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SCOPELESS QUANTIFIERS AND OPERATORS

0. OVERVIEW

The main objective of this paper is to give a characterization of those quantifiers and operators that are freely interchangeable with all other quantifiers or operators on a possibly different domain. The starting point is a slight generalization of an earlier result on unary extensional quantifiers. These are shown to be scopeless just in case they are ultrafilters with certain strong completeness properties: in many, though not all cases, a quantifier must be trivial or name-like (i.e. principal) in order to be scopeless. Which cases depends on the relative sizes of the domains of quantification.

Operators other than unary extensional quantifiers for which the notion of scopelessness also makes sense include quantifiers in three-valued logic, intensional quantifiers and propositional operators in possible worlds semantics, as well as modifiers in natural language. These operators, about which the above results have nothing to say, are the subject of Section 2.

The characterization result can in a sense, however, be extended to them. This is done in Section 3. Although the notion of a complete ultrafilter is as central in their case as it is in the case of unary extensional quantifiers, the results for the latter do not generalize as directly as one might think: scopelessness turns out to be even rarer in the more general setting.

Finally, in Section 4, we briefly turn to notions of scoplessness that do not involve variable-binding, where it can be shown that completeness, but not ultrafilterhood, is irrelevant.

1. SCOPELESS QUANTIFIERS

The aim of this section is to give a characterization of those quantifiers that show no scope interactions with any other quantifiers, i.e.

Journal of Philosophical Logic **22**: 545–561, 1993.

those F that satisfy:

$$(S_q) \qquad (Fx)(Qy)(xRy) \quad \text{iff} \quad (Qy)(Fx)(xRy),$$

for arbitrary relations R and quantifiers Q. The term *quantifier* will be used in the broad (and purely semantic) sense of a second-order set: if A is a non-empty set, then a *quantifier Q over A* is a set of subsets of A; and the bound-variable notation '$(Qx)\phi$' expresses that the set of x satisfying ϕ is in Q: $\{x \in A|\phi\} \in Q$. Since we will not assume that the quantifiers Q in (S_q) range over the same domain as F, our notion of scopelessness is a relative one: a quantifier F over A is said to be *B-scopeless* if (S_q) holds for every quantifier Q over B and every relation $R \subseteq A \times B$. If we think of such R as functions from B to A's power set, 'Ry' naturally denotes the set of y's predecessors: $\{x \in A|xRy\}$; similarly, I will use 'xR' to refer to x's successors: $\{y \in B|xRy\}$.

A brief look at the literature on generalized quantifiers already suggests a class of obvious candidates for scopelessness, viz. the name-like *principal ultrafilters over A* (*generated by $x \in A$*), i.e. the sets x^* ($= x^*_A$) of all subsets of A containing x as an element: in their logical behaviour they are hardly distinguishable from (names of) $x \in A$, for which the notion of scope does not even make sense. And, indeed, if Q is any quantifier over B and $R \subseteq A \times B$, then arbitrary $x_0 \in A$ satisfy: $(Qy)\ (x_0^* x)\ (xRy)$ iff $(Qy)\ (Ry \in x_0^*)$ iff $(Qy)\ (x_0 Ry)$ iff $x_0 \in \{x \in A|(Qy)\ (xRy)\}$ iff $(x_0^* x)\ (Qy)\ (xRy)$. Thus every principal ultrafilter over A is B-scopeless.

Principal ultrafilters are only special cases of *ultrafilters* (*over A*) in general, i.e. quantifiers F that are closed under supersets and (finite) intersections as well as *\-maximal* in the sense that $X \in F$ iff $\bar{X}$ ($= A \backslash X$) $\notin F$. It is easily seen (and well-known) that ultrafilters are exactly those quantifiers F that lack scope with respect to arbitrary truth-functional connectives ©: $[(Fz)\ z \in X$ © $(Fz)\ z \in Y]$ iff (Fz) $[z \in X$ © $z \in Y]$. Hence, due to the functional completeness of *neither–nor*, we find that F is an ultrafilter over A iff all subsets X and Y of A satisfy: $\overline{X \cup Y} \in F$ iff $[X \notin F$ and $Y \notin F]$. This characterization immediately leads to the observation that any B-scopeless quantifier must be an ultrafilter, provided that B contains more than one element. For if we pick $y_0 \in B$ and let R be $(X \times \{y_0\}) \cup$

$(Y \times B\backslash\{y_0\})$, we get: $\overline{X \cup Y} \in F$ iff $\{x \in A|(\sim\exists y)xRy\} \in F$ iff (by F's scopelessness) $(\sim\exists y)$ $(Fx)xRy$ iff $[Ry_0 \notin F$ and $(\forall y \neq y_0)Ry \notin F]$ iff $[X \notin F$ and $Y \notin F]$.

