Math. Ann. 211, 237—244 (1974)

On Rational Singularities in Dimensions > 2

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Introduction. We consider here in a rudimentary way the natural generalization of Artin's rational surface singularities [3] to higher dimensions. A point $x \in X$ is rational if $(R^i\pi_* \mathcal{O}_{\tilde{X}})_x = 0$, for $i > 0$, where $\pi : \tilde{X} \to X$ is a resolution of singularities. Basically, we extend the results of Laufer [15] to this situation, our main tool being the result of Grauert-Riemenschneider [9] that $R^i\pi_* \Omega^n_{\tilde{X}} = 0$, for algebraic singularities. Several examples are found of such singularities, especially the Arnold singularities of [1], and all quotient singularities. We consider in § 5 the relationship with $\bar{\partial}_b$-harmonic forms, though we are unable to present an appropriate "vanishing theorem for harmonic integrals" in this situation. Just what the appropriate curvature condition, if any, for $\bar{\partial}_b$-harmonic forms should be is not clear, and seems an interesting open question.

"Rational singularities" of the sort considered here should be useful as examples and in computing arithmetic genera of modular surfaces and their analogues ("local contributions come only from cusps"). They are also among the singularities for which the homology Todd class of [5] is particularly simple: if all singularities of the projective variety X are rational as above, and $\pi : \tilde{X} \to X$ is a resolution, then the homology Todd class of X is $\pi_*(\tau(\tilde{X}))$, where $\tau(\tilde{X})$ is the homology Todd class of $\tilde{X}$ (i.e., the Poincaré dual of the usual Todd class). This was pointed out to the author by Paul Baum.

Definition (1.1). Let x be a point in X, a complex analytic space. Call x a rational singular point if, given a resolution of singularities $\pi : \tilde{X} \to X$ in the sense of [10], then $(R^i\pi_* \mathcal{O}_{\tilde{X}})_x = 0$, for $i > 0$.

It follows from Hironaka's work that, 1. a resolution $\tilde{X}$ always exists [11], 2. given any two such $\tilde{X}_1$ and $\tilde{X}_2$, there is a third $\tilde{X}_3$ which dominates them both, and, therefore, 3. the condition on $R^i\pi_* \mathcal{O}_{\tilde{X}}$ is independent of the choice of $\tilde{X}$, by [10], Corollary 2, p. 153. Proposition (5.3) below gives an intrinsic analytic characterization of this property.

In the case where $\dim_x X = 2$, this is the definition given by Artin in [3]. In higher dimensions, however, the condition is in some sense less restrictive than in dimension 2, as the following example shows:

(1.2) *Example.* Let $V \subset \mathbb{P}^n$ be a smooth hypersurface of degree d, defined by the non-singular homogeneous form $F(Z)$ in the homogeneous variables $Z_1, \ldots, Z_{n+1}$. Let X be defined by $F(Z) = 0$ in $\mathbb{C}^{n+1}$; X is the

* Partially supported by a grant from the National Science Foundation.

"cone over V". Let $H_V^* \to V$ be the line bundle induced on V by the "tautological" line bundle on $\mathbb{P}^n$ (dual to the hyperplane bundle). The "tautological" map $\pi: H_V^* = \tilde{X} \to X$ is a resolution of singularities with $\pi^{-1}(0) = V$. Using this, it is easy to see that $0 \in X$ is a rational singular point iff $d \leqq n$.

In fact, by [9], we can compute $R^i \pi_* \mathcal{O}_{\tilde{X}}$ in the algebraic category. The Leray spectral sequence of π shows $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^0(X, R^i \pi_* \mathcal{O}_{\tilde{X}}) = (R^i \pi_* \mathcal{O}_{\tilde{X}})_0$, and the Leray sequence for $p: \tilde{X} \to V$ shows $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^i(V, R^0 \pi_* \mathcal{O}_{\tilde{X}}) = \bigoplus_{n \geqq 0} H^i(V, \mathcal{O}(H_V^{\otimes n}))$. All these groups vanish for $i > 0$, if $d \leqq n$. For $d = n+1$, $H^{n-1}(V, \mathcal{O})$ is one-dimensional.

