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Math. Annalen 172, 105—115 (1967)

Ascent, Descent, Nullity and Defect, a Note on a Paper by A. E. Taylor*

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1. Introduction

Throughout the present paper T will be a linear operator with domain $D(T)$ in a (real or complex) linear space X and range $R(T)$ in the same linear space.

In his paper [2], TAYLOR presents an intensive study of the relationships between the nullity $n(T)$ and defect $d(T)$ on one hand, and the ascent $\alpha(T)$ and descent $\delta(T)$ on the other. The main purpose of the present paper is to clear up a number of matters which were left open in TAYLOR's paper. In what follows, therefore, we suppose the reader familiar with the definitions and notations introduced by TAYLOR in the first section of [2].

In the second section of this paper we gather a few simple facts on linear manifolds in a linear space. We present these facts because they will be frequently used in the fourth section.

The third section is devoted to the study of the relationships between the linear subspaces $N(T^k)$, $D(T^k)$ and $R(T^k)$ for $k = 0, 1, 2, \ldots$. A number of lemmas is presented to show how these manifolds are situated in the space X. We use these results in the fourth section in order to get a better understanding of the relationships between the numbers $n(T)$, $d(T)$, $\alpha(T)$ and $\delta(T)$.

Also, in Section 4 we prove that in the case $n(T) = d(T) < \infty$ and $p = \alpha(T) = \delta(T) < \infty$, the linear space X is the direct sum of the linear manifolds $N(T^p)$ and $R(T^p)$. It is well-known that this statement is true in the case that either $D(T) = X$ or $p = 1$, but in general its truth was unknown (cf. [2], Theorem 5.5 and also the comment to this theorem).

2. Linear spaces

In the sequel we need some simple facts about linear manifolds in a linear space. We gather these facts in this section. First of all, we present the following definition.

Definition 2.1. The linear spaces X_1 and X_2 are said to be isomorphic whenever there exists a one-one linear mapping from X_1 onto X_2. For abbreviation we use the symbol $X_1 \cong X_2$ to denote that X_1 and X_2 are isomorphic.

Let M and N be linear subspaces in the linear space X. As usual $M + N$ denotes the set of all $x + y$ with $x \in M$ and $y \in N$. It is easy to show that $M + N$

* This work was supported by the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

is the smallest linear subspace which contains M and N. If, in addition, $M \cap N = \{0\}$ we write $M \oplus N$ for $M+N$. If $N \subset M$, then either M/N or

$$\frac{M}{N}$$

denotes the quotient space M modulo N.

Lemma 2.2. *Let M and N be linear subspaces in the linear space X. Then*

$$\frac{M}{M \cap N} \cong \frac{M+N}{N}.$$

Proof. Let $[x]$ denote a coset in the quotient space $(M+N)/N$. Define for each $m \in M$

$$Jm = [m].$$

Then J is a linear mapping from M into $(M+N)/N$.

If $[x]$ is an element in $(M+N)/N$, then $x = m + z$ with $m \in M$ and $z \in N$, and hence $[x] = [m]$. This shows that J is a linear mapping onto $(M+N)/N$. Combining this fact with the fact that the kernel of J is the subspace $M \cap N$, we obtain the desired result.

If X is any linear space, the dimension of X (denoted by $\dim X$) is the maximal number of linearly independent elements in X. Hence, the value of $\dim X$ can be zero, any natural number or $+\infty$.

Obviously, two isomorphic linear space have the same dimension. This statement has a partial converse as follows. If the linear space X_1 and X_2 have the same finite dimension, then X_1 and X_2 are isomorphic.

Lemma 2.3. *Let M_1, M_2 and N be linear subspaces in the linear space X. Suppose that $M_1 \subset M_2$. Then*

$$\dim \frac{M_1}{M_1 \cap N} \leqq \dim \frac{M_2}{M_2 \cap N}. \tag{2 - 1}$$

Proof. Let $[x]$ denote a coset in the quotient space $M_2/(M_2 \cap N)$. Then the mapping J defined by

$$Jm = [m]$$

for each m in M_1, is a linear mapping from M_1 into $M_2/(M_2 \cap N)$. The kernel of J is the subspace $M_1 \cap N$. Hence $M_1/(M_1 \cap N)$ is isomorphic with a subspace in $M_2/(M_2 \cap N)$. But then formula (2 - 1) is true.

