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Symmetries of Riemann Surfaces with Large Automorphism Group

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§ 1. Introduction

A Riemann surface is *symmetric* if it admits an anti-conformal involution. The basic question which we discuss in this paper is whether compact Riemann surfaces of genus $g>1$ which admit large groups of automorphisms are symmetric. As is well-known, the automorphism group of a compact Riemann surface of genus $g>1$ is finite and bounded above by $84(g-1)$. Macbeath ([12| 13]) has found infinitely many g for which this bound is attained. We show that all the surfaces found by Macbeath's methods are indeed symmetric. However, we do exhibit an example of a non-symmetric Riemann surface of genus $g=17$ which does admit $84(g-1)$ automorphisms.

We also study Riemann surfaces admitting automorphisms of large order. The order of an automorphism of a Riemann surface of genus g is bounded above by $4g+2$ and this bound is attained for every g [8]. We show that all Riemann surfaces admitting automorphisms of order greater that $2g+2$ are symmetric.

There is a close link between our work and the theory of irreflexible regular maps on surfaces. (See § 8 for definitions.) There is a connection between the groups of regular maps and large groups of automorphisms of compact Riemann surfaces. Indeed, every group of automorphisms of a Riemann surface of genus g of order greater than $24(g-1)$ is also the group of some regular map and conversely, every group of a regular map can be thought of as the group of automorphisms of a Riemann surface. The irreflexible regular maps turn out to be rather exceptional. (In fact, it was suggested in early editions of [3] that they did not exist for surfaces of genus $g>1$). We show in the above correspondence that large groups of automorphisms of non-symmetric surfaces will give rise to irreflexible regular maps, but that the converse of this fact is not always true. Thus, for example, groups of automorphisms of order greater than $24(g-1)$ of a compact non-symmetric Riemann surface of genus g are more exceptional than irreflexible regular maps.

There is another interpretation of symmetric Riemann surfaces which is of interest. Every compact Riemann surface can be obtained as the Riemann surface of an algebraic curve $f(z,w)=0$. A Riemann surface is symmetric if an only if it can be obtained as the Riemann surface of a real curve [10, p. 69]. The automorphisms of the surface correspond to birational self-transformations of the curve. Thus, for example, we can say that a curve of genus g admitting a birational self-transformation of order greater than $2g+2$ is birationally equivalent to a real curve.

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By a surface in this paper we shall mean a compact surface of genus greater than one.

§ 2. Preliminaries on NEC Groups

Let $\mathscr{L}$ denote the group of all conformal and anti-conformal homeomorphisms of the upper-half plane U and let $\mathscr{L}^+$ denote the subgroup of index 2 in $\mathscr{L}$ consisting of the conformal homeomorphisms. By a *non-Euclidean crystallographic (NEC) group* we shall mean a discrete subgroup Γ of $\mathscr{L}$ for which U/Γ is compact. If $\Gamma \subset \mathscr{L}^+$ then Γ is called a *Fuchsian* group. (We are departing from the usual terminology in that we are requiring U/Γ to be compact.) If $\Gamma \cap (\mathscr{L} - \mathscr{L}^+) \neq \emptyset$ then we shall call Γ a *proper* NEC group. Every compact Riemann surface of genus $g>1$ can be represented as U/K where K is a Fuchsian group acting without fixed points on U. This occurs if and only if K is torsion-free. In this case the group $A(S)$ of automorphisms (i.e. conformal self-homeomorphisms) of S is isomorphic to $N^+(K)/K$ where $N^+(K)$ is the normalizer of K in $\mathscr{L}^+$; the group $EA(S)$ of conformal and anticonformal homeomorphisms of S is isomorphic to $N(K)/K$, where $N(K)$ is the normalizer of K in $\mathscr{L}$. $N(K)$ is an NEC group so that any group of conformal and anti-conformal homeomorphisms of S is of the form Γ/K where Γ is an NEC group and $K \triangleleft \Gamma$. For an arbitrary NEC group Γ let $\mu(\Gamma)$ denote the non-Euclidean measure of a fundamental region for Γ. Then

$$|\Gamma/K| = \frac{\mu(K)}{\mu(\Gamma)}. \tag{2.1}$$

Associated to any Fuchsian group Γ we can assign a *signature* of the form $(g_1; m_1, \ldots, m_r)$. This means that U/Γ has genus g_1 and that the projection from U to U/Γ is branched at r points on U/Γ, the orders of branching being $m_1, \ldots, m_r$. In this case

$$\mu(\Gamma) = 2\pi\left(2g_1 - 2 + \sum_1^r \left(1 - \frac{1}{m_i}\right)\right). \tag{2.2}$$

If K is torsion free then its signature will be $(g; -)$. Such a group is called a *surface group*.

