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Order Convergence and Topological Completion of Commutative Lattice-Groups

By

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§ 1. Introduction

Various concepts of order convergence can be defined in a lattice or a lattice-ordered algebra. Cf. [1], [2, Chap. IV, § 8], [3, § 3 and § 7], [4], [6], [8], [9, § 2.1], [10, § 3], [11], [12, § 5 and note on p. 314], [15, Definition 4.1]. In lattice-groups (briefly l-groups) and Boolean algebras each such concept induces, by means of the operations "$-$" and "$+$" respectively, a corresponding concept of fundamentality (i.e. of Cauchy nets). A pseudo-uniform structure is thus introduced and our main object in this paper is the completion of commutative l-groups (in a lesser degree of Boolean algebras) relative to order-fundamental nets or sequences in particular.

Stimulation for this research grew out of some discussions with D. A. Kappos (to whom the author expresses his sincere thanks) and the study of some publications in the subject, mainly [3] and [8]. H. Löwig in [8] introduces what he calls "intrinsic convergence" for ordinary sequences in an arbitrary Boolean ring B and proves that the pertinent completion by the Cantor process requires ω_1 steps (ω_1 is the least uncountable ordinal) and that the resulting ring $B(\omega_1)$ is the minimal Boolean σ-ring over B, preserving all existing joins and meets (Thm. 143). C. J. Everett in [3] employed an essentially different (and stronger) concept of sequence convergence (called o-convergence) in commutative l-groups and investigated the completion problem again. He first studied the outcome of the Cantor method, but did not notice two remarkable features of this notion of convergence: First, that sequence completion is effected in one step and second, that the resulting extension G' is topologically invariant, relative to sequences, over the original commutative l-group G (i.e. if $x_n \in G$, $x \in G$, then $x_n \to x$ relative to G if and only if $x_n \to x$ relative to G'). We establish both these statements in the present paper (§ 4) and thus disprove two conjectures of Everett. Actually we prove somewhat sharper theorems and show that they are valid for a class of convergences of the kind defined in § 7 of [3]. Possibly these results generalize to non-commutative l-groups, by means of the technique used in [4], but we have not worked it out.

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The same results apply to Boolean algebras for this class of convergences and we formulate our exposition in § 2, § 3 and § 4 so as to reveal this analogy. Our proofs are elementary and simplify many of Everett's proofs.

The extension G' mentioned earlier is not in general a σ-extension, even if G is Archimedean. The same situation prevails with Boolean algebras. We give a necessary and sufficient condition for G' to be a σ-extension; this condition is satisfied in the simply ordered Archimedean case. If G is lattice-ordered and Archimedean, we can get its minimal σ-extension (in ω_1 steps) by employing another concept of convergence, which we call here natural convergence: This is the one defined e.g. in [10, (3.1)]; in the case of Boolean algebras (but not Boolean rings in general!) it is equivalent to Löwig's intrinsic convergence. Natural convergence in arbitrary commutative l-groups is investigated in § 6. The sequential completion requires ω_1 steps and gives rise to a topologically invariant (relative to all nets) extension $G(\omega_1)$. We show that (for ordinary sequences) natural convergence in G is the restriction to G of o-convergence in $G(\omega_1)$. If G is Archimedean, then $G(\omega_1)$ is the minimal σ-extension of G. The completion G^* of G relative to all naturally fundamental nets was determined by B. BANASCHEWSKI [1].

A net which is convergent under either o-convergence or natural convergence is necessarily eventually bounded. In § 7 we investigate a concept of convergence (which we call L-convergence) weaker than natural convergence and allowing for convergent nets which are not eventually bounded. This is a generalization of "individual convergence", introduced by H. NAKANO [11] for sequences in a σ-vector-lattice. We also touch upon a still weaker concept of convergence, which, however, we leave for future investigation. § 8 is devoted to the completion of a commutative l-group relative to L-convergence.

With respect to this convergence the analogy between Boolean rings and commutative l-groups is not complete, especially in the non-Archimedean case. For example, LÖWIG has shown that in the case of Boolean rings an extension preserving joins and meets is necessarily topologically invariant. In the case of commutative l-groups we have to impose certain restrictions, mainly as regards Archimedity (cf. Thm. 6.4 for natural convergence and Thm. 7.5 for L-convergence). Further, a Boolean ring which is saturated (complete relative to joins and meets) is topologically complete relative to Löwig's intrinsic convergence, whereas a conditionally saturated l-group may fail to be complete relative to L-convergence.

