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Semi-Ideals in Posets

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1. Introduction

In this paper we develop a theory of semi-ideals for posets (partially ordered sets). In Section 2 we summarize some results used in subsequent sections; for the proofs of these results, the reader is referred to [12]. Some concepts relating to ideals like cut-complement, comprincipal envelope and normality, as well as results concerning them carry over almost verbatim to semi-ideals. These are given in Section 3. The concepts of comprincipal ideal, cut-complement of an ideal and comprincipal envelope of an ideal were introduced by Vaidyanathaswamy [10]. The concept of comprincipal ideal is identical with that of closed ideal of Birkhoff [3, p. 59]. Section 4 is devoted to a study of normal and dense semi-ideals. In this section we obtain generalisations of some theorems of Balachandran [1] and a theorem of Pankajam [6]. The last two sections deal with prime semi-ideals. Guided by the definition of Stone's topology for prime ideals (in a distributive lattice) we introduce a topology for the prime semi-ideals in a poset, and obtain extensions (vide Section 6) of some of the results of Stone [8] and Balachandran [2]. While the above topology for semi-ideals shares some of the features of Stone's topology (cf. Theorems 28 and 32) there are also one or two points of departure. Thus the topology for the prime semi-ideals is connected and non-Hausdorff while the Stone's topology of a Boolean algebra is totally disconnected and Hausdorff. space.

2. Preliminaries

We shall denote the ordering relation in a poset by $\leqq$. The greatest and least elements of a poset, whenever they exist, will be denoted by 1 and 0 respectively. A non-null subset A of a poset P is called a semi-ideal if $a \in A$, $b \leqq a (b \in P) \Rightarrow b \in A$. A semi-ideal A of P is called an ideal if the sum of any finite number of elements of A, whenever it exists, belongs to A. The principal ideal generated by a and the principal dual ideal generated by a are denoted by $(a]$ and $[a)$ respectively. An element a of a poset P with 0 is said to have a pseudo-complement a^*, if in P, there exists an element a^* such that $(a] \cap (a^*] = (0]$ and for $b \in P$, $(a] \cap (b] = (0] \Rightarrow (b] \subseteqq (a^*]$.

Theorem A. *The set S_u of all semi-ideals of a poset P with 0, forms a complete distributive lattice closed for pseudo-complements under set-inclusion as ordering* *relation. The lattice sum and lattice product in S_μ coincide with the set union and set intersection. Similar result holds for the set S_α of dual semi-ideals of a poset with* 1.

The pseudo-complement of an element A of S_μ will be denoted by A^*.

Theorem B. *The set I_μ of all ideals of a poset P with* 0 *forms a complete lattice under set-inclusion as ordering relation; the lattice product in I_μ is the same as that in S_μ. Similar result holds for the set I_α of all dual ideals of a poset with* 1.

We shall denote the set-inclusion, set-union and set-intersection by $\subseteqq$, $\cup$ and $\cap$ respectively. The lattice sum in I_μ and I_α will be denoted by $\bigvee$.

Lemma A. *In a poset P, a lattice product $\prod_{i\in I} a_i$ $\left(\text{lattice-sum } \sum_{i\in I} a_i\right)$ exists if and only if $\cap(a_i]$ $(\cap[a_i))$ is a principal ideal (principal dual ideal). Also whenever $\Pi a_i(\sum a_i)$ exists $\cap(a_i]=(\Pi a_i]$ $(\cap[a_i)=[\sum a_i))$.*

Lemma B. *In a poset P with* 0, *the pseudo-complement a^* of an element a exists if and only if $(a]^*$ is a principal ideal. Further whenever a^* exists $(a]^* =(a^*]$.*

Theorem C. *In a poset P closed for pseudo-complements, the following results hold:*

(i) $a \leqq a^{**}$ *for every* $a \in P$.

(ii) $a \leqq b \Rightarrow a^* \geqq b^*$ *for* $a, b \in P$.