If Γ is an NEC group we let $R(\Gamma)$ denote the set of isomorphisms $r:\Gamma\to\mathcal{L}$ with the property that $r(\Gamma)$ is discrete and $U/r(\Gamma)$ is compact. $r_1, r_2\in R(\Gamma)$ are called equivalent if for all $\gamma\in\Gamma, r_1(\gamma)=gr_2(\gamma)g^{-1}$, for some $g\in\mathcal{L}$. The quotient space is denoted by $T(\Gamma)$, the Teichmüller space of Γ. It is homeomorphic to a cell of dimension $d(\Gamma)$. If Γ is a Fuchsian group with signature $(g; m_1, \ldots, m_r)$ then $d(\Gamma)=6g-6+2r$. Note that $d(\Gamma)=0$ if and only if Γ is a *triangle group*. I.e. $g=0$ and $r=3$. (There is no Fuchsian group with $g=1, r=0$.) If K is a surface group then $T(K)$ can be identified with the Teichmüller space of the Riemann surface U/K.

Every monomorphism $\alpha:\Gamma_1\to\Gamma_2$, between NEC groups induces an embedding $\alpha^*: T(\Gamma_2)\to T(\Gamma_1)$. The points in the image of this embedding correspond to groups isomorphic to Γ_1 which are contained in groups isomorphic to Γ_2. If K is a surface group and $K\lhd\Gamma$ then the points in the image of the induced embedding of $T(\Gamma)$ in $T(K)$ will correspond to surfaces admitting a group of conformal and anti-conformal homeomorphisms isomorphic to Γ/K ([8, 15]).

§ 3. Basic Lemmas

Let (G, S) be a *Riemann surface transformation* (RST) group. This means that G is a group of automorphisms of the Riemann surface S. If $S=U/K, G=\Gamma/K$ (K a surface group, Γ a Fuchsian group) then we call (Γ, U) its universal covering transformation (UCT) group. The aim of this paragraph is to prove Theorem 1 which tells us that in our context, we are only interested in RST groups (G, S) whose UCT group is (Γ, U) where Γ is a triangle group.

Our first lemma is a slightly extended version of a well-known result ([10], p. 71, [5]). For any proper NEC group Γ we will let $\Gamma^+=\Gamma\cap\mathcal{L}^+$. Γ^+ is of index 2 in Γ and is called the canonical Fuchsian group of Γ.

3.1. Lemma. *Let Γ be a proper NEC group. Then* $d(\Gamma)=\frac{1}{2}d(\Gamma^+)$.

Proof. Let K be a surface group normal in Γ. $d(\Gamma^+)$ is equal to the dimension of the real linear space Q of Γ^+/K-invariant quadratic differentials on U/K. $d(\Gamma)$ is equal to the dimension of the real linear space Q_1 of Γ/K-invariant quadratic differentials on U/K. (See [11].)

Let $t\in\Gamma/K-\Gamma^+/K$. As Γ^+/K has index two in Γ/K, $f\in Q_1$ if and only if $f\in Q$ and

$$\overline{f(tz)}\,\overline{d(tz)}^2=f(z)\,dz^2\,.$$

Let Q_2 denote the subspace of Q consisting of those Γ^+/K-invariant quadratic differentials obeying

$$\overline{f(tz)}\,\overline{d(tz)}^2=-f(z)\,dz^2\,.$$

If $f(z)\,dz^2 \in Q$ then $f = f_1 + f_2$ where

$$f_1 = \tfrac{1}{2}(f(z)\,dz^2 + \overline{f(tz)}\,d(\overline{tz})^2) \in Q_1\,,$$

$$f_2 = \tfrac{1}{2}(f(z)\,dz^2 - \overline{f(tz)}\,d(\overline{tz})^2) \in Q_2\,.$$

Thus

$$Q = Q_1 \oplus Q_2\,.$$

The map $f \to if$ is an isomorphism of Q_1 onto Q_2 and so

$$d(\Gamma) = \dim Q_1 = \tfrac{1}{2}\dim Q = \tfrac{1}{2}d(\Gamma^+)\,.$$

It follows that $d(\Gamma) < d(\Gamma^+)$ except when $d(\Gamma^+) = 0$; i.e. when Γ^+ is a triangle group.

3.2. Lemma. *Let Γ be a Fuchsian group which is not a triangle group. Then there exists a Fuchsian group Λ isomorphic to Γ such that Λ is not contained in any proper NEC group.*

Proof. First a few remarks about maximal Fuchsian groups. A Fuchsian group Γ_1 is called maximal if Γ_1 is not a proper subgroup of another Fuchsian group. If $\operatorname{Max}\Gamma_1$ denotes the subset of $T(\Gamma_1)$ consisting of the maximal groups then it is known that $\operatorname{Max}\Gamma_1$ is usually an open everywhere dense subset of $T(\Gamma_1)$. However there are some exceptional signatures. In these cases there always exists a Fuchsian group Γ_2 such that every group isomorphic to Γ_1 is contained in a group isomorphic to Γ_2. (See [7, 18].) The reason for this is as follows. If $\Gamma_1 \subsetneqq \Gamma_2$ and $d(\Gamma_2) < d(\Gamma_1)$ then the group isomorphic to Γ_1 contained in groups isomorphic to Γ_2 form a small subset of $T(\Gamma_1)$. If $d(\Gamma_1) = d(\Gamma_2)$ we obtain the exceptional signatures. Let Λ be an NEC group such that Δ^+ is isomorphic to Γ. As $d(\Delta) < d(\Gamma)$ (by 3.1), it follows that the groups isomorphic to Γ which are canonical Fuchsian groups of NEC groups form a "small" set in $T(\Gamma)$. (In fact, at most a countable union of submanifolds of smaller dimension.) If every group isomorphic to Γ is contained in a proper NEC group isomorphic to Δ_1 and Γ is not a canonical Fuchsian group, then $\Gamma \subsetneqq \Delta_1^+$, and so we have one of the exceptional signatures. Now $d(\Gamma) = d(\Delta_1^+)$ and so $d(\Delta_1) < d(\Gamma)$. We can therefore deduce the existence of a group Λ with the required properties.