We remark that all constructions concerning commutative l-groups can be applied to vector lattices as well. It is not difficult to see how scalar multiplication can be extended in each case.

Most of the content of § 2—§ 6 is a condensation of results contained in the author's doctoral dissertation, accepted by the University of Athens and published in the Greek-language section of the Bulletin of the Greek Math. Society [14], where more detailed developments and proofs are to be found.

Notation and terminology. Let L be a lattice. A (directed) *net* in L is a family $(x_\iota)_{\iota \in I}$ of elements of L, whose index domain I is directed by a partial ordering $\geqq$ (i.e. a reflexive and transitive relation) satisfying the Moore-Smith condition. We use either of the notations $(x_i)_{i\in I}$, (x_i) or x_i, $i \in I$ and adopt the terminology of KELLEY [7, Chap. 2]. The partial ordering of L will also be denoted by $\geqq$. The corresponding strict orderings will be denoted by $>$. A net (x_i) in L is *increasing* (resp. *decreasing*), if $i \geqq j$ implies $x_i \geqq x_j$ (resp. $x_i \leqq x_j$); notation: $(x_i\uparrow)_{i\in I}$ or $(x_i)\uparrow$ (resp. $(x_i\downarrow)_{i\in I}$ or $(x_i)\downarrow$).

If A is a subset of L, the supremum (or join or l.u.b.) of A in L, if it exists, will be denoted by $\bigvee^{(L)}_{a\in A} a$ or $\sup^{(L)} A$. Dually the infimum (meet, g.l.b.) will be denoted by $\bigwedge^{(L)}_{a\in A} a$ or $\inf^{(L)} A$. The corresponding notation for a family $(x_i)_{i\in I}$ in L will be: $\bigvee^{(L)}_{i\in I} x_i$ or $\sup^{(L)}\{x_i : i \in I\}$ and dually $\bigwedge^{(L)}_{i\in I} x_i$ or $\inf^{(L)}\{x_i : i \in I\}$. $x_i\uparrow^{(L)} x$ means that (x_i) is an increasing net and $x = \bigvee^{(L)}_{i\in I} x_i$. Dually $x_i\downarrow^{(L)} x$.

A sublattice L_0 of L is said to be *regular in* L (equivalently L is said to be *regular over* L_0), if $A \subseteqq L_0$, $x_0 = \sup^{(L_0)} A$ imply $x_0 = \sup^{(L)} A$, and dually. L_0 is σ*-regular in* L (equivalently L is σ*-regular over* L_0), if the above condition is satisfied for countable subsets $A \subseteqq L_0$.

L is *saturated*[1]) (resp. σ*-saturated*), if every subset of L (resp. every non-void countable subset of L) has a supremum and an infimum in L. L is *conditionally saturated* (resp. *conditionally* σ*-saturated*), if every non-void bounded subset of L (resp. every non-void countable bounded subset of L) has a supremum and an infimum in L. The *MacNeille saturation* of L is the saturation $\bar{L}$ of L by cuts $[A, B]$; (one of the two sets may be void). The *MacNeille conditional saturation* $\hat{L}$ of L consists of the "non-void" cuts ($[A, B]$ with $A \neq \emptyset$, $B \neq \emptyset$).

Set-theoretical unions and intersections will be denoted by $\cup$ and $\cap$ respectively. R denotes the real line and M the l-group of all bounded real functions on $[0, 1]$. N is the set of natural numbers 1, 2, ... and J the ordered group of all integers. Subscripts, superscripts and references are omitted whenever no confusion is likely.

§ 2. Convergence. Preliminaries

Let L be a lattice.

2.1. Definition.[2]) A net $(x_i)_{i\in I}$ in L *o-converges to* $x \in L$ *relative to* L (denoted: $o\text{-}\lim^{(L)}_{i\in I} x_i = x$), if there are an increasing net $(a_i)_{i\in I}$ and a decreasing net $(b_i)_{i\in I}$ in L (with the same index domain I as $(x_i)_{i\in I}$) such that $a_i \leqq x_i \leqq b_i$ for every $i \in I$, $a_i\uparrow^{(L)} x$ and $b_i\downarrow^{(L)} x$.

In a conditionally saturated lattice this takes the well-known form:

$$\bigwedge_{i\in I} \bigvee_{r\geqq i} x_r = \bigvee_{i\in I} \bigwedge_{r\geqq i} x_r = x\,.$$

A slight weakening of Definition 2.1 gives:

1) We do not use the term "complete" to avoid confusion with the notion of topological completeness which will be used later.