(iii) $a^{***}=a^*$ *for every* $a \in P$.

(iv) *P has the greated element* 1 *and* $1=0^*$.

Theorem D. *In a poset P closed for pseudo-complements the following results hold:*

(i) *If a finite product $a_1.a_2.\ldots.a_n$ exists in P, then so does the product $a_1^{**}.a_2^{**}.\ldots.a_n^{**}$. Further $(a_1,a_2.\ldots.a_n)^{**}=a_1^{**}.a_2^{**}.\ldots.a_n^{**}$ and $(a_1.a_2.\ldots.a_n)^* =(a_1^{**}.a_2^{**}.\ldots.a_n^{**})^*$.*

(ii) *If a sum $\sum_{i\in I} a_i$ exists in P, then the product $\prod_{i\in I} a_i^*$ exists in P and $(\sum a_i)^*=\Pi a_i^*$.*

3. Comprincipal Ideal and Cut-complement

A semi-ideal of a poset P is called a comprincipal ideal if it is a product of principal ideals. The comprincipal envelope of a semi-ideal is the product of all the principal ideals containing it. Similarly we define a comprincipal dual ideal and the comprincipal envelope of a dual semi-ideal. By the cut-complement of a semi-ideal (dual semi-ideal) A of P, we mean the set of all elements x such that $x \geqq a (x \leqq a)$ for all $a \in A$ and it is denoted by A_c. The cut-complement of A_c is denoted by A_{cc}.

The following two results (Theorem 1 and 2) are clear.

Theorem 1. *Any comprincipal ideal of a poset P is the product of all the principal ideals containing it. Similar result holds for dual ideals.*

Theorem 2. *The comprincipal envelope of a semi-ideal (dual semi-ideal) A of a poset P is the smallest comprincipal ideal (comprincipal dual ideal) containing A.*

Remark. For any semi-ideal (dual semi-ideal) A, A_c is a comprincipal dual ideal (comprincipal ideal).

Theorem 3. *The comprincipal envelope of a semi-ideal or a dual semi-ideal A of a poset P is A_{cc}.*

Proof. Let A be a semi-ideal of P. Then $A_{cc} = \bigcap_{x \in A_c} (x]$. Now the elements $x \in A_c$ are precisely those elements of P which are $\geqq a$ for every $a \in A$. It follows that A_{cc} is the product of all the principal ideals containing A. Hence the first part. The second part is proved on similar lines.

Theorem 4. *A semi-ideal or a dual semi-ideal A of a poset is comprincipal if and only if $A = A_{cc}$.*

Theorem 4 follows from Theorem 3.

4. Normal and Dense Semi-ideals

A semi-ideal (dual semi-ideal) of a poset P with $0(1)$ is called normal if it is a normal element of the lattice $S_\mu(S_\alpha)$ of all semi-ideals (dual semi-ideals) of P. Similarly we define a dense semi-ideal (dense dual semi-ideal) in a poset with $0(1)$. If the lattice of all ideals (dual ideals) $I_\mu(I_\alpha)$ of a poset P with $0(1)$ is closed for pseudo-complements, an ideal (a dual ideal) of P is called normal if it is a normal element of $I_\mu(I_\alpha)$. An ideal (a dual ideal) of a poset P with $0(1)$ is called dense if it is a dense element of $I_\mu(I_\alpha)$. We shall denote the dual ideal consisting of the dense elements of a poset by D.

Hereafter, throughout this section, unless otherwise stated, P will denote a poset closed for pseudo-complements.

Theorem 5. *Every normal semi-ideal of P is a comprincipal ideal.*

Proof. Any normal semi-ideal of P is of the form A^* for some $A \in S_\mu$. Since $A = \bigcup_{a \in A} (a]$, by Theorem D, $A^* = \bigcap_{a \in A} (a]^* = \bigcap_{a \in A} (a^*]$ by Leυma B, thus completing the proof.