Does scopelessness with respect to Boolean connectives imply scopelessness with respect to quantifiers? In other words, is every ultrafilter over a set A B-scopeless, or are the x_0^* the only quantifiers with such trivial scope behaviour? The answer to this question depends on what A and B are. Thus, e.g., if A is finite, the question is pointless because every ultrafilter is principal. But if A is infinite and, consequently, does carry non-principal ultrafilters, the latter need not be scopeless. In particular, if there exists a one–one function f from A to B, any B-scopeless quantifier F over A must contain a singleton and thus be principal: $A \in F$, because F is an ultrafilter, and $A = \{x|(\exists y)f(x) = y\}$, which means that $\{x|f(x) = y\} \in F$, for some $y \in B$. As a consequence, we get the following two conclusions concerning ordinary and polyadic quantifiers, respectively[1]:

(1) A quantifier F over a set A of size $\geqslant 2$ is A-scopeless iff F is a principal ultrafilter.
(2) A quantifier F over the Cartesian product A^n is A^m-scopeless $(n, m \geqslant 1)$ iff F is a principal ultrafilter.

(1) is immediate; (2) holds because either A, A^n, and A^m are all finite, or else of the same cardinality. What remains to be done, then, is to find a characterization of B-scopelessness where A is infinite and $|B| < |A|$. The following observation gives a clue:

(3) If F is B-scopeless over A, then F is $|B|^+$-complete,

which means that F is closed under intersections of length $\leqslant |B|$, i.e. for any $H \subseteq F$ of that size: $\cap H \in F$. Given such H, let g be a function from B onto H (i.e. $rge(g) = H$) and put: xRy iff $x \in g(y)$. Clearly, every Ry is in F: $(\forall_B y)$ $(Ry \in F)$, where $\forall_B = \{B\}$. So F's scopelessness tells us that: (Fx) $(xR \in \forall_B)$. By the definition of R, this means that $\cap rge(g) = \cap H \in F$.

Note that (3) is independent of the relative sizes of A and B. And it can be reversed, yielding the following general characterization of scopelessness:

THEOREM 1. *Let A and B be non-empty sets such that $|B| \geqslant 2$, and let F be a quantifier over A. Then F is B-scopeless iff F is a $|B|^+$-complete ultrafilter.*

I will only sketch the proof here, leaving technicalities to the appendix. Given A, B and F as described plus some $R \subseteq A \times B$ and a quantifier Q over B, we must show that (S_q) is satisfied. We can think of R as a function indexing certain 'R-definable' subsets Ry of A and then concentrate on these subsets, some of which might be in F. Let F^+ be the set of all R-definable sets in F: $F^+ = \{X \in F | (\exists y \in B)(X = Ry)\}$; similarly, F^- is the set of R-definable non-members of F. Moreover, we let F^0 be: $\cap F^+ \backslash \cup F^-$. Intuitively speaking, F^0 contains all elements of A that could – should F be principal and judging only from the R-definable section of F – be F's generator. Finally, F^B will contain those members of B that define F-members: $F^B = \{y \in B | Ry \in F\}$. Using this notation, the right half of (S_q) reads: '$F^B \in Q$'. The key to the proof of Theorem 1 lies in the following two observations about F^0:

(O1) $\quad F^0 = \{x \in A | xR = F^B\}$,

(O2) $\quad F^0 \in F$.

Let us see how we can use (O1) and (O2) to obtain (S_q). If ('$\Rightarrow$') $\{x \in A | xR \in Q\} \in F$, then so is $F^0 \cap \{x \in A | xR \in Q\}$, by (O2). So, by (O1) and the fact that $\emptyset \notin F$, there is some $x \in A$ such that $xR = F^B$ and $xR \in Q$. If ('$\Leftarrow$') $F^B \in Q$, then $xR = F^B$ always implies $xR \in Q$ and hence, using F's $\subseteq$-monotonicity and our observations, we conclude: $\{x \in A | xR \in Q\} \in F$. So it remains to be shown that (O1) and (O2) hold, which is done in the appendix.

Note that if B is finite, then $|B|^+ \in \omega$ and hence *any* ultrafilter is $|B|^+$-complete. Thus Theorem 1 implies that some non-principal ultrafilters are scopeless quantifiers. However, if B happens to be infinite, a set A carrying a non-principal $|B|^+$-complete ultrafilter would have to be very large – at least as large as the first *measurable* cardinal μ_0. And whether μ_0 exists seems to be left open by ordinary ZF set theory. So if we forget about the realm of excessively large cardinals, we can sum up our findings by the following, possibly partial characterization of scopeless quantifiers:

COROLLARY 1.1. *Let A and B be non-empty sets such that $|A| < \mu_0$ and $|B| \geqslant \omega$. Then a quantifier over A is B-scopeless iff it is a principal ultrafilter.*

COROLLARY 1.2. *If $|A| < \mu_0$, $2 \leqslant |B|$, and F is a B-scopeless quantifier but not a principal ultrafilter over A, then F is an ultrafilter and B is finite.*

2. SCOPELESSNESS GENERALIZED

The quantifiers studied in the preceding section certainly do not exhaust all notions of quantification. Moreover, the very concept of scope is not confined to quantifiers but also makes sense for other kinds of operations, whether variable-binding or not. It is therefore natural to ask whether the above characterization of scopelessness can be shown to carry over to operators other than second-order sets. In this section we will look at some particular cases for which the problem of characterizing scopelessness can be formulated without being covered by Theorem 1.