For X of dimension 2, with $\pi^{-1}(x)$ an imbedded submanifold of $\tilde{X}$, x is a rational singularity iff $\pi^{-1}(x)$ is a rational curve. That is certainly not the case for an isolated rational singularity in higher dimensions. For example, by the above, the cone over a cubic threefold $V \subset \mathbb{P}^4$ has a rational singularity at 0, and a resolution with $\pi^{-1}(0) = V$, but V is not a rational variety [6].

(2.1) Recall the point of view of [15], which we will be using below. In particular, recall the long exact sequence for "cohomology at ∞":

$$\cdots \xrightarrow{\delta} H_c^i(Y, \mathcal{F}) \to H^i(Y, \mathcal{F}) \to H_\infty^i(Y, \mathcal{F}) \xrightarrow{\delta} H_c^{i+1}(Y, \mathcal{F}) \to \cdots.$$

For Y a complex manifold, and $\mathcal{F} = \mathcal{O}(E)$ the sheaf of germs of sections of a holomorphic vector bundle E, then $H_\infty^*(Y, \mathcal{O}(E))$ may be calculated as the cohomology of the quotient complex $C^\infty(Y, E \otimes \Lambda^{0,*}) / C_c^\infty(Y, E \otimes \Lambda^{0,*})$. Here $C^\infty(Y, E \otimes \Lambda^{0,*})$ is the C^∞-Dolbeault complex, and $C_c^\infty(Y, E \otimes \Lambda^{0,*})$ the subcomplex of smooth compactly supported E-valued $(0, g)$-forms.

(2.2) Consider X a normal complex analytic space with $x \in X$ an isolated singularity. There exists in X a basis of neighborhoods U of x such that 1. $\bar{U}$ is compact, 2. ∂U is an imbedded smooth sub-manifold of X, and 3. ∂U is strictly pseudo-convex. It is easy to construct such U's, of course, by imbedding a neighborhood of x into a subdomain of $\mathbb{C}^N$, and looking at the intersections of small balls with X.

(2.3) By Artin's algebraization theorem [4], we may assume for a local question near $x \in X$, that X is a normal, affine algebraic variety. As such, by [10], we may assume that we have a resolution $\pi: \tilde{X} \to X$ so that $\tilde{X}$ is Zariski open in a smooth projective variety. In this situation, we can apply the result of Grauert and Riemenschneider [8], that $R^i \pi_* \Omega_{\tilde{X}}^n = 0$, for $i > 0$, $n = \dim \tilde{X}$.

(3.1) The rationality of an isolated singular point $x \in X$ is a local condition, so we are free to assume X is normal affine algebraic, as above, with x as its only singularity. Taking $\pi: \tilde{X} \to X$ also as in (2.3), and considering the Leray spectral sequence for π, we want $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$, $i > 0$, to show rationality of x. The Leray sequence also shows that $H^i(\tilde{X}, \Omega_{\tilde{X}}^n) = 0$, $i > 0$. By Serre duality, $H_c^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$, for $i < n$.

Proposition (3.2). *Let ω be a holomorphic n-form defined on a deleted neighborhood of $x \in X$, which is nowhere vanishing on this neighborhood. Then $x \in X$ is rational if and only if ω is square integrable in a neighborhood of x.*

$\Big($*Here square integrable means* $\int_U \omega \wedge \bar{\omega} < \infty$, *for U any sufficiently small relatively compact neighborhood of* $x \in X.\Big)$