Lemma 2.4. *Let M_1, M_2 and N be linear subspaces in the linear space X. Suppose that $M_1 \subset M_2$, and that*

$$\dim \frac{M_1}{M_1 \cap N} = \dim \frac{M_2}{M_2 \cap N} < \infty.$$

Then $M_1 + N = M_2 + N$.

Proof. Let J be defined as in the proof of the preceding lemma. Then

$$\dim \{M_1/(M_1 \cap N)\} = \dim J M_1,$$

and hence

$$\dim \{M_2/(M_2 \cap N)\} = \dim J M_1 < \infty.$$

But then J is a mapping onto $M_2/(M_2 \cap N)$.

Take an element x in M_2. Since J is a mapping onto $M_2/(M_2 \cap N)$, there exists an element m in M_1 such that $Jm = [m] = [x]$. But then $x = m + z$, with z in N. This shows that $M_1 + N \subset M_2 + N \subset M_1 + N$, and hence $M_1 + N = M_2 + N$.

3. The subspaces $N(T^k)$, $D(T^k)$ and $R(T^k)$

In this section T will be a linear operator with domain $D(T)$ in the linear space X and with range $R(T)$ in the same space. The null space of T is denoted by $N(T)$.

By induction we define the iterates T^2, T^3, ... For $n \geqq 1$, T^n is the linear operator with domain

$$D(T^n) = \{x : x, Tx, \ldots, T^{n-1}x \text{ are in } D(T)\}$$

and such that for each x in $D(T^n)$

$$T^n x = T(T^{n-1}x).$$

Also we define $T^0 = I$ is the identity operator from X into X.

Let n and m be non-negative integers. Then $x \in D(T^{n+m})$ if and only if $T^n x \in D(T^m)$, and in this case

$$T^m(T^n x) = T^{n+m}x.$$

This is easily proved by induction on m.

This section is devoted to the study of the relationships between the subspaces $N(T^k)$, $D(T^k)$ and $R(T^k)$ $(k = 0, 1, 2, \ldots)$. First of all, we have the following well-known formulas:

$$N(T^k) \subset N(T^{k+1}),\ D(T^k) \supset D(T^{k+1}),\ R(T^k) \supset R(T^{k+1}) \text{ for } k = 0, 1, 2, \ldots.$$

Furthermore, we present the following lemmas.

Lemma 3.1. *For $k = 0, 1, 2, \ldots$ and $i = 0, 1, 2, \ldots$, we have*

$$\frac{N(T^{i+k})}{N(T^i)} \cong N(T^k) \cap R(T^i).$$

Proof. Define for each x in $N(T^{i+k}) \subset D(T^i)$

$$Jx = T^i x.$$

Then J is a linear operator from $N(T^{i+k})$ into the linear space $N(T^k) \cap R(T^i)$.

Let y be an element in $N(T^k) \cap R(T^i)$. Then $y = T^i x$ for some $x \in D(T^i)$, and $T^i x \in N(T^k) \subset D(T^k)$. This implies that x belongs to $D(T^{i+k})$ and $T^{i+k}x = T^k(T^i x) = 0$. Hence $x \in N(T^{i+k})$, and $Jx = y$. But then we have proved that J is a mapping onto $N(T^k) \cap R(T^i)$.

Since the kernel of J is $N(T^i)$, the last fact implies that the quotient space $N(T^{i+k})/N(T^i)$ is isomorphic with $N(T^k) \cap R(T^i)$.

Lemma 3.2. *For $k = 0, 1, 2, \ldots$ and $i = 0, 1, 2, \ldots$, we have*

$$\frac{R(T^i)}{R(T^{i+k})} \cong \frac{D(T^i)}{\{R(T^k) + N(T^i)\} \cap D(T^i)}.$$

Proof. Let $[y]$ denote any coset in the quotient space $R(T^i)/R(T^{i+k})$. Define for each x in $D(T^i)$

$$Jx = [T^i x].$$

Obviously, J is a linear operator from $D(T^i)$ onto $R(T^i)/R(T^{i+k})$.