The most obvious generalizations of the above concept of scopelessness are obtained by varying the notion of a quantifier. I want to mention two possibilities, because they are both covered by the concept of an operator to be given in due course. The first one is trivalent quantification: *three-valued quantifiers* (*on* a universe A of at least two individuals) Q can be thought of as tri-partitions of the set of gapped predicates, i.e. as functions from gapped predicates to the set $\{\top, \bot, u\}$ of truth-values, where a *gapped* predicate is again a function from individuals to $\{\top, \bot, u\}$. If we write '$(Qx)\phi$' for the result of applying Q to that function $\hat{x}\phi$ that yields the truth-value ϕ for arbitrary $x \in A$, F may be called *scopeless* iff it satisfies:

$$(S_3) \qquad (Fx)\,(Qy)\,R(x, y) \;=\; (Qy)\,(Fx)\,R(x, y),$$

for any three-valued quantifiers Q (over A) and gapped relations R. Note that this is an absolute notion of scopelessness, so that we may expect it to be satisfied only by name-like quantifiers, i.e. those of the form $[\hat{P}\ P(x)]$, where 'P' ranges over gapped predicates and $x \in A$. This is indeed what we will be able to show.

The other notion of quantification I want to mention is intensional, in the following sense. Suppose that, in addition to our domain A of quantification, we are given a non-empty set W of possible worlds. Then a *property* (*of As*) may be thought of as a function P from W to A; and an *intensional quantifier* simply is a property of properties (of As). The appropriate notion of scopelessness becomes apparent when we take possible worlds as invisible parameters of interpretation in a language of higher-order modal logic. Thus F's satisfaction of the following formula of Montague's (1970: 384ff.) language of intensional logic offers a natural criterion of scopelessness:

$$\Box(\forall Q)\,(\forall R)F\{\hat{x}Q\{\hat{y}R\{x, y\}\}\} \leftrightarrow Q\{\hat{y}F\{\hat{x}R\{x, y\}\}\}.$$

By translation into our extensional meta-language, we get the following definition: an intensional quantifier F over A is *scopeless* iff (S_i) holds for arbitrary intensional quantifiers Q, binary properties R and possible worlds i:

$$(S_i) \qquad F_i(\hat{j}\hat{x}Q_j(\hat{k}\hat{y}R_k(x, y))) \;=\; Q_i(\hat{k}\hat{y}F_k(\hat{j}\hat{x}R_j(x, y))).$$

Again this notion of scopelessness is absolute, i.e. with respect to quantifiers over the same domain and depending on the same possible worlds; and again we will find that only name-like quantifiers of the form $[\hat{\imath}\hat{P}P_i(x)]$ (for $x \in A$) satisfy it.[2]

Judging from the variants of quantification discussed so far, it appears that the connection between (absolute) scopelessness and name-likeness is rather tight. However, facts are not always that simple. As a case in point, we may consider *propositional operators* expressing modalities, tenses, propositional attitudes, etc. Following Montague (1968), we may classify all these operators as properties of propositions, where a *proposition* is a set of *indices* (i.e. possible worlds, times, world-time pairs, etc.). Again, we find a natural criterion of scopelessness in higher-order intensional logic:

$$\Box(\forall O')\,(\forall p)\,[O({}^\wedge O'\{p\}) \leftrightarrow O'({}^\wedge O\{p\})],$$

which expresses that a scopeless propositional operator O must satisfy:

$$(S_p) \qquad O_i(\hat{j}O'_j(p)) \;=\; O'_i(\hat{j}O_j(p)),$$

for arbitrary propositional operators O', indices i and propositions p. However, it is easily seen that this criterion is only met by the trivial operator $[\hat{\imath}\hat{p}p(i)]$: just insert '$[\hat{\imath}\hat{q}p(i)]$' for '$O'$' in (S_p) and receive a point-wise definition of scopeless O.

An intensional language also allows for interactions between intensional quantifiers Q and propositional operators O, thus giving rise to more general notions of scopelessness. Using results from Section 4, we will, e.g., be able to show that, as long as the number of individuals does not exceed the number of propositions, the only intensional quantifiers that lack scope with respect to arbitrary intensional operators are name-like.

Let me finally draw attention to natural language constructions that have a flavour of scopelessness about them. For the obviously scopeless proper names[3] do not appear to be the only expressions that are immune to shiftings of quantified noun phrases. Thus, e.g., van der Does (1991) observes that certain infinitive embeddings allow for *export and import of quantifiers* as illustrated by the absence of scope ambiguities in **Caroline sees Tom eat five cakes**. Now if we wanted to use a possible world framework to analyze **see** as a ternary (intensional) relation **S** holding between two objects and a property, we could express van der Does's transportation principle by a formula of intensional logic:

$$\Box(\forall x_0)\,(\forall y_0)\,(\forall Q)\,(\forall R)\,[\mathbf{S}(x_0, y_0, \hat{y}Q\{\hat{z}R\{y, z\}\}) \leftrightarrow Q\{\hat{z}\mathbf{S}(x_0, y_0, \hat{y}\hat{R}\{y, z\})\}],$$

which means that:

$$(S_{\mathrm{S}}) \qquad [\mathbf{S}_i(x_0, y_0, \hat{\jmath}\hat{y}Q_j(\hat{k}\hat{z}R_k(y, z))) \Leftrightarrow Q_i(\hat{k}\hat{z}\mathbf{S}_k(x_0, y_0, \hat{\jmath}\hat{y}R_j(y, z)))],$$

where Q is an arbitrary intensional quantifier, i is a possible world, x_0 and y_0 are individuals and R is a binary (intensional) relation. What restrictions does (S_{S}) impose on the relation **S**? One consequence of the main result of Section 3 is that **S** cannot be *veridical*[4] in the sense of:

$$(\mathrm{V}) \qquad \mathbf{S}_i(x, y, P) \Rightarrow P_i(y).$$

But even without (V) would we get such unwelcome consequences that we may safely conclude that a Montague-style formalization of quantifier transportation is impossible. The details can be found in the appendix. Analogous remarks apply to certain modifier constructions like manner or temporal adverbs. Concerning the former, it was already pointed out in Zimmermann (1987) that there is no way of analyzing **slowly** as a (non-trivial) verb phrase operator in **Alain eats few apples slowly**, if it only modifies the main verb **eats**. And the fact that most temporal modifiers do scopally interact with even extensional quantifiers finds a natural explanation if we analyze them as propositional operators: given infinitely (but not measurably) many individuals, only designators of times (like **today** or **the Queen's birthday**) are scopeless with respect to quantifiers over individuals, as Theorem 1 shows.[5]

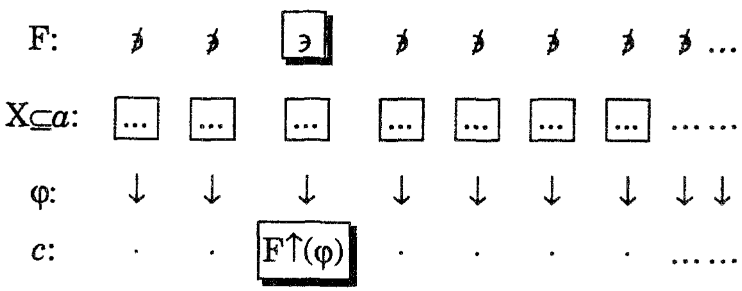
3. OPERATORS

Here is a straightforward generalization of the notion of a scopeless quantifier that covers some, but not all of the above examples: just think of the defining equivalence (S_q) as an equation in a language of functional type theory with application (from types *ab* and *a* to *b*) and abstraction (from *a* and *b* to *ab*). We thus get:

$$(S_0) \qquad \mathscr{F}(\hat{x}\mathscr{G}(\hat{y}\phi(x, y))) = \mathscr{G}(\hat{y}\mathscr{F}(\hat{x}\phi(x, y))).$$

It is readily seen that, in order for (S_0) to make sense, $\mathscr{F}$, $\mathscr{G}$, and ϕ would have to be expressions of some types $(ac)c$, $(bc)c$, and $b(ac)$, respectively, where '$\phi(x, y)$' is the same as '$\phi(y)(x)$'. This leads to a natural concept of operators: given a family of objects of some basic types (of at least two objects each), we define an *ac-operator* $\mathscr{F}$ to be a function of type $(ac)c$; and we say that $\mathscr{F}$ is *b-scopeless* if it satisfies (S_0), for any *bc*-operators $\mathscr{G}$ and functions ϕ of type $b(ac)$. It is clear that this concept is a generalization of scopelessness for quantifiers, because quantifiers are *at*-operators, where $t = \{0, 1\}$ is the type of classical truth-values. The subsumption of scopeless three-valued quantifiers under (S_0) is equally straightforward. As to intensional quantifiers, we only have to observe that, being of type $s((s(et))t)$, they stand in a one–one correspondence to $e(st)$-operators.[6] So if we manage to characterize scopeless operators, we can deal with the notions of quantification indicated in the preceding section. And in the appendix the results on operators will also be applied to scopeless natural language modifiers. Only the discussion of scopeless propositional operators will have to await Section 4.

In order to generalize the above considerations on quantifiers to operators, we must first find a generalization of the notion of an ultrafilter. In the case of principal ultrafilters x^*, the obvious candidates are those *ac*-operators that are *reducible to* x, i.e. of the form $[\hat{\phi}\phi(x)]$. Indeed, replacing sets by characteristic functions turns a principal ultrafilter x^* into a reducible *at*-operator $x\uparrow$. If we think of characteristic functions ϕ as bi-partitions, then $x\uparrow$ assigns to ϕ that truth-value that ϕ assigns to the cell belonging to x^*. This correspondence between x^* and $x\uparrow$ can be generalized to ultrafilters and (certain) operators in general. Given an ultrafilter F and a function ϕ of type *ac*, we define $F\uparrow(\phi)$ as that element of c whose ϕ-predecessors form a filter set:



Obviously, the existence of $F\uparrow$ depends on there being a unique F-member X_0 in the partition P determined by ϕ. Now whereas X_0's uniqueness is directly implied by F's filter properties, its existence in general amounts to F being $|P|^+$-complete.[7] Since $|P|$ is at worst $|c|$, we find that an ultrafilter (of type $(at)t$) can be *extended* to an *ac*-operator $F\uparrow$ iff F is $|c|^+$-complete. This means that, in effect, either $|c|$ is finite or F is principal (or $|a| \geqslant \mu_0$). But restricted to those F that can be extended, the $\uparrow$-operation is obviously a one–one correspondence between them and their operator-counterparts matching principal ultrafilters with reducible operators. We will now see that it also preserves scope behaviour.

