rangle$. The curvature transformation $R_{XY} (X, Y \in \mathfrak{X}(M))$ is defined by the formula

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for $X, Y \in \mathfrak{X}(M)$. (Note that some authors define the curvature transformation to be the negative of ours.) We define a double form on M of type $(2, 2)$ by the formula

$$R(W, X)(Y, Z) = \langle R_{WX} Y, Z \rangle.$$

We call this double form the *curvature double form*, and more generally we call $R^p = R \wedge \cdots \wedge R$ (p-times) the pth *curvature double form*.

It is well known that the curvature transformation, and hence the corresponding double form, satisfy certain identities, including the first and second identities of Bianchi. We shall find it useful to consider other double forms which satisfy the same identities.

Definitions. A *Riemannian double form* is a symmetric double form A on M of type (p, p) with the following properties:

$$A' = 0 ; \tag{2.10}$$

$$DA = 0 . \tag{2.11}$$

Note that for $A = R$, (2.10) and (2.11) reduce to the first and second Bianchi identities respectively. In passing, we remark that the requirement that a Riemannian double form be symmetric is redundant because of the following proposition.

Proposition (2.2). *Let A be a double form of type (p, p) on M which satisfies (2.10). Then A is symmetric.*

Proof. This is well known for the ordinary curvature form of type (2, 2); a nice proof has been given by Milnor [6, p. 54]. In fact this proof can be generalized to the case at hand. Instead of an octohedron, one just uses a generalized octohedron in $\boldsymbol{R}^p$.

It is very easy to form new Riemannian double forms from old because of the following proposition.

Proposition (2.3). *Let A_1 and A_2 be Riemannian double forms on M. Then any homogeneous polynomial $P(A_1, A_2)$ with constant coefficients is a Riemannian double form.*

Proof. This is an easy consequence of Proposition (2.1) and Eqs. (2.8) and (2.9).

Proposition (2.3) shows that any power of a Riemannian double form is a Riemannian double form. In particular R^p is a Riemannian double form for all p. (Thorpe [9] has shown that R^p satisfies (2.10).)

3. Examples of Riemannian Double Forms

Of course the curvature double form and its powers are the principal examples of Riemannian double forms. We next describe several ways of constructing Riemannian double forms of type (2, 2). First let f be a 2-form on M; then we set

$$A_f(W, X)(Y, Z) = 2f(W, X)\, f(Y, Z) + f(W, Y)\, f(X, Z) - f(W, Z)\, f(X, Y). \tag{3.1}$$

Next let g and h be symmetric bilinear forms on M. We may regard g and h as double forms of type (1, 1). Then

$$\begin{aligned}(g \wedge h)(W, X)(Y, Z) = {} & g(W, Y)\, h(X, Z) - g(W, Z)\, h(X, Y) \\ & + g(X, Z)\, h(W, Y) - g(X, Y)\, h(W, Z).\end{aligned} \tag{3.2}$$

(Here we have written $g(W, X)$ for $g(W)(X)$, etc.) Note that

$$(Dg)(W, X)(Y) = \nabla_W(g)(X, Y) - \nabla_X(g)(W, Y). \tag{3.3}$$

Proposition (3.1). *Let f be a 2-form on M, and let g and h be symmetric bilinear forms on M. Then*

(i) *if $\nabla_X(f)=0$ for all $X \in \mathfrak{X}(M)$, then A_f is a Riemannian double form;*
(ii) *if $Dg=Dh=0$, then $g \wedge h$ is a Riemannian double form.*

We omit the proof, which is a straightforward calculation from (3.1), (3.2), and (3.3).

Some examples of Riemannian double forms on M (besides the curvature double form R) which have special geometric interest will now be given. In what follows we write $g=\langle\ ,\ \rangle$, where it is convenient.

Example 1. Let K be a constant and set

$$B(K)=R-\frac{K}{2}g \wedge g.$$

Then $B(K)$ is a Riemannian double form (because $Dg=0$) which measures how much the curvature tensor of M differs from that of a space of constant curvature K.