Proof. Choose a relatively compact neighborhood U of x as in (2.2), so that ω is defined on $U - \{x\}$. Let $\tilde{U} = \pi^{-1}(U)$. Just as in (3.1), we have $H^i(\tilde{U}, \Omega^n_{\tilde{X}}) = H_c^{n-i}(\tilde{U}, \mathcal{O}_{\tilde{X}})^* = 0$, for $i > 0$. If x is rational, we also get $H_c^{n-i}(\tilde{U}, \Omega^n_{\tilde{X}}) = 0$, $i > 0$, and hence, the exact sequence:

$$0 \to H^0(\tilde{U}, \Omega^n_{\tilde{X}}) \to H^0_\infty(\tilde{U}, \Omega^n_{\tilde{X}}) \to 0$$

and hence $\pi^* \omega$ has an extension to all of $\tilde{U}$, and for

$$\int_U \omega \wedge \bar{\omega} = \int_{\tilde{U}} \pi^* \omega \wedge \pi^* \bar{\omega} < \infty .$$

Conversely, if $\infty > \int_U \omega \wedge \bar{\omega} = \int_{\tilde{U}} \pi^* \omega \wedge \pi^* \bar{\omega}$, then ω has an extension $\tilde{\omega}$ to $\tilde{U}$ as a holomorphic form [15]. In this case, consider the following commutative diagram, where the vertical arrows are cupping, or wedging, with $\tilde{\omega}$:

$$\begin{array}{ccccccc}
\cdots \to H_c^i(\tilde{U}, \Omega^n_{\tilde{U}}) & \to & H^i(\tilde{U}, \Omega^n_{\tilde{U}}) & \to & H^i_\infty(\tilde{U}, \Omega^n_{\tilde{U}}) & \to \cdots \\
\uparrow & & \uparrow & & \uparrow \wr & \\
\cdots \to H_c^i(\tilde{U}, \mathcal{O}_{\tilde{U}}) & \to & H^i(\tilde{U}, \mathcal{O}_{\tilde{U}}) & \to & H^i_\infty(\tilde{U}, \mathcal{O}_{\tilde{U}}) & \to \cdots .
\end{array}$$

The right hand arrow is an isomorphism, since "at ∞" $\tilde{\omega} = \omega$ doesn't vanish. $H_c^i(\tilde{U}, \mathcal{O}_{\tilde{U}}) = 0$ for $i < n$, and $H^i(\tilde{U}, \Omega^n_{\tilde{U}}) = 0$, $i > 0$, so $H^i(\tilde{U}, \mathcal{O}_{\tilde{U}}) = 0$, for $0 < i < n$. For $i = n$, $H^i(\tilde{U}, \mathcal{O}_{\tilde{U}})$ is always trivial for $\tilde{U}$ open.

Corollary (3.3). *Let X be locally a complete intersection with isolated singular point x, defined locally by equations* $f_1, \ldots, f_k$ *in* $\mathbb{C}^{n+k}$. *Then x is rational iff*

$$\omega = dx_1 \wedge \cdots \wedge dx_n \Big/ \frac{\partial(f_1, \ldots, f_k)}{\partial(x_{n+1}, \ldots, x_{n+k})}$$

is locally square-integrable at x on X.

$\left(\textit{Here } \dfrac{\partial(f_1, \ldots, f_k)}{\partial(x_{n+1}, \ldots, x_{n+k})} \textit{ is the Jacobian determinant.}\right)$

Proof. It is easy to check that ω is a well-defined nowhere vanishing n-form on a deleted neighborhood of x in X.

(3.4) *Example*. Arnold has recently classified those isolated hypersurface singularities whose versal deformations contain only finitely many analytically inequivalent singularities [1]. They are surprisingly direct generalizations of the rational double points of dimension two, and the rationality of these Arnold singularities can be deduced from the 2-dimensional case by (3.3).