If $Jx = 0$, then $T^i x = T^{i+k} z$ for some $z \in D(T^{i+k})$, and hence $x - T^k z \in N(T^i)$. This shows that

$$N(J) \subset \{R(T^k) + N(T^i)\} \cap D(T^i).$$

Conversely, if $x \in \{R(T^k) + N(T^i)\} \cap D(T^i)$, then $T^i x$ belongs to $R(T^{i+k})$, and hence $Jx = 0$. This shows

$$N(J) \supset \{R(T^k) + N(T^i)\} \cap D(T^i).$$

But then $N(J) = \{R(T^k) + N(T^i)\} \cap D(T^i)$ and

$$R(T^i)/R(T^{i+k}) \cong D(T^i)/N(J).$$

This completes the proof.

Lemma 3.3. *For* $k = 0, 1, 2, \ldots$ *and* $i = 0, 1, 2, \ldots$, *we have*

$$\frac{D(T^i)}{D(T^{i+k})} \cong \frac{R(T^i)}{D(T^k) \cap R(T^i)}.$$

Proof. Let $[y]$ denote any coset in $R(T^i)/\{D(T^k) \cap R(T^i)\}$. Define for each x in $D(T^i)$

$$Jx = [T^i x].$$

Then it is easy to show that J is a linear operator from $D(T^i)$ onto $R(T^i)/\{D(T^k) \cap R(T^i)\}$ with $N(J) = D(T^{i+k})$. Hence

$$\frac{D(T^i)}{D(T^{i+k})} \cong \frac{D(T^i)}{N(J)} \cong R(J) \cong \frac{R(T^i)}{D(T^k) \cap R(T^i)}.$$

Lemma 3.4. *For* $i = 0, 1, 2, \ldots$, *we have*

$$\frac{N(T^{i+1})}{\{N(T^i) + R(T)\} \cap N(T^{i+1})} \cong \frac{N(T) \cap R(T^i)}{N(T) \cap R(T^{i+1})}. \tag{3 - 1}$$

Proof. Let $[y]$ denote any coset in the quotient space

$$\frac{N(T) \cap R(T^i)}{N(T) \cap R(T^{i+1})}. \tag{3 - 2}$$

Define for each x in $N(T^{i+1})$

$$Jx = [T^i x].$$

Then J is a linear operator from $N(T^{i+1})$ into the space (3 - 2). Hence in order to prove (3 - 1), it will suffice to show that J is a mapping onto (3 - 2), and that

$$N(J) = \{N(T^i) + R(T)\} \cap N(T^{i+1}).$$

If $[y]$ is in (3 - 2), then $y = T^i x \in N(T)$ for some x in $D(T^i)$. But then x belongs to $N(T^{i+1})$ and $Jx = [T^i x] = [y]$. This shows that J is a mapping onto (3 - 2).

Take x in $N(J)$, then $T^i x \in N(T) \cap R(T^{i+1})$, and $T^i x = T^{i+1} z$ for some z in $D(T^{i+1})$. But then $x - Tz$ is in $N(T^i)$, and so $x \in N(T^i) + R(T)$. This shows

$$N(J) \subset \{N(T^i) + R(T)\} \cap N(T^{i+1}). \tag{3-3}$$

Conversely, take x in $\{N(T^i) + R(T)\} \cap N(T^{i+1})$. Then $x = n + Tz$ for some $n \in N(T^i)$ and some $z \in D(T)$. Since $Tz = x - n \in N(T^i) \subset D(T^i)$, we have $z \in D(T^{i+1})$. But then $T^i x = T^i n + T^{i+1} z = T^{i+1} z$, and hence $Jx = 0$. Combining this with (3 - 3), we obtain $N(J) = \{N(T^i) + R(T)\} \cap N(T^{i+1})$. This completes the proof.