Example 2. More generally let h be a symmetric bilinear form on M with $Dh=0$. Set

$$B(h,K)=R-\tfrac{1}{2}(h \wedge h+Kg \wedge g);$$

then $B(h,K)$ is a Riemannian double form. Note that a hypersurface of a space of constant curvature has such a bilinear form h, namely the second fundamental form. The condition $Dh=0$ is equivalent to the Codazzi equation (see [4], for example). Thus $B(h,K)$ measures how much the curvature tensor of M differs from that of a hypersurface with second fundamental form h of a space of constant curvature K. (Compare Chern [1].)

Example 3. Let M be a Kähler manifold and let F be the Kähler form of M. Define

$$D(K)=R-\frac{K}{8}(g \wedge g+2A_F)$$

where K is a constant. Then $D(K)$ is a Riemannian double form which measures how much the curvature of M differs from that of a Kähler manifold of constant holomorphic curvature K.

Example 4. A double form C can be formed from the Weyl conformal tensor in the same way that one is formed from the curvature tensor. Thus

$$C=R+\frac{R_0}{2(n-1)(n-2)}g \wedge g-\frac{1}{(n-2)}g \wedge k$$

where $n=\dim M \geqq 3$, k is the Ricci curvature of M, and $R_0 \in \mathscr{F}(M)$ is the Ricci scalar curvature of M. If M is locally symmetric, then C is a Riemannian double form.

It is quite natural to generalize Examples 1, 2, and 3 by defining Riemannian double forms of type $(2p, 2p)$ which measure how much the pth curvature double form differs from that of a manifold whose pth curvature double form is particularly simple.

Example 5. Let K be a constant and set

$$B_p(K) = R^p - 2^{-p} K^p g^{2p}.$$

Then $B_p(K)$ is a Riemannian double form of type $(2p, 2p)$ which measures how much R^p differs from the pth curvature double form of a manifold of constant sectional curvature.

Example 6. Let h be a symmetric bilinear form on M with $Dh = 0$. Set

$$B_p(h, K) = R^p - 2^{-p}(h^2 + K g^2)^p.$$

Then $B_p(h, K)$ is a Riemannian double form of type $(2p, 2p)$ which measures how much R^p differs from the curvature double form of a hypersurface with second fundamental form h in a space of constant sectional curvature K.

Example 7. Let M be a Kähler manifold and let F be the Kähler form of M. Let K be a constant and set

$$D_p(K) = R^p - 8^{-p}(g^2 + 2 A_F)^p.$$

Then $D_p(K)$ measures how much R^p differs from the curvature double form of a Kähler manifold of constant holomorphic curvature.

For geometric interpretations one further property of Riemannian double forms is useful. For $m \in M$, a Riemannian double form A determines a tensor A_m on the tangent space M_m in the obvious way.

Proposition (3.2). *Let A be a Riemannian double form of type (p, p) on M. Suppose that*

$$A_m(x_1, \ldots, x_p)(x_1, \ldots, x_p) = 0 \tag{3.4}$$

for all $x_1, \ldots, x_p \in M_m$. Then $A_m = 0$.

Proof. Linearization of (3.4) shows that

$$A(x_1, \ldots, x_p)(y_1, x_2, \ldots, x_p) = 0$$

for all $x_1, \ldots, x_2, y_1 \in M_m$. Suppose that

$$A(x_1, \ldots, x_p)(y_1, \ldots, y_{q-1}, x_q, \ldots, x_p) = 0 \tag{3.5}$$

for all $x_1, \ldots, x_p, y_1, \ldots, y_{q-1} \in M_m$. Let $x_{p+1} = y_1$ and let $y_q \in M_m$. Then by (2.10) we have

$$\begin{aligned} 0 &= \sum_{j=1}^{p+1} (-1)^{j+1} A(x_1, \ldots, \hat{x}_j, \ldots, x_{p+1})(x_j, y_2, \ldots, y_q, x_{q+1}, \ldots, x_p) \\ &= A(x_1, \ldots, x_p)(y_1, \ldots, y_q, x_{q+1}, \ldots, x_p) \\ &\quad - \sum_{j=1}^{q} A(x_1, \ldots, x_{j-1}, y_1, x_{j+1}, \ldots, x_p)(x_j, y_2, \ldots, y_q, x_{q+1}, \ldots, x_p) \\ &= (q+1) A(x_1, \ldots, x_p)(y_1, \ldots, y_q, x_{q+1}, \ldots, x_p). \end{aligned}$$

Thus by induction (3.5) is true for $q = 1, \ldots, 2p$. Hence Proposition (3.2) follows.