As a first step towards the characterization of scopeless operators we can prove that, just as scopelessness implies ultrafilterhood in the case of quantifiers, so must scopeless operators always extend ultrafilters. In order to show this, we will first study the effects of scopeless operators on characteristic functions. The first observation is that whichever members $\top$ and $\bot$ of c we might declare our *ad hoc* truth-values, a scopeless *ac*-operator will assign one of the two to any characteristic function $X_\bot^\top$ (of $X \subseteq a$). In fact, one can prove a more general

PROPOSITION. *If* $\mathscr{F}$ *is a b-scopeless operator, then* $\mathscr{F}(\phi) \in rge(\phi)$, *for any* ϕ *in* $\mathscr{F}$*'s range.*

For the limiting case of constant *bc*-functions $[\hat{x}z]$, this is easily established: $\mathscr{F}(\hat{x}z) = \mathscr{F}(\hat{x}\,[\hat{\psi}z]\,(\hat{y}\chi(x, y))) = [\hat{\psi}z]\,(\hat{y}\mathscr{F}(\hat{x}\chi(x, y))) = z$. The general case then follows if we assume that $\mathscr{F}(\phi) \notin rge(\phi)$ and let $\mathscr{F}$ interact with the *bc*-operator $\{\hat{y}\mathscr{F}(\phi)\}_\bot^\top$ on the matrix $\phi(x)$. I leave the details to the reader.

So for any choice of $\top$ and $\bot$, *b*-scopeless $\mathscr{F}$'s restriction to characteristic functions is (the characteristic function of) a quantifier $\mathscr{F}\downarrow$. Certainly, $\mathscr{F}\downarrow$ is *b*-scopeless. And one can also show that it does not depend on $\top$ and $\bot$: given an alternative set $\{+, -\}$ of truth-values, we can define a 'mixed' existential operator $\mathscr{G}_\exists$ taking characteristic functions $X_\bot^\top$ and yielding a $\{+, -\}$-value:

$$\mathscr{G}_\exists(\psi) = \begin{cases} +, \text{ if } \psi(y) = \top, & \text{for some } y \in b. \\ - \text{ otherwise.} & \end{cases}$$

It is then readily seen that, for any x of type a, $x \in X$ iff $\mathscr{G}_\exists(\hat{y}X_\bot^\top(x)) = +$; in other words, $[\hat{x}\mathscr{G}_\exists(\hat{y}X_\bot^\top(x))] = X_-^+$. Using $\mathscr{F}$'s scopelessness, we may therefore conclude:

$$\begin{aligned} & \mathscr{F}(X_-^+) = + \\ \text{iff } & \mathscr{F}(\hat{x}\mathscr{G}_\exists(\hat{y}X_\bot^\top(x))) = + \\ \text{iff } & \mathscr{G}_\exists(\hat{y}\mathscr{F}(\hat{x}X_\bot^\top(x))) = + \\ \text{iff } & (\exists y \in b)\mathscr{F}(\hat{x}X_\bot^\top(x)) = \top \\ \text{iff } & \mathscr{F}(X_\bot^\top) = \top, \end{aligned}$$

which means that $\mathscr{F}\downarrow$ does not depend on our choice of truth-values. Moreover, by studying $\mathscr{F}$'s interaction with the *bc*-operator $\{\hat{y}\mathscr{F}(\phi)\}_{\perp}^{\mathscr{F}(\phi)}$, we find that $\mathscr{F}$ extends $\mathscr{F}\downarrow$: $\mathscr{F}\downarrow\uparrow = \mathscr{F}$. Again I leave the details to the reader.

So a scopeless operator always extends a scopeless quantifier, i.e. an ultrafilter. As was already mentioned, the simplest scopeless operators are the reducible ones extending principal ultrafilters. One might expect that their role as scopeless operators is analogous to that of the principal ultrafilters as scopeless quantifiers. And, in a sense, this is so. However, we must keep in mind that not every scopeless quantifier can be extended to an operator. In particular, then, we should not expect the existence of scopeless, irreducible operators to be independent of the choice of *c*. For if $|c| \geqslant |a|$, scopeless $\mathscr{F}$ would have to extend a principal ultrafilter, because only $|c|^+$-complete ultrafilters can be extended to *ac*-operators; thus $\mathscr{F}$ is reducible, even if *b* happens to be finite. In general, then, if a scopeless quantifier fails to be $|c|^+$-complete, we will not be able to extend it to a scopeless operator. However, in all other cases we will, as is apparent from the

MAIN LEMMA. *Any ac-operator extending a $|b|^+$-complete ultrafilter is b-scopeless.*

The proof can be found in the appendix. We just draw the main conclusion:

THEOREM 2. *An ac-operator $\mathscr{F}$ is b-scopeless iff $\mathscr{F}\downarrow$ is a $|b|^+$-complete ultrafilter.*