Theorem 1. *Let (G, S) be a RST group and let (Γ, U) be its UCT group. If Γ is not a triangle group then there exists a RST group (G_1, S_1) where G_1 is isomorphic to G and S_1 is a non-symmetric Riemann surface homeomorphic to S.*

Proof. Let $S = U/K$ where K is a normal surface subgroup of Γ. By Lemma 3.2 we can find a Fuchsian group Λ isomorphic to Γ with the property that Λ is not contained in a proper NEC group. Λ then contains a normal surface subgroup K_1 such that $S_1 = U_1/K_1$, $G_1 = \Lambda/K_1$ will have the required properties. (For if S_1 is symmetric $N(K_1)$ will be a proper NEC group containing Λ.)

§ 4. Triangle Groups

From Theorem 1 we see that in order to be able to discuss symmetry properties of Riemann surfaces derivable from knowledge of their automorphism group, we have to turn our attention to RST groups (G, S) whose UCT groups are (Γ, U), where Γ is a triangle group. We will denote the signature $(0; l, m, n)$ by (l, m, n) and by $\Gamma(l, m, n)$ we will mean a triangle group with signature (l, m, n). It has the presentation

$$\{x, y \mid x^l = y^m = (xy)^n = 1\} \tag{4.1}$$

and as $\Gamma(l, m, n)$ is Fuchsian $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$.

This group can be constructed as follows. Consider a non-Euclidean triangle with angles $\pi/l, \pi/m, \pi/n$. Let $\Gamma^*[l, m, n]$ be the NEC group generated by the three reflections a, b, c in the sides of this triangle. Then $(\Gamma^*[l, m, n])^+$ is a triangle group with signature (l, m, n). [It is well-known that all triangle groups with a given signature are conjugate in $\mathscr{L}$-indeed, this is what we mean by saying that $d(\Gamma) = 0$, for triangle groups Γ.] Thus the triangle group $\Gamma(l, m, n)$ is the canonical Fuchsian group of a triangle group of the form $\Gamma^*[l, m, n]$.

$\Gamma^*[l, m, n]$ will have the presentation

$$\{a, b, c \mid a^2 = b^2 = c^2 = (ab)^l = (bc)^m = (ac)^n = 1\} \tag{4.2}$$

and the canonical Fuchsian group is generated by $x = ab$, $y = bc$.

From a study of the classification of NEC groups [14] we see that there is one other type of NEC group whose canonical Fuchsian is a triangle group. This group has the presentation

$$\{c, x \mid c^2 = x^m = (cxcx^{-1})^n = 1\}\,. \tag{4.3}$$

(In Macbeath's notation [14] it has NEC signature $(0, +, [m]\ \{(n)\})$.) Here, c is a reflection and x is elliptic. The canonical Fuchsian group is a triangle group with signature (m, m, n). It is generated by x^{-1} and cxc. Thus if a triangle group has two equal periods it is the canonical Fuchsian group of two non-isomorphic proper NEC groups.

Let (G, S) be a RST group with UCT group (Γ, U) where Γ is a triangle group. Then there exists an epimorphism $\theta: \Gamma \to G$ whose kernel is a surface group. Such a homomorphism is called a surface-kernel homomorphism and it is characterized by the fact that it preserves the orders of elements of finite order in Γ. In this case, (where Γ is a triangle group) we shall call G an *L-automorphism group.* We will be interested in § 8 in the case where one of the periods of Γ is 2. Then we shall call G a *K-automorphism group.* The case which has received most study is when the periods of Γ are 2, 3 and 7. In this case G will be called an *H-automorphism group.* We shall sometimes talk about H-groups, by which we mean a group G which acts as an H-group on some Riemann surface. (With a similar remark about K-groups and L-groups.) H-groups are usually called *Hurwitz groups* in the literature. From (1.1), (1.2) we deduce the following. Let (G, S) be a RST group and let S have genus g. Then

(i) If $|G| > 12(g-1)$ then G is an L-automorphism group.

(ii) If $|G| > 24(g-1)$ then G is a K-automorphism group.

(iii) G is an H-automorphism group if and only if $|G| = 84(g-1)$.