The $2p$th *sectional curvature* γ_{2p} of Thorpe [7] is defined as follows. Let $m \in M$ and let P be a $2p$-plane in M_m. Then

$$\gamma_{2p}(m, P) = 2^p((2p)!)^{-1} R^p(e_1, \ldots, e_{2p})(e_1, \ldots, e_{2p}),$$

where $\{e_1, \ldots, e_{2p}\}$ is any orthonormal basis of P. (Our coefficient differs from Thorpe's because of our use of shuffle permutations.) Obviously γ_2 is the ordinary sectional curvature of M.

Using the notion of $2p$th sectional curvature together with Proposition (3.2), we obtain the following proposition.

Proposition (3.3). (i) *$B_p(K) \equiv 0$ on M if and only if $\gamma_{2p}(m, M) = K^p$ for all $m \in M$;*

(ii) *$B_p(h, K) \equiv 0$ on M if and only if $\gamma_{2p}(m, M)$ for each $m \in M$ has the same properties as the $2p$th sectional curvature of a hypersurface of a manifold of constant sectional curvature γ_2;*

(iii) *$D_p(K) \equiv 0$ on M if and only if $\gamma_{2p}(m, M)$ for each $m \in M$ has the same properties as the $2p$th sectional curvature of a manifold of constant holomorphic curvature.*

4. Spaces of Nullity of Riemannian Double Forms

Definition. Let $m \in M$ and let A be a Riemannian double form of type $(2p, 2p)$ on M. We set

$$\mathcal{N}_A(m) = \{x_1 \in M_m : A(x_1, x_2, \ldots, x_p) = 0 \quad \text{for all} \quad x_2, \ldots, x_p \in M_m\}$$

and we denote by $\mathcal{N}_A$ the distribution $m \to \mathcal{N}_A(m)$. We call $\mathcal{N}_A(m)$ the *space of nullity of A at m*, $\mathcal{N}_A$ the *field of nullity of A*, and $\mu_A(m) = \dim \mathcal{N}_A(m)$ the *index of nullity of A at m*.

These notions were defined for the case $p = 1$ in [5]. We show in this section that the result of [5] generalize completely to the case of arbitrary p.

The following propositions will be useful; we omit the proofs, which are straightforward.

Proposition (4.1). *Let A be a Riemannian double form of type (p, p) on M. Then for each $m \in M$ either $\mu_A(m) = \dim M$ or $\mu_A(m) \leqq \dim M - p$.*

Proposition (4.2). *Let A be a Riemannian double form of type (p, p) on M, and let $X_1, \ldots, X_{p+1} \in \mathfrak{X}(M)$. Then*

$$\sum_{j=1}^{p+1} (-1)^{j+1} \nabla_{X_j}(A(X_1, \ldots, \hat{X}_j, \ldots, X_{p+1})) = \sum_{1 \leqq j < k \leqq p+1} (-1)^{j+k+1} A([X_j, X_k], X_1, \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_{p+1}). \tag{4.1}$$

We remark that Eq. (4.1) is just a generalization of a well-known formula for closed differential forms.

We are now in a position to prove very simply the key result of this paper.

Theorem (4.3). *Let U be an open subset of M on which the index of nullity μ_A of A is constant. Then the distribution $\mathcal{N}_A$ is integrable on U.*

Proof. Let X_1 and X_2 be vector fields in $\mathcal{N}_A$. From (4.1) it follows that $[X_1, X_2]$ is in $\mathcal{N}_A$.