The direction '$\Leftarrow$' is the Main Lemma, the other direction is a combination of our earlier observation and Theorem 1. In view of the fact that not every ultrafilter can be extended, we get the following corollaries, the first of which covers the above-mentioned characterization of scopeless intensional and multi-valued quantifiers as name-like:

COROLLARY 2.1. *If $|a| \leqslant |b|$ or $|a| \leqslant |c|$, then the b-scopeless ac-operators are precisely the reducible ones.*

COROLLARY 2.2. *If* $|a| < \mu_0$ *and* $\mathscr{F}$ *is a b-scopeless, irreducible ac-operator, then* $\mathscr{F}$ *extends a non-principal ultrafilter and both b and c are finite.*

4. SCOPELESSNESS WITHOUT BINDING

Even within the type-theoretic framework sketched at the beginning of the preceding section, the above considerations do not cover all phenomena falling under the intuitive notion of scopelessness. The aim of the present section is to fill one obvious gap. Let us first observe that the results obtained in Section 2 immediately carry over to multiple variable binding: we can always identify multiple binding with unary binding of variables ranging over a Cartesian product. Thus, without loss of generality, we have restricted our attention to those instances of the scheme $(S_{n,m})$ in which $n = m = 1$:

$$(S_{n,m}) \quad \mathscr{F}(\hat{x}_1 \ldots \hat{x}_n \mathscr{G}(\hat{y}_1 \ldots \hat{y}_m \chi(x_1, \ldots, x_n, y_1, \ldots, y_m))) = \mathscr{G}(\hat{y}_1 \ldots \hat{y}_m \mathscr{F}(\hat{x}_1 \ldots \hat{x}_n \chi(x_1, \ldots, x_n, y_1, \ldots, y_m))).$$

This trick only works if there actually are variables to be bound by $\mathscr{F}$ and $\mathscr{G}$, i.e. if both n and m are at least 1. However, $(S_{n,m})$ also makes (type-theoretic) sense for cases in which n or m (or both) are 0, i.e. where no variables are bound by one or both of the operators. It is easily seen that the only $\mathscr{F}$ satisfying $(S_{0,0})$ or $(S_{0,m})$ is the *identity function* $[\hat{z}z]$. But $(S_{n,0})$ is different. One of its instances is the case where c contains exactly two members. In this case $(S_{n,0})$ essentially says that $\mathscr{F}$ is a quantifier lacking scope with respect to all unary truth-functions, i.e. it is \-maximal and contains the universe.[8] As a case in point we may consider the quantifier $\exists^{>50\%}$ $(=\{X \subseteq a \mid |X| > |a|/2\})$ on a finite domain of an odd number of individuals; note that $\exists^{>50\%}$ is not an ultrafilter.

The above discussion may give the impression that satisfaction of $(S_{n,0})$ – and hence scopelessness without variable-binding in general – is quite unrelated to the notions of scopelessness discussed in the previous two sections. However, the case $|c| = 2$ is somewhat misleading: for larger $|c|$, the kind of scopelessness defined by $(S_{n,0})$ turns out to be a variant of what we have already seen. To see this, we should recall that every ultrafilter lacks scope with respect to (classical) truth-functional connectives ©. Since the arguments of such © can be recategorized as sequences (rather than tuples) of truth-values, binary connectives may be classified as *tt*-operators so that the ultrafilters turn out to be precisely the *t*-scopeless *at*-operators. This recategorization is, of course, perfectly general and can also be applied to operators with ranges other than *t*:

PROPOSITION. *An ac-operator $\mathcal{F}$ satisfies (S_2) (for arbitrary $\mathcal{H}$ of type c(cc) and ϕ and ψ of type ac) iff $\mathcal{F}$ is t-scopeless*:

$$(S_2) \qquad \mathcal{F}(\hat{x}\mathcal{H}(\phi(x), \psi(x))) = \mathcal{H}(\mathcal{F}(\hat{x}\phi(x)), \mathcal{F}(\hat{x}\psi(x))).$$

The proof can be given by directly applying the above-mentioned recategorization.

There is a tight connection between (S_2) and the kind of scopelessness that we are after, i.e. $(S_{n,0})$: if a type *c* covers some set *d* as well as the Cartesian product d^2 $(= d \times d)$, the functions from *d*-pairs to *d* are contained in the type *cc* and hence scopelessness with respect to the latter implies scopelessness with respect to the former: $(S_{n,0}) \Rightarrow (S_2)$. Instead of going into the details of this argument, I will merely indicate how it helps characterizing scopelessness of the $(S_{n,0})$ kind. The trick is that scopelessness never depends on the internal structure of the argument type *a* of an *ac*-operator, so that it is preserved under type-isomorphisms. So the condition that $d \cup d^2 \subseteq c$ is not essential for getting from $(S_{n,0})$ to (S_2).[9] In view of the above proposition and the Boolean scopelessness of ultrafilters, we thus obtain the direction '$\Rightarrow$' of

THEOREM 3. *Let a and c be types such that $|c| \geqslant 5$. Then an ac-operator $\mathcal{F}$ satisfies $(S_{n,0})$ iff $\mathcal{F}$ extends an ultrafilter on a.*

The other direction ('$\Leftarrow$') is not related to the discussion in the present section. If $\mathcal{F}{\downarrow}$ exists and $\mathcal{G}$ and χ are as required by $(S_{n,0})$, we pick $z_0 \in c$ and let $\mathcal{G}^*$ be the *cc*-operator $[\hat{\phi}\mathcal{G}(\phi(z_0))]$. Using $\mathcal{F}$'s *c*-scopelessness guaranteed by Theorem 2, we immediately find that it satisfies $(S_{n,0})$.