Arnold's singularities are given by the following list:

$$A_n(m):\quad x_1^2+x_2^{n+1}+x_3^2+\cdots+x_{m+1}^2=0\,;\qquad n\geqq 1,\ m\geqq 2$$

$$D_n(m):\quad x_1(x_1^{n-2}+x_2^2)+x_3^2+\cdots+x_{m+1}^2=0\,;\qquad n\geqq 4,\ m\geqq 2$$

$$E_6(m):\quad x_1^3+x_2^4+x_3^2+\cdots+x_{m+1}^2=0\,;\qquad m\geqq 2$$

$$E_7(m):\quad x_1(x_1^2+x_2^3)+x_3^2+\cdots+x_{m+1}^2=0\,;\qquad m\geqq 2$$

$$E_8(m):\quad x_1^3+x_2^5+x_3^2+\cdots+x_{m+1}^2=0\,;\qquad m\geqq 2\,.$$

Lemma (3.5). $\int\limits_{|x|\leqq 1}\frac{dx}{|x^2-\lambda^2|}\leqq c\log\left(\frac{1}{|\lambda|}\right)$, *where λ is a sufficiently small complex parameter. (Here dx stands for Lebesgue measure in $\mathbb{C}$.)*

Proof.

$$\int\limits_{|x|\leqq 1}\frac{dx}{|x^2-\lambda^2|}=\int\limits_{|x|\leqq\frac{1}{|\lambda|}}\frac{dx}{|x^2-1|}\leqq c\int\limits_{1\leqq|x|\leqq\frac{1}{|\lambda|}}\frac{dx}{|x|^2}=c'\log\left(\frac{1}{|\lambda|}\right).$$

Lemma (3.6). *Let $f(x_2,\ldots,x_n)$ be a continuous function near $0\in\mathbb{C}^{n-1}$, with $f(0)=0$. Then, if $\int\limits_{\Delta_\varepsilon}\frac{dx_2\ldots dx_n}{|f(x_2,\ldots,x_n)|}<\infty$, for ε sufficiently small, then $\int\limits_{\Delta_\varepsilon}\frac{dx_1\ldots dx_n}{|x_1^2+f(x_2,\ldots,x_n)|}<\infty$, for ε sufficiently small. ($\Delta_\varepsilon=$ the polydisc of radius ε about 0 in the appropriate $\mathbb{C}^k$'s.)*

Proof. Use Fubini's theorem and (3.5):

$$\begin{aligned}\int\limits_{\Delta_\varepsilon}\frac{dx_1\ldots dx_n}{|x_1^2+f(x_2,\ldots,x_n)|}&=\int\limits_{\Delta_\varepsilon}\int\limits_{\Delta_\varepsilon}\frac{dx_1}{|x_1^2+f(x_2,\ldots,x_n)|}dx_2\ldots dx_n\\&\leqq c\int\limits_{\Delta_\varepsilon}\log\left(\frac{1}{|f(x_2,\ldots,x_n)|}\right)dx_2\ldots dx_n\\&\leqq c'\int\limits_{\Delta_\varepsilon}\frac{dx_2\ldots dx_n}{|f(x_2,\ldots,x_n)|}<\infty\,.\end{aligned}$$

To prove the rationality of Arnold's singularities we proceed by induction on m. For $m=2$, the singularities are known to be rational, and each has the form $f(x_1,x_2)+x_3^2=0$. By (3.3) or [15], $\omega=dx_1\wedge dx_2/2x_3$ must be locally square integrable on the variety in question. Considering the variety a branched double covering over (x_1,x_2)-space, we have

$$\int\limits_{\Delta_\varepsilon}\frac{dx_1\,dx_2}{|f(x_1,x_2)|}<\infty,\quad\text{for }\varepsilon\text{ small}.$$

For larger values of m, one considers the variety in question as a branched double cover of $(x_1, \ldots, x_m)$-space, and we will want

$$\int_{\Delta_\varepsilon} \frac{dx_1 \ldots dx_m}{|x_m^2 + g(x_1, \ldots, x_{m-1})|} < \infty .$$

Induction implies $\int_{\Delta_\varepsilon} \frac{dx_1 \ldots dx_{m-1}}{|g(x_1, \ldots, x_{m-1})|} < \infty$, and (3.6) says we are done.