Thus these groups are, in some sense, "large" groups of automorphisms. The following lemma gives us an easy method of deciding whether a surface admitting an L-automorphism group is symmetric.

4.1. Lemma. *Let (G, S) be a RST group and (Γ, U) its UCT group where Γ is a triangle group. Let $S = U/K$, K a surface group, and $\theta : \Gamma \to \Gamma/K = G$ the canonical homomorphism. Let Γ^* be a proper NEC group with $(\Gamma^*)^+ = \Gamma$. Suppose G^* is a group which contains G with index 2 and $\theta^* : \Gamma^* \to G^*$ is an epimorphism with $\theta^* | \Gamma = \theta$. Then S is symmetric.*

Proof. Γ^* contains a reflection c and $\Gamma^* = \Gamma + c\Gamma$. $\theta^*(c)$ is not the identity otherwise $G = G^*$ and so $\theta^*(c)$ is an involution. As $\theta^* | \Gamma = \theta$, $\ker \theta^* \subsetneqq K$. Moreover, if $g \in \ker \theta^*$, $g \notin K$ then $g = ct$ where $t \in \Gamma$. Therefore $\theta^*(c) = \theta^*(t^{-1}) \in G$ and again $G = G^*$. Hence $g \in K$ and so $\ker \theta^* = K$. Thus $K \lhd \Gamma^*$ and $\theta^*(c)$ is a symmetry of S.

§ 5. *L*-Groups

Theorem 2. *Let G be an L-group of automorphisms of a Riemann surface S generated by X, Y obeying*

$$X^l = Y^m = (XY)^n = I$$

(i.e. there is an epimorphism from $\Gamma(l, m, n)$, with presentation 4.1 to G, defined by $x \to X$, $y \to Y$ and S is the quotient of the kernel). Then S is symmetric if and only if there is an automorphism $\alpha : G \to G$ obeying either

(i) $\alpha(X) = X^{-1}$, $\alpha(Y) = Y^{-1}$ *or*

(ii) $\alpha(X) = Y^{-1}$, $\alpha(Y) = X^{-1}$.

(Note that α has period 2.)

Proof. Suppose that an automorphism α exists obeying (i). Assume that it is an outer automorphism. Then there exists a Z_2-extension G^* of G and an element $T \in G^*$ obeying $T^2 = I, TXT^{-1} = X^{-1}, TYT^{-1} = Y^{-1}$ i.e.

$$T^2 = (XT)^2 = (TY)^2 = X^l = Y^m = (XY)^n = I .$$

Then there is an epimorphism

$$\phi : \Gamma^*[l, m, n] \to G^*$$

defined by

$$\phi(a) = XT, \quad \phi(b) = T, \quad \phi(c) = TY$$

and

$$\phi((\Gamma^*[l, m, n])^+) = G .$$

By Lemma 4.1, S is symmetric.

If α is an outer automorphism obeying (ii) then there is a Z_2-extension G^* of G and an element $T \in G^*$ which satisfies

$$T^2 = I, \quad TXT^{-1} = Y^{-1} .$$

Let Γ^* be an NEC group with presentation (4.3). Then there exists an epimorphism

$$\phi : \Gamma^* \to G^*$$

defined by

$$\phi(c) = T, \quad \phi(x) = X$$

and

$$\phi((\Gamma)^+) = G .$$

By Lemma 4.1, S is symmetric.

If α is an inner automorphism then the element T above lies in G. Let $G^* = Z_2 \times G$ where $Z_2 = \{V \mid V^2 = I\}$.

Then $G^* = ((V, W), (I, W); W \in G)$ and G can be identified with $((I, W); W \in G)$.

Suppose that α obeys (i). Then if we let

$$T_1 = (V, T), \quad X_1 = (I, X), \quad Y_1 = (I, Y)$$

then

$$T_1^2 = (X_1 T_1)^2 = (T_1 Y_1)^2 = X^l = Y^m = (XY)^n = I$$

$(T_1 \notin G)$, and we have reduced this case to the preceeding case. If α is an inner automorphism obeying (ii) then we proceed similarly.

Converse. As S is symmetric, G is contained in G^* with index 2 and there is a proper NEC group Γ^* and an epimorphism $\theta^* : \Gamma^* \to G^*$ such that $U/\ker \theta^* = S$. From (2.1), (2.2), Γ^* contains $\Gamma(l, m, n)$ with index 2 and hence $(\Gamma^*)^+ = \Gamma(l, m, n)$. Γ^* thus has presentation (4.2) or (4.3). If it has presentation (4.2) then $\theta^*(b)$ induces the required automorphism by conjugation and if it has presentation (4.3), $\theta^*(c)$ does.

Corollary. *If S admits an abelian L-automorphism group then S is symmetric.*

Proof. Mapping every element of an abelian group to its inverse gives us an automorphism of the group.