As a corollary, Theorem 3 gives us the announced characterization of those intensional quantifiers that lack scope with respect to arbitrary propositional operators:

COROLLARY 3.1. *If there are at least three possible worlds, a is a type and* $|a| \leqslant |st|$, *then an intensional quantifier Q is scopeless with respect to intensional operators iff Q is of the form* $[\hat{\imath}\hat{P}P_i(x)]$, *for some* $x \in a$.

The relevant notion of scopelessness is, of course, satisfaction of

$$\text{(i)} \qquad O_i(\hat{\jmath}Q_j(\hat{k}\hat{x}P_k(x))) = Q_i(\hat{k}\hat{x}O_k(\hat{\jmath}P_j(x))),$$

for arbitrary $O \in s((st)t)$, $P \in s(at)$ and worlds $i \in s$. We have already mentioned that intensional quantifiers Q directly correspond to $a(st)$-operators. Moreover, satisfaction of (i) obviously amounts to $(S_{n,0})$-scopelessness. The assumption about $|s|$ guarantees that $|st| > 4$, so that Theorem 3 applies, informing us about $Q\downarrow$'s existence. But the latter must be $|st|^+$-complete and hence principal, because $|a| \leqslant |st|$. So Q is of the form indicated. The other direction is trivial.[10]

APPENDIX: SOME PROOFS

(O1): '$\Rightarrow$': We assume that $x_0 \in \bigcap F^+ \setminus \bigcup F^-$. But if $x_0 Ry$, then $Ry \in F$: otherwise we would have $Ry \in F^-$ and hence: $x_0 \in \bigcup F^-$. So $x_0 R \subseteq F^B$. If, on the other hand, $y \in F^B$, $Ry \in F^+$ so that $x_0 \in Ry$ (because $x_0 \in \bigcap F^+$). We thus have: $x_0 R = F^B$, as required.

'$\Leftarrow$': If (!) $x_1 R = F^B$, we would like to conclude that (a) $X_1 \in F^+$ implies $x_1 \in X_1$, and (b) $X_0 \in F^-$ implies $x_1 \notin X_0$. But $X_1 \in F^+$ means: $X_1 = Ry_1 \in F$ (for some y_1), i.e.: $y_1 \in F^B$. So, by (!), $x_1 \in Ry_1 = X_1$. For (b), we assume that $x_1 \in X_0 \in F^-$ and derive a contradiction: $X_0 \in F^-$ tells us that X_0 is of the form Ry_0 but $Ry_0 \notin F$. On the other hand, we have: $x_1 \in X_0$ iff $x_1 Ry_0$ iff $y_0 \in x_1 R$ iff $y_0 \in F^B$, by (!). But then $Ry_0 \in F$, contrary to what we have just found.

(O2): Let us first note that, because of F's $|B|^+$-completeness, $\bigcap F^+$ must be in F, because each member is R-defined by some $y \in B$. Proceeding indirectly, let us now assume that $F^0 \notin F$, i.e.: $\overline{\bigcap F^+} \cup$

$\bigcup F^- \in F$. Using F's filter properties, we can thus conclude that $\overline{[\bigcap F^+ \cup \bigcup F^-]} \cap \bigcap F^+ \subseteq \bigcup F^- \in F$, i.e. that $\overline{\bigcup F^-} = \bigcap_{X \in F^-} \bar{X} \notin F$, because F is \-maximal. But certainly, for any $X \in F^-$, $\bar{X} \in F$ (because $X \notin F$). Moreover, $|F^-| \leqslant |B| < |B|^+$. So, by F's $|B|^+$-completeness, we should also have: $\bigcap_{X \in F^-} \bar{X} \in F$, a contradiction.

Incompatibility of (S_S) *and* (V):

We show that, under (S_S) and (a certain instance of) (V), Barwise sees Perry wink if and only if Perry winks, i.e. that (1) and (2) are equivalent:

(1) $\mathbf{S}_i(\mathbf{b}, \mathbf{p}, \mathbf{W})$,

(2) $\mathbf{W}_i(\mathbf{p})$.

Now, (1) is the same as $\mathbf{Q}_i(\mathbf{W})$, where $\mathbf{Q}$ is the intensional quantifier that yields $\mathbf{S}_j(\mathbf{b}, \mathbf{p}, P)$ when applied to indices j and properties P. According to (S_S), $\mathbf{Q}$ is scopeless and hence of the form $[\hat{j}\hat{P}P_j(x_0)]$, by Theorem 2. So (1) just says that x_0 winks:

(!) $(1) \Leftrightarrow \mathbf{Q}_i(\mathbf{W}) \Leftrightarrow [\hat{j}\hat{P}P_j(x_0)](i)(\mathbf{W}) \Leftrightarrow \mathbf{W}_i(x_0)$.