Remark. If $A, A^* \in I_\mu$ then clearly $A^* = A^{\circledast}$($A^{\circledast}$ denotes the pseudo-complement of A in I_μ).

Theorem 6. *I_μ is closed for pseudo-complements.*

Proof. Let $A \in I_\mu$. Then clearly $A \in S_\mu$ and so A^* exists. By Theorem 5, $A^* \in I_\mu$. Hence the result follows by the above remark.

Corollary 1. *Any normal semi-ideal A of P is a normal ideal of P.*

Proof. Clearly $A = B^*$ for some $B \in S_\mu$. Hence by Theorem C, $A = B^{***}$. Also, by Theorem 5, $B^{**} \in I_\mu$. Now the result follows by the remark under Theorem 5.

Corollary 2. *Every normal ideal A of a complete lattice L closed for pseudo-complements is principal.*

Proof. From Theorem 5 it follows that A is of the form $\bigcap_{i \in I} (a_i^*]$. Since L is a complete lattice, by Lemma A, $\cap(a_i^*] = (\Pi a_i^*]$. Hence the result.

Theorem 7. *The pseudo-complement of a semi-ideal A of P is identical with that of its comprincipal envelope.*

Proof. Since A^{**} is comprincipal (Theorem 5) and A_{cc} is the smallest comprincipal ideal containing A (Theorem 2), we have $A \subseteqq A_{cc} \subseteqq A^{**}$. By Theorem C, it follows that $A^* \supseteqq (A_{cc})^* \supseteqq A^{***} = A^*$. Hence $A^* = (A_{cc})^*$.

Theorem 8. *The dense semi-ideals of P are precisely those whose cut-complement is contained in D.*

Proof. Let A be a semi-ideal of P with $A^* = (0]$. Then if $x \in A_c$, $(x] \supseteqq A$ and so by Theorem C and Lemma B, we have $(x^*] \subseteqq A^* = (0]$. Hence $x \in D$ and so $A_c \subseteqq D$.

Conversely, suppose A is a semi-ideal such that $A_c \subseteqq D$. If $x \in A^*$, by Theorem C and Lemma B, we have $(x^*] \supseteqq A^{**} \supseteqq A$. It follows that $x^* \in A_c \subseteqq D$. Hence $x \leqq x^{**} = 0$ and so $A^* = (0]$.

Theorem 9. *If in a poset P (not necessarily closed for pseudo-complements) with* 0, 1, $D = [1)$, *then any dense semi-ideal A has P for its comprincipal envelope.*

Proof. If $(a]$ is any principal ideal containing A, $(a]^* \subseteqq A^* = (0]$. Hence $a \in D$. As $D = [1)$, it follows that $(1]$ is the only principal ideal containing A, whence the result follows.

Theorem 10. *In a complete lattice L closed for pseudo-complements the dense semi-ideals are precisely those whose comprincipal envelope is a principal ideal having non-void intersection with D.*

Proof. Since L is a complete lattice, if A is any semi-ideal of L, A_{cc} is principal, say $A_{cc} = (t]$. If $A^* = (0]$, by Theorem 7, $(t]^* = (0]$. Hence $t \in D$. Thus $t \in A_{cc} \cap D$, proving thereby $A_{cc} \cap D$ is non-null. Converse is got by retracing the steps.

Theorem 11. *If every element of P is normal the set of normal ideals coincides with the set of comprincipal ideals.*

Proof. In view of Theorem 5, it suffices to prove that every comprincipal ideal $\bigcap_{i \in I} (a_i]$ of P is normal. Since every element of P is normal, $\bigcap_{i \in I} (a_i] = \bigcap_{i \in I} (a_i^{**}] = \left(\bigvee_{i \in I} (a_i^*]\right)^{\circledast}$ by Theorem D (as I_μ is closed for pseudo-complements by Theorem 6). Hence the result.

Remark. Theorems 7, 8, 9, and 10 generalise the corresponding results of Balachandran [1] about ideals in distributive lattices. Theorem 11 is a generalisation of the following result of Pankajam [6]: In a Boolean algebra the set of comprincipal ideals is identical with the set of normal ideals.