Theorem 3. *If S admits an L-automorphism group isomorphic to $PSL(2, q)$ then S is symmetric.*

Proof. The following statement follows immediately from [12], Theorem 3. Let $(X, Y), (X_1, Y_1)$, be two sets of generators for $G = PSL(2, q)$ obeying $\operatorname{tr} X = \operatorname{tr} X_1, \operatorname{tr} Y = \operatorname{tr} Y_1, \operatorname{tr} XY = \operatorname{tr} X_1 Y_1$. Then there exists an extension $\bar{G}$ of G and an element $U \in \bar{G}$ such that $UXU^{-1} = X_1$, $UYU^{-1} = Y_1$. As $\operatorname{tr} X = \operatorname{tr} X^{-1}, \operatorname{tr} Y = \operatorname{tr} Y^{-1}, \operatorname{tr} XY = \operatorname{tr} X^{-1} Y^{-1}$, our theorem follows from Theorem 2.

Note. For each g there is a Riemann surface of genus g admitting $8(g+1)$ automorphisms and for infinitely many g this is the largest group acting on a surface of genus g (Maclachlan [16], Accola [1]). In the former paper the following presentation for these groups of order $8(g+1)$ is given

$$\{X, Y \mid X^4 = Y^{2(g+1)} = (XY)^2 = (X^{-1}Y)^2 = I\}.$$

As the map $X \to X^{-1}$, $Y \to Y^{-1}$, extends to an automorphism of the group the corresponding surfaces are symmetric.

§ 6. Riemann Surfaces Admitting Automorphisms of Large Order

In this paragraph we show that Riemann surfaces that admit automorphisms of high order are symmetric. Before we do this we discuss some results of Harvey [8] on cyclic automorphism groups. As we have seen, a necessary and sufficient condition for the cyclic group Z_n to be a group of automorphisms of a surface of genus g, is that there exists a surface-kernel homomorphism $\phi: \Gamma \to Z_n$ for some Fuchsian group Γ. Let Γ have signature $(h; m_1, \ldots, m_r)$. Then we have

Theorem (Harvey [8]). Let $M = \text{l.c.}m\{m_1, \ldots, m_r\}$. Then there exists a surface-kernel homomorphism $\phi: \Gamma \to Z_N$ if and only if the following conditions are satisfied.

(i) $\text{l.c.}m\{m_1, \ldots, \hat{m}_i, \ldots, m_r\} = M$ for all i, where $\hat{m}_i$ denotes the omission of m_i.

(ii) If $2 \mid M$ then the number of periods divisible by the maximum power of 2 dividing M is even.

(iii) $M \mid N$ and if $h = 0$, $M = N$.

(iv) $r \neq 1$ and if $h = 0$, $r \geqq 3$.

From this result Harvey deduced a result of Wiman that the largest order of a cyclic group of automorphisms of a Riemann surface of genus g is $4g+2$, and that this bound is attained for every g. (Γ in this case has signature $(2, 2g+1, 4g+2)$.)

Theorem 4. *Let S be a non-symmetric Riemann surface of genus g which admits an automorphism of order $N > 2g$. Then g is even and $N = 2g + 2$.*

From this theorem the following corollaries are immediate.

Corollary 1. Let S be a Riemann surface of genus g which admits an automorphism of order N > 2g + 2. Then S is symmetric.

Corollary 2. Let S be a Riemann surface of odd genus g. If S admits an automorphism of order N > 2g then S is symmetric.

Proof. Let $S = U/K$ (K a surface group), be a non-symmetric Riemann surface of genus g which admits Z_N, $N > 2g$, as an automorphism group. If the UCT group of (Z_N, S) is (Γ, U), then by the corollary to Theorem 2 Γ is not a triangle group. We now find all signatures of Fuchsian groups Γ, which do not define triangle groups, obey the conditions of Harvey's theorem and for which $|\Gamma/K| > 2g$.

By (2.1), (2.2), we obtain $\mu(\Gamma) < 2\pi$ and hence by (2.2)

$$2h - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) < 1 . \tag{5.1}$$

Hence $h \geqq 1$ and if $h = 1, r = 1$. By (iii), (iv) (of Harvey's theorem) $h = 0$, $M = N$. Thus we want to find all signatures which obey (5.1) and the conditions of Harvey's theorem. It is easy to see that no signatures with $h = 0$, $r \geqq 5$ can occur so our problem reduces to finding all signatures $(0; m_1, m_2, m_3, m_4)$ which obey the conditions of Harvey's theorem and

$$\sum_{i=1}^{4} \frac{1}{m_i} > 1 .$$

Write the signature $(0; m_1, m_2, m_3, m_4)$ as (m_1, m_2, m_3, m_4). Then an arithmetical calculation gives us all possibilities. These are

A. $(3, 3, 3, 3)$ giving $g = 2$, $N = 3$,

B. $(3, 3, 4, 4)$ giving $g = 6$, $N = 12$,

C. $(3, 3, 5, 5)$ giving $g = 8$, $N = 15$,

D. $(2, 5, 5, 6)$ giving $g = 15$, $N = 30$,

E. $(2, 3, 3, 6)$ giving $g = 3$, $N = 6$,

F. $(2, 3, 4, 12)$ giving $g = 6$, $N = 12$,

G. $(2, 3, 5, 30)$ giving $g = 15$, $N = 30$,

H. $(2, 2, m, m)$ giving $g = \frac{m}{2}$, $N = m$,
(m even)

I. $(2, 2, m, m)$ giving $g = m - 1$, $N = 2m$.
(m odd)

I. gives the only example when $N > 2g$. In this case $N = 2g + 2$ and g is even.

Note. Theorem 1 implies that the bounds given in the corollaries are sharp. For every even g there is a non-symmetric Riemann surface of genus g admitting an automorphism of order $2g + 2$ and for every odd g there is a non-symmetric Riemann surface of genus g admitting an automorphism of order $2g$.