But who is x_0? Whoever he is, he is self-identical, i.e. the property $[\hat{j}\hat{x}\ x = x_0]$ belongs to $\mathbf{Q}$ at index i. If we now express $\mathbf{Q}$ in terms of $\mathbf{S}$ again, we find that Barwise sees Perry being identical to x_0:

$$\begin{aligned} &\mathbf{Q}_i(\hat{j}\hat{x}\ x = x_0) \\ \Leftrightarrow\ &[\hat{j}\hat{P}\mathbf{S}_j(\mathbf{b}, \mathbf{p}, P)](i)(\hat{j}\hat{x}\ x = x_0) \\ \Leftrightarrow\ &\mathbf{S}_i(\mathbf{b}, \mathbf{p}, \hat{j}\hat{x}\ x = x_0), \end{aligned}$$

which means that x_0 *is* Perry, if $\mathbf{S}$ satisfies at least this instance of (V)! We may thus add '$\Leftrightarrow$ (2)' to (!).

MAIN LEMMA. Let us assume, for contradiction, that $\mathscr{F}$ extends a $|b|^+$-complete ultrafilter but $(\top :=)\mathscr{F}(\hat{x}\mathscr{G}(\hat{y}\chi(x, y))) \neq \mathscr{G}(\hat{y}\mathscr{F}(\hat{x}\chi(x, y)))$. We now define a set M as: $\{x \in a | \mathscr{G}(\hat{y}\chi(x, y)) = \top\}$. Thus $M \in \mathscr{F}\downarrow$. Moreover, for any $x_0 \in M$, we have: $\mathscr{G}(\hat{y}\chi(x_0, y)) = \top \neq \mathscr{G}(\hat{y}\mathscr{F}(\hat{x}\chi(x, y)))$. So, in particular:

(+) $[\hat{y}\chi(x_0, y)] \neq [\hat{y}\mathscr{F}(\hat{x}\chi(x, y))]$, whenever $x_0 \in M$. If we now we put $K_y := \{x_0 \in a | \chi(x_0, y) = \mathscr{F}(\hat{x}\chi(x, y))\}$, for arbitrary $y \in b$, then all K_y must be in $\mathscr{F}\downarrow$, because $\mathscr{F}$ extends $\mathscr{F}\downarrow$. But $\mathscr{F}\downarrow$ is $|b|^+$-complete and so $\bigcap_{y \in b} K_y$ must also be in $\mathscr{F}\downarrow$. Now, any x_0 in that intersection satisfies: $\chi(x_0, y) = \mathscr{F}(\hat{x}\chi(x, y))$, whichever y we pick. Hence, in view of (+), $\bigcap_{y \in b} K_y$ and M would have to be disjoint, which they cannot be because both are in $\mathscr{F}\downarrow$.

NOTES

[1] See van Benthem (1989) for the notion of polyadic quantification in general and Zimmermann (1987: 85) and van Benthem (*ibid.*, 454f.) for different proofs of (1); the question whether (3) holds was brought to my attention by Jaap van der Does (p.c.).

[2] A reduction of this result to Theorem 1 had already been given in Zimmermann (1987: 88). But it was rather *ad hoc* and does not generalize to the other examples discussed in the present section.

[3] 'Obviously', because at least semanticists working in possible worlds frameworks tend to accept Kripke's (1972) thesis about the rigidity of proper names.

[4] Unless it is the trivial relation $[\hat{\imath}\hat{P}\hat{y}\hat{x}P_i(y)]$, that is. Van der Does claims non-veridicality for **see**, too, but for empirical reasons. I should perhaps mention that we get the same results under a 'small clause' analysis of **see** as a binary relation between seers and propositions.

[5] The reason is that any propositional operator O can be re-written as a time-dependent quantifier over times, so that scopelessness amounts to each time slice's of O being D-scopeless, where D is the set of individuals.

[6] Incidentally, we may note that intensional quantifiers can be thought of as many-valued quantifiers, where propositions act as (Boolean) truth-values.

[7] See, e.g., Chang & Keisler (1973: 180f.) for a proof of the fact that an ultrafilter F over a set A is κ-complete iff for every partition $P \subseteq \mathscr{P}A$ such that $|P| < \kappa$, $P \cap F \neq \emptyset$.

[8] Using the obvious isomorphism between the set of quantifiers over a singleton B and the type tt of unary truth-functions, we may also conclude that a quantifier over a set A is B-scopeless iff it is \-maximal and contains A.

[9] But c would have to contain at least 5 elements in order to cover a minimal set of the form $d \cup d^2$ (where d is, say $\{0, (0, 0)\}$). Whether this restriction is essential I do not know.

[10] The above is a shortened version of a paper written in the academic year 1989/90 that I spent as a visitor to the linguistics department of the University of Massachusetts at Amherst; I would like to take this opportunity to thank everybody in the department for their hospitality and help. Comments from Ulf Friedrichsdorf, Fritz Hamm, Angelika Kratzer, Michael Morreau, Roger Schwarzschild, and three *JPL* referees helped to improve the style and content of this paper.

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