Recall that (1.2) these are not the only isolated rational hypersurface singularities.

Proposition (4.1). *Let M be a complex manifold, and Γ a properly discontinuous group of automorphisms of M. Then $X = M/\Gamma$ has only rational singularities.*

Proof. If x is a singular point of X, then the statement is local about x, and we can therefore assume

(4.2)
1) M = the unit ball about the origin in $\mathbb{C}^n$,
2) Γ = a finite group of unitary linear transformations,
3) no element $\gamma \in \Gamma$ fixes, pointwise, a hyperplane in $\mathbb{C}^n$.

By assumption 3), M is locally isomorphic to X, outside a subvariety of M of Codimension 2 (the union of the fixed point sets for $\gamma \in \Gamma$). We will proceed by induction on the dimension of M.

For dimension 2, the result is known, and can be derived directly from [15], using the following lemma:

Lemma (4.3). *Every holomorphic n-form ω defined on the regular points of X is locally square integrable.*

Proof. We only have to prove this in a neighborhood of the point $0 \in X$ mapped onto by $0 \in M \subset \mathbb{C}^n$, under the Assumptions (4.2). In this case, $\int_{V/\Gamma} \omega \wedge \bar{\omega} = \frac{1}{g} \cdot \int_V p^*\omega \wedge p^*\bar{\omega}$, where V may be taken, say, as the ball of radius $\frac{1}{2}$, p is the quotient map $M \to X$, and $g = \operatorname{ord} \Gamma$. By (4.2) 3) and the comment after it, $\pi^*\omega$ is holomorphic except on a subvariety of codimension 2, hence it extends to all of M, and the integrals in question are finite.

(4.4) Next, assuming $\dim X > 2$, we first show that if $\pi: \tilde{X} \to X$ is a resolution of X, then $R^i\pi_* \mathcal{O}_{\tilde{X}}$ can be supported only at $0 \in X$. To see this, take y a point of x different from 0, and $y' \in M$ a point over y. Let Γ_y = the stabilizer of y' in Γ. If y is singular, $\Gamma_y \neq \{e\}$, and since Γ acts linearly, Γ_y fixes the line through 0 and y, call it $L(y)$. Since Γ acts unitarily, Γ_y fixes $L(y)^\perp$, the orthogonal complement to $L(y)$ in $\mathbb{C}^n$. It is easy to see that a neighborhood of y in X is isomorphic to a neighborhood of 0 in $L(y)\times(L(y)^{\perp}/\Gamma_y)$. By induction, $L(y)^{\perp}/\Gamma_y$ has a rational singularity at 0, and hence, so does $L(y)\times(L(y)^{\perp}/\Gamma_y)$.

(4.5) Under Assumptions (4.2), we can assume X is an open subset of an affine algebraic variety, and hence we can find a resolution $\pi:\tilde{X}\to X$ where $\tilde{X}$ is an open subset of a projective manifold. By (4.2) 1), X is Stein, and hence, to show $R^i\pi_*\mathcal{O}_{\tilde{X}}=0$, it suffices to show $H^i(\tilde{X},\mathcal{O}_{\tilde{X}})=0$, $i>0$. The Leray spectral sequence and (4.4) show that these spaces are finite dimensional, and hence we may use Serre duality to conclude they are zero iff $H_c^{n-i}(\tilde{X},\Omega_{\tilde{X}}^n)=0$, $i>0$. Using the Leray spectral sequence for H_c^*, and $R^k\pi_*\Omega_{\tilde{X}}^n=0$, $k>0$, we see $H_c^i(\tilde{X},\Omega_{\tilde{X}}^n)=H_c^i(X,\pi_*\Omega_{\tilde{X}}^n)$. As noted in [15], $\pi_*\Omega_{\tilde{X}}^n$ can be characterized as the sheaf of germs of holomorphic n-forms on the regular points of X which are locally square-integrable on X. It remains to compute $H_c^i(X,\pi_*\Omega_{\tilde{X}}^n)$ by means of the map p.