§ 7. *H*-Groups

We now discuss symmetry properties of Riemann surfaces admitting H-automorphism groups. That is, Riemann surfaces of genus g admitting $84(g-1)$ automorphisms. In [13], Macbeath found an infinite number of H-groups of the form $PSL(2, q)$. By Theorem 3, all the corresponding surfaces are symmetric. In [12], he described a method of obtaining an infinite number of H-groups from a given one. It is as follows. Let $S = U/K$, K a surface group, admit an H-automorphism group. Then $K \lhd \Gamma(2, 3, 7)$. Let K_1 be a characteristic subgroup of finite index in K. (An infinite number of these exist.) Then $K_1 \lhd \Gamma(2, 3, 7)$ and so U/K admits an H-automorphism group. If U/K is symmetric then $K \lhd \Gamma^*[2, 3, 7]$ and hence $K_1 \lhd \Gamma^*[2, 3, 7]$. Therefore U/K_1 is symmetric. Thus all Riemann surfaces found by Macbeath's methods are symmetric.

The first four values of g for which there exist surfaces admitting $84(g-1)$ automorphisms are as follows.

Genus	Group
3	PSL(2, 7)
7	PSL(2, 8)
14	PSL(2, 13)
17	A non-split extension of Z_2^3 by PSL(2, 7) of order 1344.

The first three values of g correspond to symmetric surfaces. The group of order 1344 was discovered by Sinkov [19].

Theorem 5. *Riemann surfaces of genus* 17 *admitting* 1344 *automorphisms are not symmetric.*

Proof. Let H be an H-group of order 1344. Then there are epimorphisms $\theta : \Gamma(2, 3, 7) \to H$, $\phi : H \to \mathrm{PSL}(2, 7)$. The kernel of ϕ is a normal subgroup N of H isomorphic to Z_2^3. As H/N is simple, it follows that N is the unique normal subgroup of H isomorphic to Z_2^3. (If N_1 is another one then $|N_1 N| \leqq 64$ and so $N_1 N \neq H$. Thus $N_1 N = N = N_1$.) Hence N

is characteristic in H. Let S be a Riemann surface of genus 17 admitting H as automorphism group. Suppose that S was symmetric. Then as $\Gamma^*[2,3,7]$ is the unique proper NEC group containing $\Gamma(2,3,7)$, there sould have to be a group H^* containing H with index 2 and an epimorphism $\theta: \Gamma^*[2,3,7] \to H^*$ such that $\theta^* | \Gamma(2,3,7) = \theta$. As $N \lhd H^*$ there exists a homomorphism $\psi: H^* \to \mathrm{Aut}\, N \simeq \mathrm{GL}(3,2) \simeq \mathrm{PSL}(2,7)$, ψ being induced by conjugation. (That is, $\psi(h) = \alpha$ where $\alpha(n) = hnh^{-1}$.) As $\ker \psi | H = N$, $\psi(H) = \mathrm{Aut}\, N$ and thus ψ is onto.

We now have an epimorphism

$$\chi: \Gamma^*[2,3,7] \to \mathrm{PSL}(2,7)$$

defined by

$$\chi = \psi\theta^* .$$

As all normal subgroups of index greater than two in $\Gamma^*[2,3,7]$ are surface groups and hence torsion free, it follows that χ maps finite subgroups of $\Gamma^*[2,3,7]$ isomorphically into PSL(2, 7). Thus PSL(2, 7) contains a dihedral group of order 14. This is seen to be a contradiction as follows. Z_7 is a Sylow subgroup of PSL(2, 7). By the Sylow theorems there are 8 Z_7's and they are all conjugate. The normalizer of a Z_7 then has order $168/8 = 21$. Thus PSL(2, 7) cannot contain a dihedral subgroup of order 14.

$g = 17$ gives us the least value of g for which there exists a non-real curve admitting $84(g-1)$ birational self-transformations. An infinite family of non-symmetric Riemann surfaces admitting H-groups does in fact exist. The corresponding H-groups are the Ree groups $G_2^*(q)$ of order $q^3(q^3+1)(q-1)$, where $q = 3^p$, p a prime greater than 3. The corresponding values of g are rather large; the smallest being greater than 3^{30}. The non-symmetry of the surfaces follows from Theorem 2. For we can find epimorphisms from $\Gamma(2,3,7)$ to these Ree groups with the property that the image of the element of order 3 is not mapped to its inverse under any automorphism of the group (Sah [17], pp. 30–31).