2) Cf. [15, Def. 4.1, p. 110] for the case of sequences. See also [2, p. 60, 3rd line] which deals with a saturated lattice.

2.2. Definition. *$(x_i)_{i\in I}$ τ-converges to x relative to L* $\left(\text{denoted } \tau\text{-}\lim_{i\in I}{}^{(L)} x_i = x\right)$, if there is an index $i_0 \in I$ such that the net $(x_i)_{i\in I_0}$, where $I_0 = \{i \in I : i \geqq i_0\}$, o-converges to x.

For bounded nets o- and τ-convergence are equivalent. Hence they are equivalent for sequences.

2.3. Definition. *$(x_i)_{i\in I}$ converges naturally to x relative to L* $\left(\text{denoted } \nu\text{-}\lim_{i\in I}{}^{(L)} x_i = x\right)$, if there are two nets $(a_\gamma)_{\gamma\in\Gamma}$, $(b_\delta)_{\delta\in\Delta}$ (not necessarily with the same index domain as $(x_i)_{i\in I}$), with $a_\gamma\uparrow x$, $b_\delta\downarrow x$ and such that, given $\gamma\in\Gamma$ and $\delta\in\Delta$, the relation $a_\gamma \leqq x_i \leqq b_\delta$ is eventually true relative to $(x_i)_{i\in I}$.

An element $a \in L$ is a *subelement* (resp. *superelement*) of $(x_i)_{i\in I}$ in L if $a \leqq x_i$ (resp. $a \geqq x_i$) eventually; cf. [8, Defs. 1 and 2]. Let V be the set of all subelements and U the set of all superelements of $(x_i)_{i\in I}$ in L. One can easily prove:

2.4. Proposition. *Definition 2.3 is equivalent to each of the following assertions:*

(i) *Same as Def. 2.3 with the requirement that $\Gamma = \Delta$.*

(ii) *There are two sets A, B with $\emptyset \neq A \subseteq V$, $\emptyset \neq B \subseteq U$ and such that A is (Moore-Smith) directed upwards, B is directed downwards and* $\sup A = \inf B = x$.

(iii) *Same as* (ii) *without the requirement that A and B be directed.*

(iv) $V \neq \emptyset$, $U \neq \emptyset$ *and* $\sup V = \inf U = x$.

A number of authors have used one or the other of these versions. See e.g. [10, 3.1, p. 15], [9, 2.1, p. 113], [8, Thms. 24, 19 and 22].

2.5. Theorem. *Let L be a lattice, $\bar{L}$ the MacNeille saturation and $\hat{L}$ the MacNeille conditional saturation of L. Further, let (x_i) be a net in L and $x \in L$. Then the following assertions are equivalent:*

(i) $\nu\text{-}\lim^{(L)} x_i = x$

(ii) $o\text{-}\lim^{(\bar{L})} x_i = x$

(iii) $\tau\text{-}\lim^{(\hat{L})} x_i = x$.

Proof. (i) implies (ii). Defining $\bar{a}_i = \bigwedge_{r\geqq i}^{(\bar{L})} x_r$, $\bar{b}_i = \bigvee_{r\geqq i}^{(\bar{L})} x_r$ we have $\bar{a}_i \leqq x_i \leqq \bar{b}_i$ for every $i \in I$. If v is an arbitrary subelement and u an arbitrary superelement of (x_i) in L, then $v \leqq \bar{a}_i \leqq \bar{b}_i \leqq u$ in $\bar{L}$ eventually. We easily deduce that $\bar{a}_i\uparrow^{(\bar{L})} x$, $\bar{b}_i\downarrow^{(\bar{L})} x$.

That (ii) implies (iii) is easy to prove.