Lemma 3.5. *For $i = 0, 1, 2, \ldots$, we have*

$$\dim \frac{N(T)}{N(T) \cap R(T^i)} = \dim \frac{N(T^i)}{R(T) \cap N(T^i)}.$$

Proof. First of all, we observe that it follows from Lemma 2.2 and from the preceding lemma that

$$\frac{N(T^{i+1}) + R(T)}{N(T^i) + R(T)} \cong \frac{N(T) \cap R(T^i)}{N(T) \cap R(T^{i+1})}$$

for $i = 0, 1, 2, \ldots$. But then

$$\begin{aligned} \dim \frac{N(T)}{N(T) \cap R(T^i)} &= \sum_{k=0}^{i-1} \dim \frac{N(T) \cap R(T^k)}{N(T) \cap R(T^{k+1})} \\ &= \sum_{k=0}^{i-1} \dim \frac{N(T^{k+1}) + R(T)}{N(T^k) + R(T)} = \dim \frac{N(T^i) + R(T)}{R(T)}, \end{aligned}$$

and so, once again by Lemma 2.2,

$$\dim \frac{N(T)}{N(T) \cap R(T^i)} = \dim \frac{N(T^i)}{R(T) \cap N(T^i)}.$$

If for some non-negative integer i

$$\dim N(T)/\{N(T) \cap R(T^i)\} < \infty,$$

then the preceding lemma implies that

$$\frac{N(T)}{N(T) \cap R(T^i)} \cong \frac{N(T^i)}{R(T) \cap N(T^i)}. \tag{3-4}$$

Using ZORN's lemma, it is possible to prove that formula (3 - 4) is always true.

4. Ascent, descent, nullity and defect

This section is devoted to the study of the relationships between the numbers $n(T)$, $d(T)$, $\alpha(T)$ and $\delta(T)$. For the definition of these numbers, we refer to Taylor's paper ([2], Section 1).

Some statements which appear in this section are already proved by TAYLOR. Since our methods differ considerably with those of [2], we present all theorems with full proof.

Theorem 4.1. *Suppose that $p = \alpha(T)$ and $q = \delta(T)$ are finite. Then*

$$\alpha(T) \leqq \delta(T), \tag{4-1}$$

and we have equality in (4 - 1) *if and only if T has the additional property*

$$D(T^p) \subset R(T) + D(T^q). \tag{4 - 2}$$

Proof. Suppose that $p = \alpha(T) > \delta(T) = q$. Then $R(T^p) = R(T^q)$, and hence we have by Lemma 3.1

$$\begin{aligned} 0 &= \dim\{N(T^{p+1})/N(T^p)\} = \dim\{N(T) \cap R(T^p)\} \\ &= \dim\{N(T) \cap R(T^q)\} = \dim\{N(T^{q+1})/N(T^q)\}. \end{aligned}$$

But this implies $p = \alpha(T) \leqq q$, contradicting the assumption $p > q$. Hence, we must have $p \leqq q$.

If $p = q$, then trivially $D(T^p) \subset R(T) + D(T^q)$.

Conversely, suppose that T has the additional property (4 - 2). Since $q = \delta(T) < \infty$, Lemma 3.2 implies that

$$D(T^q) \subset R(T) + N(T^q).$$

Combining this fact with (4 - 2), we obtain

$$D(T^p) \subset R(T) + N(T^q).$$

Since $p \leqq q$, the null space $N(T^p) = N(T^q)$, and so

$$D(T^p) \subset R(T) + N(T^p).$$

But then, by Lemma 3.2,

$$\frac{R(T^p)}{R(T^{p+1})} \cong \frac{D(T^p)}{\{R(T) + N(T^p)\} \cap D(T^p)} = (0),$$

and hence $R(T^p) = R(T^{p+1})$. This implies $p \geqq q$. Combining this fact with $p \leqq q$, we obtain $\alpha(T) = p = q = \delta(T)$.

Note that in the case $D(T) = X$, the linear operator T always satisfies the condition (4 - 2).