We next show that any integral manifold of the field of nullity $\mathcal{N}_A$ of a Riemannian double form A is totally geodesic. Let L be a Riemannian manifold isometrically imbedded in another Riemannian manifold M. Let $\bar{\mathfrak{X}}(L)=\{X|L:X\in\mathfrak{X}(M)\}$; then we write $\bar{\mathfrak{X}}(L)=\mathfrak{X}(L)\oplus\mathfrak{X}(L)^{\perp}$, where $\mathfrak{X}(L)^{\perp}$ is the collection of vector fields normal to L. Let $P:\bar{\mathfrak{X}}(L)\to\mathfrak{X}(L)$ be the natural projection. For $X, Y\in\mathfrak{X}(L)$ we denote the Riemannian connection and curvature operator of L by δ_X and r_{XY}, respectively. The *configuration tensor* [4] *of* L in M is an $\mathscr{F}(L)$-linear map $t:\mathfrak{X}(L)\times\bar{\mathfrak{X}}(L)\to\bar{\mathfrak{X}}(L)$ defined

$$t_X Y=\nabla_X Y-\delta_X Y(X, Y\in\mathfrak{X}(L)) \quad \text{and} \quad t_X Z=P\,\nabla_X Z(X\in\mathfrak{X}(L), Z\in\mathfrak{X}(L)^{\perp}).$$

The configuration tensor vanishes if it vanishes on either $\mathfrak{X}(L)$ or $\mathfrak{X}(L)^{\perp}$, and so it is equivalent to the second fundamental form h [4]. We say that L is *totally geodesic* in M if and only if $t\equiv 0$ on L.

Theorem (4.4). *Let L be an integral manifold of $\mathcal{N}_A$; then L is totally geodesic in M.*

Proof. Let $X_1, Y_1\in\mathfrak{X}(L)$ and $X_2,\ldots,X_{p+1}, Y_2,\ldots,Y_p\in\mathfrak{X}(L)^{\perp}$. Then

$$\begin{aligned}0&=(DA)(X_1,\ldots,X_{p+1})(Y_1,\ldots,Y_p)\\&=\sum_{j=1}^{2p+1}(-1)^{j+1}\,\nabla_{X_j}(A(X_1,\ldots,\hat{X}_j,\ldots,X_{p+1}))(Y_1,\ldots,Y_p)\\&=A(X_2,\ldots,X_{p+1})(\nabla_{X_1}Y_1,Y_2,\ldots,Y_p)\\&=A(X_2,\ldots,X_{p+1})(t_{X_1}Y_1,Y_2,\ldots,Y_p).\end{aligned}$$

Thus $t_{X_1}Y_1$ is in $\mathcal{N}_A$ and so $t_{X_1}Y_1=0$.

The spaces of nullity of the Riemannian double forms $B_p(K)$, $B_p(h,K)$, and $D_p(K)$ of Examples 5, 6, and 7 of § 3 have particularly nice geometric interpretations. By Theorem (4.3) $\mathcal{N}_{B_p(K)}(m)$ is the maximal subspace of M_m on which the pth curvature double form behaves like that of a space of constant curvature $\gamma_2=K$. Similar interpretations of the spaces of nullity of $B_p(h,K)$ and $D_p(h,K)$ can be given. The next theorem shows that the pth curvature double form of an integral manifold of one of the distributions $\mathcal{N}_{B_p(K)}$, $\mathcal{N}_{B_p(h,K)}$, or $\mathcal{N}_{D_p(K)}$ is what one would expect it to be.

Theorem (4.5). (i) *if L is an integral manifold of $\mathcal{N}_{B_p(K)}$, then the pth curvature double form r^p of L is given by*

$$r^p=2^{-p}K^p g^{2p};$$

(ii) *if L is an integral manifold of $\mathcal{N}_{B_p(h,K)}$, then*

$$r^p=2^{-p}(h^2+Kg^2)^p;$$

(iii) *if L is an integral manifold of $\mathcal{N}_{D_p(K)}$, then L is a Kähler submanifold of M and*

$$r^p = 8^{-p} K^p (g^2 + 2A_F)^p .$$

Proof. This follows from the Gauss equation (see [4]):

$$P R_{XY} = r_{XY} - [t_X, t_Y] .$$

We next generalize a result of [5]. Let A be a Riemannian double form. If $\nabla_X(A) = \alpha(X) A$ for some 1-form α and all $X \in \mathfrak{X}(M)$, we say that A is *recurrent*. If in addition $\alpha = 0$, we say that A is *parallel*. It is easy to see that if A is recurrent, then so is A^q, and the associated 1-form is $q\alpha$.