5. Prime Semi-ideals in Posets

A semi-ideal (dual semi-ideal) A of a poset P, $(A \neq P)$ is called prime if $(a] \cap (b] \subseteq A \Rightarrow (a] \subseteq A$ or $(b] \subseteq A$ $([a) \cap [b) \subseteq A \Rightarrow [a) \subseteq A$ or $[b) \subseteq A)$. An ideal (a dual ideal) of a poset is called prime if it is prime as a semi-ideal (dual semi-ideal). It is clear that in a lattice our definitions of prime ideal and prime dual ideal coincide with the usual definitions.

A prime semi-ideal (prime ideal) A of a poset P is called minimal prime if A does not contain any other prime semi-ideal (prime ideal). A semi-ideal (ideal) A of P is called completely meet-irreducible if A is not the product of any family of semi-ideals (ideals) which does not contain A as a member. By the corollary under Theorem 12, it follows that a completely meet-irreducible semi-ideal of a poset is prime.

Theorem 12 (cf. [7, Theorem 16] and [4, Corollary 1 under Theorem 8]). *Given a semi-ideal A of a poset P and $b \notin A (b \in P)$, among all the semi-ideals containing A and not containing b, there exists a maximal one, and it is prime.*

Proof. Since the set-union of any family of semi-ideals is a semi-ideal and is the *l.u.b* of the family, first part follows by Zorn's Lemma.

Now let B be a semi-ideal which is maximal among all the semi-ideals containing A and not containing b. Suppose B is not prime. Then there exist $x, y \in P$ such that $(x] \cap (y] \subseteq B$ and $(x], (y] \nsubseteq B$. The maximal property of B implies that $b \in B \cup (x], B \cup (y]$. Consequently $b \in (x], (y]$ so that $b \in (x] \cap (y]$ which is a contradiction. Hence B is prime.

Corollary. *Any semi-ideal of a poset is the product of all the prime semi-ideals containing it.*

Theorem 13. *If a prime semi-ideal of a poset contains the product of a finite number of semi-ideals, then it contains at least one of them.*

The proof of the above theorem is similar to that of the corresponding known result concerning prime ideals in a lattice (vide [9, Theorem 8]).

Corollary. *If the product of a finite number of semi-ideals of a poset P is* $(0]$, *then any prime semi-ideal of P contains at least one of them.*

Theorem 14. *A prime semi-ideal A of a poset with 0 is either normal or dense.*

Proof. $A^* \cap A^{**} = (0]$ and so by the corollary under Theorem 13, $A \supseteq A^*$ or $A \supseteq A^{**}$. Hence the result.

Theorem 15. *The set complement cA of a prime semi-ideal A of a poset P is a dual ideal.*

Proof. Clearly cA is a dual semi-ideal. Further if $x_1, x_2, \ldots, x_n \in cA$ and $x_1 . x_2 . \ldots . x_n$ exists, by Lemma A, $(x_1 . x_2 . \ldots . x_n] = (x_1] \cap (x_2] \cap \cdots \cap (x_n] \nsubseteq A$, since $(x_i] \nsubseteq A$ for $i = 1, 2, \ldots, n$ and A is prime. Hence $x_1 . x_2 . \ldots . x_n \in cA$ and so cA is a dual ideal.

Theorem 16 (cf. [2, 3.2.6]). *The set complement cM of a maximal principal dual ideal $M = [a)$ of a poset with 0 is a normal prime semi-ideal.*

Proof. Clearly cM is a semi-ideal. Now if $(x], (y] \nsubseteq cM$, then $x, y \geqq a$ and so $a \in (x] \cap (y]$. Hence $(x] \cap (y] \nsubseteq cM$, proving that cM is prime. To prove cM is normal, in view of Theorem 14, it suffices to show that $(cM)^* \neq (0]$. Now if $b \in cM$, then $b \notin [a)$ and so, as $[a)$ is maximal $a \cdot b = 0$. Hence $a \in (cM)^*$. Consequently $(cM)^* \neq (0]$, completing the proof.