Theorem 4.2. *Suppose that either $n(T)$ or $d(T)$ is finite, and that $p = \alpha(T) < \infty$. Then*

$$n(T) \leqq d(T), \tag{4 - 3}$$

and we have equality in (4 - 3) *if and only if T has the additional property*

$$X = R(T) + N(T^p). \tag{4 - 4}$$

Proof. Since $p = \alpha(T) < \infty$, it follows from Lemma 3.1 that

$$N(T) \cap R(T^p) = (0).$$

But then Lemma 3.5 implies

$$n(T) = \dim \frac{N(T)}{N(T) \cap R(T^p)} = \dim \frac{N(T^p)}{R(T) \cap N(T^p)},$$

and so, by Lemma 2.3,

$$n(T) = \dim \frac{N(T^p)}{R(T) \cap N(T^p)} \leqq \dim \frac{X}{R(T) \cap X} = d(T). \tag{4 - 5}$$

This shows that $n(T) \leqq d(T)$.

Now suppose that we have equality in (4 - 3). Then we have also equality in (4 - 5), and hence

$$\dim \frac{N(T^p)}{R(T) \cap N(T^p)} = \dim \frac{X}{R(T) \cap X} < \infty .$$

But then Lemma 2.4 implies that $X = R(T) + N(T^p)$.

Conversely, suppose that T has the additional property (4 - 4). Then by Lemmy 2.2,

$$\frac{X}{R(T)} \cong \frac{R(T) + N(T^p)}{R(T)} \cong \frac{N(T^p)}{R(T) \cap N(T^p)} .$$

But then we have equality in (4 - 5), and hence $n(T) = d(T)$.

Theorem 4.3. *Suppose that* $n(T) = d(T) < \infty$, *and that* $p = \alpha(T) < \infty$. *Then*

(i) $\delta(T) = \alpha(T)$,

(ii) $n(T^i) = d(T^i) < \infty$ *for* $i = 0, 1, 2, \ldots,$

(iii) $X = R(T^p) \oplus N(T^p)$.

Proof. (i) From the preceding theorem it follows that

$$X = R(T) + N(T^p) .$$

This implies, by Lemma 3.2, that

$$\frac{R(T^p)}{R(T^{p+1})} \cong \frac{D(T^p)}{\{R(T) + N(T^p)\} \cap D(T^p)} = (0) ,$$

and hence $\delta(T) \leqq \alpha(T) < \infty$. Combining this fact with the result in Theorem 4.1, we obtain $\delta(T) = \alpha(T)$.

(ii) Let k be a non-negative integer. From

$$X \supset R(T^k) \supset R(T^{k+1})$$

it follows that

$$d(T^{k+1}) = \dim \frac{X}{R(T^{k+1})} = \dim \frac{X}{R(T^k)} + \dim \frac{R(T^k)}{R(T^{k+1})}$$
$$= d(T^k) + \dim \{R(T^k)/R(T^{k+1})\} .$$

Since $d(T) < \infty$, we have $d(T^k) \leqq d(T^{k+1}) < \infty$ (cf. [2], Lemma 3.3 (b)), and so

$$d(T^{k+1}) - d(T^k) = \dim \{R(T^k)/R(T^{k+1})\} .$$

Then we deduce from Lemma 3.2 that

$$d(T^{k+1}) - d(T^k) = \dim \frac{D(T^k)}{\{R(T) + N(T^k)\} \cap D(T^k)} .$$

Observe that $R(T) + N(T^k) + D(T^k) = R(T) + D(T^k)$, and apply Lemma 2.2, then

$$d(T^{k+1}) - d(T^k) = \dim \frac{R(T) + D(T^k)}{R(T) + N(T^k)} .$$

Since $n(T) = d(T) < \infty$, and since $p = \alpha(T) < \infty$, the preceding theorem implies that $X = N(T^p) + R(T)$. Observe that $N(T^p) \subset D(T^k)$. Then

$$X = R(T) + N(T^p) \subset R(T) + D(T^k) \subset X ,$$

and hence

$$d(T^{k+1})-d(T^k)=\dim X/\{R(T)+N(T^k)\}. \tag{4 - 6}$$

From $X\supset N(T^k)+R(T)\supset R(T)$, it follows that

$$\infty > d(T)=\dim \frac{X}{N(T^k)+R(T)}+\dim \frac{N(T^k)+R(T)}{R(T)}.$$

Combining this with formula (4 - 6), we obtain

$$d(T^{k+1})-d(T^k)=d(T)-\dim\{N(T^k)+R(T)\}/R(T). \tag{4 - 7}$$

Observe that the Lemmas 2.2 and 3.5 imply that

$$\begin{aligned}&\dim\{N(T^k)+R(T)\}/R(T)\\ &=\dim \frac{N(T^k)}{R(T)\cap N(T^k)}=\dim \frac{N(T)}{N(T)\cap R(T^k)}.\end{aligned}$$