Proposition (4.6). *Let A be a Riemannian double form on M and let G be the set on which the index of nullity μ_A assumes its minimum value λ. Then μ_A is upper semicontinuous, and the set G is open.*

Proof. It suffices to prove that for any $m \in M$ there exists a neighborhood U of m such that $\mu(m') \leqq \mu(m)$ for $m' \in U$, but this is obvious.

Theorem (4.7). *Assume that M is a complete Riemannian manifold and that A is a recurrent double form of type (p, p) on M. Then each integral manifold L of $\mathcal{N}_A$ on G is complete.*

Proof. If $\lambda = \dim M$, the proof is trivial, so we assume $\lambda < n$. Let $\gamma : [0, b) \to L$ be a unit speed geodesic. Since M is complete we may extend γ to a geodesic $\gamma : [0, \infty) \to M$. Let $Z, X_1, \ldots, X_p, Y_1, \ldots, Y_p \in \mathfrak{X}(M)$ be such that for $0 \leqq t \leqq b$, $Z_{\gamma(t)} = \gamma'(t)$, each $X_{i\gamma(t)}$ and $Y_{i\gamma(t)}$ is perpendicular to $\mathcal{N}_A(\gamma(t))$, and each X_i and Y_i is parallel on $\gamma | [0, b]$. Define $f : [0, b] \to R$ by

$$f = A(X_1, \ldots, X_p)(Y_1, \ldots, Y_p) \circ \gamma .$$

Then

$$\begin{aligned} f' &= \nabla_Z(A)(X_1, \ldots, X_p)(Y_1, \ldots, Y_p) \circ \gamma \\ &= \{\alpha(Z) A(X_1, \ldots, X_p)(Y_1, \ldots, Y_p)\} \circ \gamma = \gamma^*(\alpha) f, \end{aligned}$$

and so $f(t) = f(0) \exp \int_0^t \gamma^*(\alpha)(t)\, dt$ for $0 \leqq t \leqq b$. It follows that if $f(0) \neq 0$ then $f(t) \neq 0$ for $0 \leqq t \leqq b$. Since $X_1, \ldots, X_p, Y_1, \ldots, Y_p$ are arbitrary, we must have $\mu_A(\gamma(b)) = \lambda$. Therefore $\gamma(b) \in G$, and so there exists $c > b$ such that $\gamma([0, c)) \subset G$. Hence every geodesic in L is infinitely extendable (in L) and so L is complete.

Thorpe [7] has shown that if γ_{2p} vanishes on M for some p, then γ_{2q} vanishes on M for all $q \geqq p$. We generalize this result as follows.

Theorem (4.8). *Let A_1 and A_2 be any two Riemannian double forms on M. Then for all $m \in M$ we have*

$$\mathcal{N}_{A_1}(m) \cap \mathcal{N}_{A_2}(m) \subseteqq \mathcal{N}_{A_1 \wedge A_2}(m) .$$

Proof. This is an obvious consequence of (2.2) and the definition of the spaces of nullity.

Corollary (4.9). *Let A be a Riemannian double form on M. Then*

$$\mathcal{N}_A(m) \subseteqq \mathcal{N}_{A^p}(m)$$

for all integers p and all $m \in M$.

The main geometric interest of this corollary is in the case when $A = R^q$ for some q. In this situation we also have the following result.

Theorem (4.10). *Let P be a $2r$-plane in M_m such that for some $p \leqq r$ we have*

$$P \cap \mathcal{N}_{R^p}(m) \neq 0 .$$

Then $\gamma_{2r}(m, P) = 0$.

Proof. Let $\{e_1, \ldots, e_{2r}\}$ be an orthonormal basis of P such that $e_1 \in \mathcal{N}_{R^p}(m)$. There exists a number λ such that

$$\gamma_{2r}(m, P) = \lambda \Sigma \varepsilon_\varrho \varepsilon_\sigma P^p(e_1, e_{\varrho_2}, \ldots, e_{\varrho_{2p}})(e_{\sigma_1}, \ldots, e_{\sigma_{2p}}) \cdot R^{r-p}(e_{\varrho_{2p+1}}, \ldots, e_{\varrho_{2r}})(e_{\sigma_{2p+1}}, \ldots, e_{\sigma_{2r}}) .$$

Here the sum is over all $\varrho, \sigma \in Sh(p, r-p)$ such that $\varrho_1 = 1$. From this formula the theorem is immediate.