§ 8. Regular Maps on Surfaces

(See [3].) Let S be a compact orientable (topological) surface. A map on S is a partitioning of S into simply-connected non-overlapping regions called faces by means of line segments called edges. The intersection of the edges are called vertices. An automorphism of a map is a partitioning of its elements that preserve incidence. If amongst the automorphisms there is a element R that cyclically permutes the edges surrounding a face of the map and another element S which cyclically permutes the edges meeting at a vertex of this face, then the map is called *regular*. Then the same number, say p, of vertices, surround each face and the same number, say q, of edges meet at each vertex. The map is said to be of type $\{p, q\}$. The group G generated by R and S is called the *group* of the map. R and S obey

$$R^p = S^q = (RS)^2 = I. \tag{8.1}$$

If there is an automorphism R_1 which interchanges two vertices of a face but leaves the two bordering faces fixed then we say that the regular map is reflexible. $R_1 \notin G$ but G is a subgroup of index 2 in a group G^* generated by $R_1, R_2 = R_1 R, R_3 = R_2 S$ satisfying

$$R_1^2 = R_2^2 = R_3^2 = (R_1 R_2)^p = (R_2 R_3)^q = (R_1 R_3)^2 = I. \tag{8.2}$$

Now suppose that we have a finite group G generated by R and S obeying (8.1). Then there exists a regular map of type $\{p, q\}$ whose group is G. This map is constructed as follows. Let K be the kernel of the canonical homomorphism from $\Gamma(2, p, q)$ to G. U has a $(2, p, q)$ tesselation defined on it. If we project from U to U/K then this tesselation will project to the required map on U/K. The map is reflexible if and only if $K \lhd \Gamma^*[2, p, q]$; i.e. if $G \lhd G^*$ with index 2 where G^* has generators R_1, R_2, R_3 obeying (8.2). We sum up in

8.1. Lemma. *Let G act as a K-automorphism group of a Riemann surface S, G being a surface-kernel homomorphic image of $\Gamma(2, p, q)$. Then there exists a regular map of type $\{p, q\}$ on the topological surface underlying S. If this map is reflexible then S is a symmetric Riemann surface. Conversely, if G is the group of a map of type $\{p, q\}$ on a surface S, then G is a K-automorphism group of a Riemann surface homeomorphic to S.*

In the converse the symmetry of the Riemann surface need not imply the reflexibility of the map. From the above discussion it follows that if G admits an outer automorphism α obeying condition (i) of Theorem 2, then the map is reflexible. If α obeys condition (ii) of Theorem 2 and not condition (i) then the map is irreflexible but the corresponding surface is symmetric. However if $p \neq q$ then the symmetry of the Riemann surface does imply the reflexibility of the map.

Irreflexible maps of genus $g > 1$ seem rather exceptional. Garbe [6] has shown that for $2 \leqq g < 7$ irreflexible maps do not exist. This yields

Theorem 6. *Let S be a Riemann surface of genus g, $2 \leqq g < 7$. If S admits a K-automorphism group (and a fortiori if S admits more than $24(g-1)$ automorphisms) then S is symmetric.*

We now discuss the known example of an irreflexible map on a surface of genus 7. This is Edmonds map of type {7,7} described in [2]. We will show that the corresponding Riemann surface is symmetric (and also give an algebraic proof of the irreflexibility). First of all we will find the group of the map, which from now on will be denoted by G. G is a homomorphic image of $\Gamma(2,7,7)$ and as the kernel is a surface group of genus 7, it has order 56.

8.2. Lemma. $G \simeq \mathrm{Aff}(1,8)$, *the one dimensional affine group over a field with* 8 *elements.*

Proof. Aff(1,8) is the subgroup of PSL(2,8) consists of matrices of the form

$$V(a,b)=\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \quad a,b \in GF(8), \quad a \neq 0.$$

If $a=1$, $V(a,b)$ has order 2, otherwise $V(a,b)$ has order 7. It is then easy to see that there is an epimorphism from (2,7,7) to Aff(1,8).

Now we show that G must be Aff(1,8). First of all, G is non-abelian, being an image of $\Gamma(2,7,7)$. If G contains a normal subgroup of order 7 then there would be an epimorphism from $\Gamma(2,7,7)$ to a group of order 8 which is impossible. Sylow's theorem now implies that there are 8 Z_7's and a normal subgroup N of order 8. Suppose, if possible, that G contains an element W of order 4. Let $V \in G$ have order 7. Then $V^i W V^{-i}$ $(i=0,1,\dots,5)$ have order 4 and hence at least two of them must be equal. We now see that an element of order 7 commutes with W and G contains a cyclic subgroup of order 28. It is easy to see that this is impossible for a group of order 56 containing 8 Z_7's. Thus $N \simeq Z_2 \times Z_2 \times Z_2$. Now suppose that G_1 and G_2 are two groups of order 56 containing a group isomorphic to N as a normal subgroup. Consider the Holomorph of N, HolN.