Since $\dim N(T)=n(T)<\infty$, we have

$$\dim \frac{N(T)}{N(T)\cap R(T^k)}=n(T)-\dim\{N(T)\cap R(T^k)\}.$$

Combining these facts with formula (4 - 7) and using the hypothese $n(T)=d(T)$, we obtain

$$d(T^{k+1})-d(T^k)=\dim\{N(T)\cap R(T^k)\},$$

and hence, by Lemma 3.1

$$d(T^{k+1})-d(T^k)=\dim\{N(T^{k+1})/N(T^k)\}.$$

Since $N(T^k)\subset N(T^{k+1})$, and since $\dim N(T^{k+1})=n(T^{k+1})<\infty$, it follows that

$$d(T^{k+1})-d(T^k)=n(T^{k+1})-n(T^k). \tag{4 - 8}$$

Formula (4 - 8) holds for each non-negative integer k. Hence $d(T^0)=n(T^0)=0$ implies

$$d(T^i)=\sum_{k=0}^{i-1}\{d(T^{k+1})-d(T^k)\}=\sum_{k=0}^{i-1}\{n(T^{k+1})-n(T^k)\}=n(T^i).$$

This completes the proof of (ii).

(iii) Since $p=\alpha(T)<\infty$, Lemma 3.1 implies

$$N(T^p)\cap R(T^p)=(0). \tag{4 - 9}$$

But then

$$n(T^p)=\dim \frac{N(T^p)}{R(T^p)\cap N(T^p)}\leq \dim \frac{X}{R(T^p)}=d(T^p).$$

By (ii) we have $n(T^p)=d(T^p)<\infty$. Hence

$$\dim \frac{N(T^p)}{R(T^p)\cap N(T^p)}=\dim \frac{X}{R(T^p)}<\infty,$$

and so, by Lemma 2.4, we have $X=R(T^p)+N(T^p)$. Combining this with (4 - 9), we obtain

$$X=R(T^p)\oplus N(T^p).$$

In the particular case $D(T)=X$ and $p=\alpha(T)=\delta(T)<\infty$, it is possible to prove that

$$X=R(T^p)\oplus N(T^p),$$

without using the hypothesis $n(T)=d(T)<\infty$ (cf. [1], § 1, Hilfssatz 8). But in general Theorem 4.3 (iii) does not hold, if we omit the assumption that $n(T)=d(T)<\infty$, as is seen from the following example.

Let D be a linear manifold in the infinite dimensional linear space X such that $\dim X/D=1$. Suppose that T is the restriction of the null operator from X into X to D. Then $D=N(T)$ and $R(T)=(0)$, and hence

$$n(T)=d(T)=+\infty, \quad \alpha(T)=\delta(T)=1.$$

But $X \neq D=N(T)\oplus R(T)$.

It is interesting to note that in the case $n(T)=d(T)<\infty$ and $p=\alpha(T)=\delta(T)<\infty$, it is not necessary that $D(T^p)=D(T^{p+1})$. In order to show this, we present the following theorem.