Theorem 17 (cf. [2, Corollary under 3.2.2]). *The pseudo-complement of a semi-ideal A of a poset P with 0 is the product of all the prime semi-ideals not containing A.*

Proof. Let $B =$ product of all the prime semi-ideals not containing A. From the corollary under Theorem 13, it follows that $B \supseteqq A^*$. If possible suppose $B \neq A^*$. Then there exists $x \in P$ such that $x \in B$, $x \notin A^*$. So, for some $y \in A$, $(x] \cap (y] \neq (0]$. Hence by Theorem 12, there exists a prime semi-ideal C such that $(x] \cap (y] \nsubseteq C$. Clearly $(x], (y] \nsubseteq C$. Consequently $A, B \nsubseteq C$ which is a contradiction to the choice of B. Hence $B = A^*$.

Theorem 18. *The set-union of any family of prime semi-ideals of a poset P is a prime semi-ideal. The set-intersection of any lower-directed family of prime semi-ideals is a prime semi-ideal.*

Proof. The first part is obvious.

Now let $F = \{A_i \mid i \in I\}$ be a lower-directed family of prime semi-ideals of P. Let $B = \bigcap_{i \in I} A_i$. Suppose B is not prime. Then there exist $x, y \in P$ such that $(x] \cap (y] \subseteqq B$ and $(x], (y] \nsubseteq B$. Hence $(x] \nsubseteq A_i$, $(y] \nsubseteq A_j$ for some $i, j \in I$. As F is lower-directed there exists a $k \in I$ such that $A_k \subseteqq A_i, A_j$. Clearly $(x], (y] \nsubseteq A_k$. But $(x] \cap (y] \subseteqq B \subseteqq A_k$. This contradicts the fact that A_k is prime. Hence B is prime.

6. A Topology for the Prime Semi-ideals

Most of the topological concepts used in this section are found in [5] and [11]. However, we recall them for convenience.

Let T be a topological space. T is called, as usual, T_0 if distinct points of T have distinct closures. A point p of T is called a T_1 point if the closure of p contains no point other than p; a point p of T is called an anti-T_1 point if the closure of no point other than p contains p. T is called T_1 if every point of T is T_1. T is called T_2 if any two distinct points of T have disjoint neighbourhoods.

A closed (open) sub-set of T is called a closed domain (open domain) if it is identical with the closure of its interior (interior of its closure). A closed (open) sub-set of T is called semi-regular if it is an intersection (union) of closed domains (open domains). A closed (open) sub-set of T is called regular if it is an intersection (union) of closed domains (open domains) whose interiors (closures) contain (are contained in) it. T is called semi-regular (regular) if every open sub-set of T is semi-regular (regular). The regularity of T may alternatively be expressed as follows: Given a non-void closed sub-set C of T and a point $p \notin C$, we can find closed sub-sets C_1, C_2, of T containing C, p respectively such that $C_1 \cap (p) = \emptyset = C \cap C_2$ and $C_1 \cup C_2 = T$ ($\emptyset$ denotes the empty set). A T_1 regular space is called, as usual, a T_3-space. A sub-set A of T is called compact, if from every open covering of A we can extract a finite covering.

By the Hausdorff-residue of a sub-set A of T we mean $R(A)$ where $R(A) = ff(A)$, $f(A) =$ closure of $A - A$.

Throughout this section, unless otherwise stated, P denotes a poset with 0 and $\mathscr{P}$ the set of all prime semi-ideals of P. The set of all prime semi-ideals containing a semi-ideal A is denoted by $F(A)$ and the set-complement of $F(A)$ in $\mathscr{P}$ by $F'(A)$.

Theorem 19 (cf. [2, 2.3.1 to 2.3.4]).