In fact, by (4.3) and its proof, we have that $(p_*\Omega_M^n)^\Gamma=\pi_*\Omega_{\tilde{X}}^n$, where the $(\cdot)^\Gamma$ denotes Γ-invariants with respect to the natural action of Γ on $p_*\Omega_M^n$. It is easy to check that, if

$$0\to\Omega_M^n\to\mathcal{E}_M^{n,0}\xrightarrow{\bar\partial}\mathcal{E}_M^{n,1}\to\cdots$$

is the fine Dolbeault resolution of Ω_M^n, then

$$0\to(p_*\Omega_M^n)^\Gamma\to(p_*\mathcal{E}_M^{n,0})^\Gamma\xrightarrow{\bar\partial}\cdots$$

is a fine resolution on X. Hence, taking compactly supported sections,

$$0\to\Gamma_c(X,(p_*\mathcal{E}_M^{n,0})^\Gamma)\xrightarrow{\bar\partial}\Gamma_c(X,(p_*\mathcal{E}_M^{n,1})^\Gamma)\to\cdots$$

is the same thing as

$$0\to\Gamma_c(M,\mathcal{E}_M^{n,0})^\Gamma\xrightarrow{\bar\partial}\Gamma_c(M,\mathcal{E}_M^{n,1})^\Gamma\to\cdots.$$

Thus, we have

$$H_c^i(X,\pi_*\Omega_{\tilde{X}}^n)=H_c^i(X,(p_*\Omega_M^n)^\Gamma)=H_c^i(M,\Omega_M^n)^\Gamma=0,\qquad i<n,$$

since $H_c^i(M,\Omega_M^n)=0$, because M is Stein.

(5.1) It is clear that our previous considerations should be related to the $\bar\partial_b$-cohomology of Kohn and Rossi. In the case of an isolated 2-dimensional singularity, the sole obstruction to rationality was $H_c^1(\tilde{X},\Omega_{\tilde{X}}^2)=0$, i.e., all 2-forms near $\partial\tilde{X}$ should extend to $\tilde{X}$. For $\tilde{X}$ of $\dim n$, we want $H_c^i(\tilde{X},\Omega_{\tilde{X}}^n)=H_\infty^{i-1}(\tilde{X},\Omega_{\tilde{X}}^n)$ to be zero as well $(1<i<n)$, and it is these invariants we show are directly related to the $\bar\partial_b$-complex.

We recall for convenience the definition of the $\bar\partial_b$-complex. Let M be a complex manifold, with smooth boundary ∂M, which we'll assume is an open set in some slightly larger manifold M'. Assume ∂M defined by $r=0$, where $dr\neq0$ on ∂M, and $M=\{m\in M'\,|\,r(m)<0\}$. For V a holomorphic vector bundle, define vector spaces as follows:

$$\mathscr{A}^q(V) = C^\infty(M, V \otimes \Lambda^{0,q})$$

$$\bar{\mathscr{A}}^q(V) = \{\varphi \in \mathscr{A}^q(V) \text{ which extend smoothly across } \partial M\}$$

$$\mathscr{C}^q(V) = \{\varphi \in \bar{\mathscr{A}}^q(V) \text{ such that } \bar{\partial} r \wedge \varphi|_{\partial M} = 0\}.$$

$\bar{\partial}$ on $\bar{\mathscr{A}}^*(V)$ preserves $\mathscr{C}^*(V)$, and so passes to $\mathscr{B}^*(V) = \bar{\mathscr{A}}^*(V)/\mathscr{C}^*(V)$. $\mathscr{B}^*(V)$ is the $\bar{\partial}_b$-complex associated with V.