Theorem 4.4. *Suppose that* $n(T)=d(T)<\infty$, *and that* $\alpha(T)=\delta(T)<\infty$. *Then, for* $i=0,1,2,\ldots$,

$$\frac{D(T^i)}{D(T^{i+1})}\cong\frac{X}{D(T)}.$$

Proof. Let i be some non-negative integer. Since $p=\delta(T)<\infty$, we have $R(T^p)\subset R(T^i)$. By the preceding theorem, $p=\alpha(T)=\delta(T)<\infty$ implies that $X=R(T^p)\oplus N(T^p)$. Combining these facts with $N(T^p)\subset D(T)$, we obtain

$$X=R(T^p)+N(T^p)\subset R(T^i)+N(T^p)\subset R(T^i)+D(T)\subset X,$$

and hence $X=R(T^i)+D(T)$. Then, by Lemma 2.2,

$$\frac{X}{D(T)}\cong\frac{D(T)+R(T^i)}{D(T)}\cong\frac{R(T^i)}{D(T)\cap R(T^i)}.$$

But then, as a consequence of Lemma 3.3,

$$\frac{X}{D(T)}\cong\frac{D(T^i)}{D(T^{i+1})}.$$

We proceed with an investigation of the case $\delta(T)<\infty$.

Theorem 4.5. *Suppose that either* $n(T)$ *or* $d(T)$ *is finite, and that* $q=\delta(T)<\infty$. *Then*

$$d(T)\leqq n(T)+\dim X/\{D(T^q)+R(T)\}, \tag{4-10}$$

and we have equality in (4 - 10) *if* T *has the additional property*

$$N(T)\cap R(T^q)=(0). \tag{4-11}$$

In the particular case that also $d(T)<\infty$, *we have equality in* (4 - 10) *if and only if* T *has the additional property* (4 - 11).

Proof. Since

$$X \supset D(T^q) + R(T) \supset N(T^q) + R(T) \supset R(T),$$

we have

$$d(T) = \dim \frac{X}{D(T^q) + R(T)} + \dim \frac{D(T^q) + R(T)}{N(T^q) + R(T)} + \dim \frac{N(T^q) + R(T)}{R(T)}.$$

By Lemma 3.2, $q = \delta(T) < \infty$ implies that

$$D(T^q) \subset R(T) + N(T^q),$$

and so

$$\dim \frac{D(T^q) + R(T)}{N(T^q) + R(T)} = 0.$$

Furthermore, it follows from Lemmas 2.2 and 3.5 that

$$\dim \frac{R(T) + N(T^q)}{R(T)} = \dim \frac{N(T^q)}{R(T) \cap N(T^q)}$$
$$= \dim N(T)/\{R(T^q) \cap N(T)\}.$$

Combining these facts, we obtain

$$d(T) = \dim \frac{X}{D(T^q) + R(T)} + \dim \frac{N(T)}{R(T^q) \cap N(T)}, \tag{4 - 12}$$

and hence

$$d(T) \leqq n(T) + \dim X/\{D(T^q) + R(T)\}.$$

If, in addition, $N(T) \cap R(T^q) = (0)$, then

$$n(T) = \dim N(T)/\{R(T^q) \cap N(T)\}.$$

But then formula (4 - 12) implies that we have equality in (4 - 10).

Conversely, suppose that

$$\infty > d(T) = n(T) + \dim X/\{D(T^q) + R(T)\}.$$

Then, by formula (4 - 12),

$$\infty > n(T) = \dim N(T)/\{R(T^q) \cap N(T)\},$$

and hence $R(T^q) \cap N(T) = (0)$.

Theorem 4.6. *Suppose that $n(T) = d(T) < \infty$, and that $q = \delta(T) < \infty$. Then*

$$\alpha(T) = \delta(T)$$

if and only if $X = D(T^q) + R(T)$.

Proof. In the case $X = D(T^q) + R(T)$, our hypotheses imply that we have equality in formula (4 - 10), and hence, since $d(T) < \infty$, we have $N(T) \cap R(T^q) = (0)$. But then, as a consequence of Lemma 3.1, we have $p = \alpha(T) \leqq q = \delta(T) < \infty$. Since

$$D(T^p) \subset X = D(T^q) + R(T),$$

Theorem 4.1 implies that $\alpha(T) = \delta(T)$.

Conversely, suppose that $n(T)=d(T)<\infty$ and $q=\alpha(T)=\delta(T)<\infty$. Then, by Theorem 4.3 (iii), we have $X=N(T^q)\oplus R(T^q)$. Since $N(T^q)\subset D(T^q)$ and $R(T^q)\subset R(T)$, this implies $X=D(T^q)+R(T)$.

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(Received October 20, 1965)