(i) $F\left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I} F(A_i)$,

(ii) $F(A_1 \cap A_2 \cap \cdots \cap A_n) = F(A_1) \cup F(A_2) \cup \cdots \cup F(A_n)$,

(iii) $F(P) = \emptyset$,

(iv) $F((0]) = \mathscr{P}$.

Proof. (i) It suffices to observe that a semi-ideal $B \supseteq \bigcup_{i\in I} A_i \Leftrightarrow B \supseteq A_i$ for every $i \in I$.

(ii) It is clear from the definition of F that $A \subseteq B \Rightarrow F(A) \supseteq F(B)$. Hence $F(A_1 \cap A_2 \cap \cdots \cap A_n) \supseteq F(A_i)$ for $i = 1, 2, \ldots, n$ and so $F(A_1 \cap A_2 \cap \cdots \cap A_n) \supseteq F(A_1) \cup F(A_2) \cup \cdots \cup F(A_n)$. Also from Theorem 13 it follows that $F(A_1 \cap A_2 \cap \cdots \cap A_n) \subseteq F(A_1) \cup F(A_2) \cup \cdots \cup F(A_n)$. Hence (ii).

(iii) and (iv) are obvious.

Theorem 19 shows that we can introduce a (unique) topology T on $\mathscr{P}$ whose closed sub-sets are precisely the sets $F(A)$. We shall denote $\mathscr{P}$ with topology T again by $\mathscr{P}$.

Since $F'(A) = \mathscr{P} - F(A)$, the following result follows from Theorem 19.

Theorem 20. (i) $F'\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} F'(A_i)$,

(ii) $F'(A_1 \cap A_2 \cap \cdots \cap A_n) = F'(A_1) \cap F'(A_2) \cap \cdots \cap F'(A_n)$,

(iii) $F'(P) = \mathscr{P}$,

(iv) $F'((0]) = \emptyset$.

Corollary. *The lattice of all open (closed) sub-sets of $\mathscr{P}$ is isomorphic (dually isomorphic) to S_μ and the mapping associates unrestricted lattice-sums with the corresponding set-unions (set-intersections).*

Proof. Since by the corollary under Theorem 12, $A \subseteqq B \Leftrightarrow F(A) \supseteqq F(B)$, the result follows from Theorems 19 and 20.

We shall denote the closure, interior and set-complement of a subset X of $\mathscr{P}$ by cl. X, int. X and cX respectively.

Theorem 21. *If X is a sub-set of $\mathscr{P}$, then cl. $X = F(X_0)$ where X_0 is the product of all the members of X.*

Proof. Clearly $F(X_0)$ is a closed sub-set of $\mathscr{P}$ containing X. Also if $F(Y_0)$ is a closed sub-set of $\mathscr{P}$ containing X, each member of X contains Y_0 and so $X_0 \supseteqq Y_0$. Consequently $F(X_0) \subseteqq F(Y_0)$, whence the result follows.

Theorem 22. (i) cl. $F'(A) = F(A^*)$.

(ii) int. $F(A) = F'(A^*)$.

Proof. (i) follows from Theorems 21 and 17.

(ii) int. $F(A) = c$ cl. $F'(A) = F'(A^*)$ by (i).

Theorem 23. *$\mathscr{P}$ is T_0.*

Proof. From Theorem 21, it follows that the closure of a single point is the set of all prime semi-ideals containing it. Clearly of any two distinct (prime) semi-ideals at least one does not contain the other. Hence distinct points of $\mathscr{P}$ have distinct closures. Thus $\mathscr{P}$ is T_0.

Theorem 24. *When P is a poset with 0, 1, $\mathscr{P}$ is T_1 if and only if P is the chain of two elements.*

Proof. Clearly the only T_1 point of $\mathscr{P}$ is the semi-ideal consisting of all the elements of P other then 1. Hence the result.