We conclude this section by generalizing another result of Thorpe [7], which states that if M has constant $2p$th sectional curvature K_{2p} and constant $2q$th sectional curvature K_{2q}, then M has constant $2(p+q)$th sectional curvature $K_{2p} K_{2q}$.

Theorem (4.11). *Let A_1 and A_2 be Riemannian double forms of types $(2p, 2p)$ and $(2q, 2q)$, respectively, on M. Let P be a $(2p+2q)$-plane in M_m and assume that*

$$\dim(P \cap \mathcal{N}_{R^p - A_1}(m)) \geqq 2q + 1 ,$$
$$\dim(P \cap \mathcal{N}_{R^q - A_2}(m)) \geqq 2p + 1 .$$

Then the $(2p+2q)$th sectional curvature of P is given by the formula

$$\gamma_{2p+2q}(m, P) = 2^{p+q}((2p+2q)!)^{-1} (A_1 \wedge A_2)(e_1, \ldots, e_{2p+2q})(e_1, \ldots, e_{2p+2q})$$

where $\{e_1, \ldots, e_{2p+2q}\}$ is any orthonormal basis of P.

Proof. This is a direct calculation from the definition of $\gamma_{2p+2q}(m, P)$.

5. Applications to the Computation of the Pontryagin Classes and Euler Characteristic

As pointed out in the introduction the topological conclusions of this paper result from Theorem (4.3) and Theorem (4.5). We now give the proofs of the theorems stated in the introduction.

Theorem (5.1). *Suppose M is compact and μ_{R^p} is constant on M and $2p \leqq \mu_{R^p}$. Then $\chi(M) = 0$.*

Proof. If $\mathcal{T}(M)$ denotes the tangent bundle of M, we have a Whitney direct sum

$$\mathcal{T}(M) = \mathcal{N}_{R^p} \oplus \mathcal{N}_{R^p}^{\perp} .$$

Here $\mathcal{N}_{R^p}$ and $\mathcal{N}_{R^p}^{\perp}$ are vector bundles because μ_{R^p} is constant. The Riemannian connection of $\mathcal{T}(M)$ induces a Riemannian connection on the vector bundle $\mathcal{N}_{R^p}$; furthermore the curvature double form of this connection is the restriction of the curvature double form of $\mathcal{T}(M)$ to $\mathcal{N}_{R^p}$. Thus from the definition of $\mathcal{N}_{R^p}$ it follows that the Euler class $\chi(\mathcal{N}_{R^p})=0$. Thus

$$\chi(\mathcal{T}(M))=\chi(\mathcal{N}_{R^p})\,\chi(\mathcal{N}_{R^p}^{\perp})=0\,,$$

and the theorem follows.

Theorem (5.2). *Suppose M is compact and $\mu=\mu_{B_p}(h,K)$ is constant on M and $2p\leqq\mu$. Then*

$$P_k(M)=0 \quad \text{for} \quad k\geqq p+\tfrac{1}{4}(n-\mu)$$

where $n=\dim M$ and $P_k(M)\in H^{4k}(M,\mathbf{R})$ denotes the kth Pontryagin class.

Proof. Let $\mathcal{N}=\mathcal{N}_{B_p(h,K)}$. Just as in Theorem (5.1) we have

$$\mathcal{T}(M)=\mathcal{N}\oplus\mathcal{N}^{\perp}$$

and so

$$P(M)=P(\mathcal{T}(M))=P(\mathcal{N})\,P(\mathcal{N}^{\perp}),$$

where $P(\mathcal{N})$, etc., denotes the total Pontryagin class. A calculation similar to that given in [7] shows that

$$P_k(\mathcal{N})=0 \quad \text{for} \quad k\geqq p\,.$$

Obviously $P_k(\mathcal{N}^{\perp})=0$ for $k\geqq\frac{1}{4}(n-\mu)$ and so the theorem follows.

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