$$\mathscr{A}_c^q(V) = \{\varphi \in \mathscr{A}^q(V) \text{ with compact support}\}$$

$$\mathscr{A}_\infty^q(V) = \mathscr{A}^q(V)/\mathscr{A}_c^q(V)$$

$$\bar{\mathscr{A}}_\infty^q(V) = \bar{\mathscr{A}}^q(V)/\mathscr{A}_c^q(V).$$

We would like to consider the following diagram:

$$\begin{array}{ccccccccc}
0 \to & \mathscr{A}_c^*(V) & \to & \mathscr{A}^*(V) & \to & \mathscr{A}_\infty^*(V) & \to 0 \\
& \| & & \uparrow \alpha & & \uparrow \beta & \\
0 \to & \mathscr{A}_c^*(V) & \to & \bar{\mathscr{A}}^*(V) & \to & \bar{\mathscr{A}}_\infty^*(V) & \to 0 \\
& \downarrow \gamma & & \| & & \downarrow \delta & \\
0 \to & \mathscr{C}^*(V) & \to & \bar{\mathscr{A}}^*(V) & \to & \mathscr{B}^*(V) & \to 0
\end{array}$$

The maps are all inclusions or restrictions.

Assume, now, that ∂M is strictly pseudo-convex. In cohomology in dimensions $\geqq 1$, α induces an isomorphism, which may be seen by Theorem 3.4.8 of [12], or by restricting both spaces of forms to $M^\varepsilon = \{m \in M' \mid r(m) < -\varepsilon < 0\}$ for $\varepsilon > 0$ sufficiently small, and using the Leray sequence to see that in dimensions > 1 in cohomology this gives an isomorphism. Consequently, β induces an isomorphism on cohomology in positive dimensions.

Next, consider the map γ. The i^{th} cohomology of $\mathscr{A}_c^*(V)$ is $H_c^i(M, \mathcal{O}(V))$ which is dual to $H^{n-i}(M, \mathcal{O}(\Omega_M^n \otimes V^*))$, for $i < n$, by the integration pairing. By 5.15 of [7], $H^i(\mathscr{C}^*(V))$ is also dual to $H^{n-i}(M, \mathcal{O}(\Omega_M^n \otimes V^*))$, for $i < n$, again by integration pairing. Since the map induced by γ on cohomology is obviously compatible with these pairings, this map is an isomorphism for $i < n$. Consequently, δ induces an isomorphism of $H^i(\bar{\mathscr{A}}_\infty^*(V))$ with $H^i(\mathscr{B}^*(V))$ for $i < n-1$. But $H^i(\mathscr{B}^*(V))$ is $H_b^i(V|_{\partial M})$.

Proposition (5.2). *For M, V as above, there is a natural isomorphism $H_\infty^i(M, \mathcal{O}(V)) \xrightarrow{\sim} H_b^i(V|_{\partial M})$, $0 < i < n-1$.*

Now $H_b^i(V|_{\partial M})$ for $0 < i < n-1$ may be represented by forms harmonic with respect to the (subelliptic) Laplacian constructed from $\bar{\partial}_b$ and Hermitian metrics on M and V. It would be interesting to know whether this Laplacian has a decomposition as in the work of Bochner, Kodaira, and others on classical Laplacians, and whether such a decomposition might yield vanishing theorems for harmonic sections. Summing up, we have:

Proposition (5.3). *Let $x \in X$ be an isolated singularity, and let ∂X be smooth and strictly pseudo-convex. Then x is a rational singularity iff*

1) *every holomorphic n-form ω on $X - x$ is L^2 at x,*

2) $H_b^i(\Omega_X^n|_{\partial X}) = 0$, $0 < i < n-1$.

Remarks. a) It would be interesting to know an explicit way of stating 1) in terms of the values of ω on ∂X, i.e., as a section of $\Omega_X^n|_{\partial X}$.

b) A. Ogus has shown, using local duality and the result of Grauert and Riemenschneider used above, that projective rational singularities are Cohen-Macaulay in characteristic zero.

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(Received January 25, 1974)