Theorem 25. *When P is a poset with 0, 1, $\mathscr{P}$ is compact and non-regular.*

Proof. Let $\mathscr{P} = \bigcup_{i\in I} F'(A_i)$. Then by (i) of Theorem 20, $\mathscr{P} = F'\left(\bigcup_{i\in I} A_i\right)$. Since $\mathscr{P} = F'(P)$, it follows that $P = \bigcup_{i\in I} A_i$. As $P = (1]$, $1 \in A_j$ for some $j \in I$. Consequently $P = A_j$ so that $\mathscr{P} = F'(A_j)$. Hence $\mathscr{P}$ is compact.

Now let $C = F(A)$ be a non-empty closed sub-set of $\mathscr{P}$ and $p \in \mathscr{P} - C$. Suppose $\mathscr{P}$ is regular. Then there exist closed sub-sets $C_1 = F(A_1)$, $C_2 = F(A_2)$ containing C and p respectively such that $C \cap C_2 = \emptyset = C_1 \cap (p)$ and $C_1 \cup C_2 = \mathscr{P}$. Since $C \cap C_2 = \emptyset$ it follows that $A \cup A_2 = P$. Consequently as $1 \in P$, $A = P$ or $A_2 = P$. So $C = \emptyset$ or $C_2 = \emptyset$ which is a contradiction to our hypothesis. Hence the result.

Theorem 26. *A closed sub-set $F(A)$ of P is a closed domain if and only if A is a normal semi-ideal.*

Proof. By Theorem 22, cl. int. $F(A) =$ cl. $F'(A^*) = F(A^{**})$. It follows that $F(A)$ is a closed domain if and only if $A = A^{**}$, thus proving the result.

Corollary. *An open sub-set $F'(A)$ of $\mathscr{P}$ is an open domain if and only if A is normal.*

Since the closed domains and the open domains of $\mathscr{P}$ are mutually complementary, the above result follows from Theorem 26.

Theorem 27. *An open sub-set $F'(A)$ of $\mathscr{P}$ is semi-regular if and only if A is a sum of normal semi-ideals.*

Proof. By the corollary under Theorem 26, $F'(A)$ is semi-regular if and only if $F'(A) = \bigcup_{i\in I} F'(N_i)$ where the N_i are normal semi-ideals. By (1) of Theorem 20, $\bigcup_{i\in I} F'(N_i) = F'\left(\bigcup_{i\in I} N_i\right)$. Hence by the corollary under Theorem 12, it follows that $F'(A)$ is semi-regular if and only if $A = \bigcup_{i\in I} N_i$, thus proving the result.

Corollary. *A closed sub-set $F(A)$ of $\mathscr{P}$ is semi-regular if and only if A is a sum of normal semi-ideals of P.*

The following result is clear from Theorem 27.

Theorem 28. (cf. [2, 5.2.2]). *$\mathscr{P}$ is semi-regular if and only if every semi-ideal of P is a union of normal semi-ideals.*

Theorem 29. *A closed sub-set $F(A)$ of $\mathscr{P}$ is non-dense if and only if A is a dense semi-ideal.*

Proof. By Theorem 22, int. $F(A) = F'(A^*)$. Hence $F(A)$ is non-dense if and only if $F'(A^*) = \emptyset$. But $F'((0]) = \emptyset$. Now the result follows from the corollary under Theorem 12.

Corollary. *An open sub-set $F'(A)$ of $\mathscr{P}$ is dense if and only if A is a dense semi-ideal.*

Theorem 30. *The only closed sub-sets of $\mathscr{P}$ which are also open are the sets $\emptyset$ and $\mathscr{P}$. Hence the space is connected.*

Proof. Clearly the sets $\emptyset$ and $\mathscr{P}$ are closed. Now any closed sub-set of $\mathscr{P}$ is of the form $F(A)$, $A \in S_\mu$ and int. $F(A) = F'(A^*)$, so that, if $F(A)$ is open, $F(A) = F'(A^*)$. Hence $F'(A \cup A^*) = \mathscr{P}$ and so $A \cup A^* = P$. As $1 \in P$, it follows that $A = P$ or $A^* = P$. Consequently $A = P$ or $A = (0]$. Hence $F(A) = \emptyset$ or $F(A) = \mathscr{P}$, thus proving the result.