HolN is a split extension of N by AutN and has order 1344. (It is *not* the same group as in Theorem 5.) There are embeddings of G_1 and G_2 in HolN. We can write $G_1=S_1N$, $G_2=S_2N$ respectively. S_1 and S_2 are Sylow subgroups af HolN and therefore are conjugate; i.e. there exists $X \in$ HolN such that $XS_1X^{-1}=S_2$. We now have

$$XG_1X^{-1}=XS_1NX^{-1}=XS_1X^{-1}N=S_2N=G_2$$

and thus G_1 and G_2 are isomorphic. This proves that Aff(1,8) is the unique group of order 56 with the required properties.

As the map we are considering turns out to be irreflexible the corresponding subgroup of $\Gamma(2,7,7)$ is not normal in $\Gamma^*[2,7,7]$ but has one other conjugate subgroup. As part of the next theorem we will show that there are exactly 2 subgroups of $\Gamma(2,7,7)$ corresponding to the map and so the map is essentially unique.

Theorem 7. (i) *Corresponding to the maps of type* $\{7,7\}$ *on a surface of genus 7 these correspond exactly 2 subgroups of* $\Gamma(2,7,7)$.

(ii) *These subgroups are conjugate in* $\mathscr{L}^+$ *and hence correspond to a unique Riemann surface. This Riemann surface is symmetric whilst the map is irreflexible.*

Proof. (i) We will first of all show that there are only two possible kernels for a homomorphism from $\Gamma(2,7,7)$ onto G. Consider the set of all pairs of elements (X,Y) in $G\times G$ which generate G and obey $X^2=Y^7=(XY)^7=1$. Call two such pairs (X_1,Y_1), (X_2,Y_2) equivalent if there is an automorphism $\alpha: G\to G$ such $\alpha(X_1)=X_2, \alpha(Y_1)=Y_2$. Then the number of kernels is equal to the number of equivalence classes. Any pair is equivalent to a pair (X,Y) where $X=V(1,1)$, $Y=V(a,0)$ under an inner automorphism. There are also three automorphisms induced by the automorphisms of the Galois group of GF(8) over GF(2). As there are 6 possible values of a, there are 2 equivalence classes and hence 2 kernels K_1, K_2.

(ii) It can easily be seen that $\Gamma(2,7,7)\lhd\Gamma(2,4,7)$ with index 2 by considering the homomorphism from $\Gamma(2,4,7)$ onto Z_2. We will show that K_1 is not normal in $\Gamma(2,4,7)$. Hence it must have another conjugate subgroup in $\Gamma(2,4,7)$ and this must be K_2. Then K_1, K_2 being conjugate it $\mathscr{L}^+$ must represent the same Riemann surface.

To show that $K_1 \ntriangleleft \Gamma(2,4,7)$ we will first show that the only Z_2-extension of G is $G\times Z_2$.

Suppose that G admits an outer automorphism τ, $\tau^2=1$. Let $G^*=\{G,\tau\}$. As N is characteristic in G, $N\lhd G^*$ and there is a homomorphism $\theta: G^*\to \mathrm{GL}(3,2)\simeq \mathrm{PSL}(2,7)$. Suppose $\theta(\tau)$ has order 2. Then G^*/N is isomorphic to a subgroup of order 14 in PSL(2, 7) which is impossible. (See end of proof of Theorem 5.) Thus $\theta(\tau)=1$ and so τ centralizes N. Therefore

$$\langle N,\tau\rangle \simeq Z_2\times Z_2\times Z_2\times Z_2 = M$$

say. $M\lhd G^*$, $G^*/M\simeq Z_7$ and $G^*=MS_1$ where $S_1\simeq Z_7$. M is a 4-dimensional space over $GF(2)$ and N is a 3-dimensional subspace. S_1 acts as a group of linear transformations of M and N in an invariant subspace. Maschke's theorem [4] now implies that there exists an S_1-invariant complement, i.e.

$$M=N\oplus P$$

where $|P|=2$.

We now have $P = Z(G^*)$ and so

$$G^* = G \times P \simeq G \times Z_2,$$

proving our claim.

There is no homomorphism from $\Gamma(2,4,7)$ onto G (as all elements of order 2 lie in N) and hence to $Z_2 \times G$. Thus $K_1 \ntriangleleft \Gamma(2,4,7)$ and as shown above this means that K_1 and K_2 are conjugate in $\Gamma(2,4,7)$.

Thus there is a unique Riemann surface S of genus 7 admitting G as an automorphism group. By Theorem 5, a Riemann surface of genus 7 admitting PSL(2, 8) as automorphism group is symmetric (see § 7), and as $G \subset \mathrm{PSL}(2,8)$, S is symmetric. [A simple proof can also be obtained from Theorem 2 by constructing an automorphism obeying (ii) of that theorem.]

The irreflexibility of Edmonds map also follows from these ideas as there is no epimorphism from $\Gamma^*[2,7,7]$ onto $G \times Z_2$. If there was one, then we could generate G by 3 involutions which is impossible as all the involutions lie in N.

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