Theorem 31. *An open sub-set $F'(A)$ of $\mathscr{P}$ is compact if and only if A is a union of a finite number of principal ideals.*

Proof. Suppose $A = (a_1] \cup (a_2] \cup \cdots \cup (a_n]$ and $F'(A) \subseteqq \bigcup_{i\in I} F'(A_i)$. Then it follows that $A \subseteqq \bigcup_{i\in I} A_i$ and so each $a_j \in A_{i_j}$ for $j = 1, 2, \ldots, n$. Hence $A \subseteqq A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_n}$, so that, $F'(A) \subseteqq F'(A_{i_1}) \cup F'(A_{i_2}) \cup \cdots \cup F'(A_{i_n})$. Thus $F'(A)$ is compact.

Conversely, suppose $F'(A)$ is compact. Now $F'(A) = F'\left(\bigcup_{a\in A} (a]\right) = \bigcup_{a\in A} F'((a])$ $= F'((a_1]) \cup F'((a_2]) \cup \cdots \cup F'((a_n])$, $a_1, a_2, \ldots, a_n \in A$ as $F'(A)$ is compact. It follows that $A = (a_1] \cup (a_2] \cup \cdots \cup (a_n]$.

Since the principal ideals form an additive basis for S_μ, the above theorem implies

Theorem 32 (cf. [2, 2.3.10]). *The compact open sub-sets of $\mathscr{P}$ form a basis of open sub-sets.*

Theorem 33 (cf. [2, 2.3.40]). *The completely meet-irreducible semi-ideals of P are identical in their totality with points of $\mathscr{P}$ which have null Hausdorff-residue.*

Proof. Suppose $A \in \mathscr{P}$ and A is completely meet-irreducible. Then $A \subset B$ where $B =$ product of all the prime semi-ideals strictly containing A. Hence cl.$(A) - (A) = F(B)$ which is a closed sub-set of $\mathscr{P}$. It follows that $R(A) = \emptyset$. The converse is got by retracing the steps.

Theorem 34 (cf. [8, Theorem 6] and [2, 2.3.41]). *A necessary and sufficient condition for a prime semi-ideal A of P to be an isolated point of $\mathscr{P}$ is that A is normal and completely meet-irreducible.*

Proof. Suppose A is an isolated point of $\mathscr{P}$. Then $(A) =$ int. $(A) = c$ cl. $c(A) = F'(B)$ where $B =$ product of all the prime semi-ideals of P other than A. Hence

$$A \nsupseteq B \tag{1}$$

$F(A) =$ cl. $(A) =$ cl. $F'(B) = F(B^*)$ (Theorem 22) and so by the corollary under Theorem 12, $A = B^*$. Thus A is normal. Suppose A is not completely meet-irreducible. Then $A = A_1$ where $A_1 =$ product of all the prime semi-ideals of P strictly containing A. So $A = A_1 \supseteq B$ which contradicts (1). Hence A is completely meet-irreducible.

Conversely, suppose A is normal and completely meet-irreducible. Then by Theorem 14, $A^* \neq (0]$ and so

$$A \nsupseteq A^* \tag{2}$$

If $A_1 =$ product of all the prime semi-ideals strictly containing A and B $=$ product of all the prime semi-ideals other than A, as A is completely meet-irreducible, we have

$$A \nsupseteq A_1 . \tag{3}$$

Since A is prime, from (2), (3) and Theorem 13, it follows that $A \nsupseteq A^* \cap A_1 = B$ by Theorem 17. Hence $(A) = F'(B)$ and so A is an isolated point.

The following result is clear.

Theorem 35. *The minimal prime semi-ideals of P are precisely the anti-T_1 points of $\mathscr{P}$.*

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