(iii) implies (i). Let $(\hat{a}_i)_{i\geqq i_0}$, $(\hat{b}_i)_{i\geqq i_0}$ $(i_0\in I)$ in $\hat{L}$ be such that $\hat{a}_i \leqq x_i \leqq \hat{b}_i$ for all $i \geqq i_0$, $\hat{a}_i\uparrow^{(\hat{L})} x$, $\hat{b}_i\downarrow^{(\hat{L})} x$. For each $\hat{a}_i$ the set $A_i = \{a \in L : a \leqq \hat{a}_i\}$ is non-void by the definition of $\hat{L}$ and $\hat{a}_i = \sup^{(\hat{L})} A_i$. We define $A = \bigcup_{i\geqq i_0} A_i$; then $x = \bigvee_{i\geqq i_0}^{(\hat{L})} \hat{a}_i = \bigvee_{i\geqq i_0}^{(\hat{L})} \sup^{(\hat{L})} A_i = \sup^{(\hat{L})} A$ and since $A \subseteq L$, $x \in L : x = \sup^{(L)} A$. Clearly every element of A is a subelement of (x_i). Dually we define a set $B \subseteq L$ consisting of superelements of (x_i) and such that $\inf^{(L)} B = x$. By Prop. 2.4 ((iii)), $\nu\text{-}\lim^{(L)} x_i = x$. The proof is complete.

Thus in the case of a lattice L natural convergence in L is the restriction to L of o-convergence in $\bar{L}$. Natural convergence in the latter form was proposed by G. BIRKHOFF [2, p. 60].

We turn now to the particular cases that will interest us in the following pages, namely commutative l-groups and Boolean algebras. The fundamental properties of these structures are assumed known; the reader is referred e.g. to [2, Chap. X, XIV, XV]. The absolute value in an l-group is defined by $|x| = x \vee -x$. Free use will be made of the inequalities $|x \vee a - y \vee a| \leqq |x - y|$, $|x \wedge a - y \wedge a| \leqq |x - y|$ and $|x + y| \leqq |x| + |y|$. The first two are valid in any l-group; the third if and only if the l-group is commutative. "Positive" will mean $\geqq 0$, "strictly positive": > 0.

We shall later need a few facts about the Everett extension of a commutative l-group. If G is such an l-group and $\hat{G}$ is the MacNeille conditional saturation of G and if for any two elements $\hat{x} = \sup^{(\hat{G})} A$, $\hat{y} = \sup^{(\hat{G})} A'$ $(A, A' \subseteqq G)$ of $\hat{G}$ we define $\hat{x} + \hat{y} = \sup^{(\hat{G})}(A + A')$, then $\hat{G}$ is made into a commutative semigroup (cf. [3, § 5], [1, p. 54]; for the case of Archimedean l-groups see [2, Thm. 17, p. 229]). Every element $\hat{x}$ of $\hat{G}$ has the form $\hat{x} = \sup^{(\hat{G})} A = \inf^{(\hat{G})} B$, where $\emptyset \neq A \subseteqq G$, $\emptyset \neq B \subseteqq G$.

2.6. Proposition. ([3, Thm. 6].) *The element* $\hat{x} = \sup^{(\hat{G})} A = \inf^{(\hat{G})} B$ *has an inverse in the semigroup* $\hat{G}$, *if and only if* $\inf^{(G)}\{b - a : b \in B, a \in A\} = 0$.

Following Banaschewski [1] we denote by G^* the set of all elements of $\hat{G}$ which have an inverse in $\hat{G}$. G^* is a commutative l-group, regular over G. Further, $G^* = \hat{G}$ (equivalently G^* is conditionally saturated), if and only if G is Archimedean ([3, Thm. 7]). We shall call G^* the *Everett extension of* G.

The operations $+, -, \vee, \wedge$ and $|x|$ in an l-group, as well as the operations $\vee$, $\wedge$, x' (= complement of x), $\dotplus$ (symmetric difference) and $-$ in a Boolean algebra are continuous relative to each of the three concepts of convergence introduced above. This follows without difficulty from the infinite distributive laws. $o\text{-}\lim x_i = x$ is equivalent to $o\text{-}\lim |x_i - x| = 0$ in an l-group and to $o\text{-}\lim(x_i \dotplus x) = 0$ in a Boolean algebra. Similarly with τ- and natural convergence.

Let G be a commutative l-group and B a Boolean algebra.

2.7. Definition. The net $(x_i)_{i \in I}$ in G is *o-fundamental*, *τ-fundamental* or *naturally fundamental*, if and only if the net $x_i - x_j$, $(i, j) \in I \times I$ o-converges, τ-converges or converges naturally to 0, respectively. Here $I \times I$ is directed by the Cartesian (coordinatewise) ordering.

2.7a. Definition. Same as 2.7, with B in place of G and $x_i \dotplus x_j$ in place of $x_i - x_j$.

Let now F be an l-subgroup of G and A a Boolean subalgebra of B. Motivated by classical uniform convergence in the space of real functions we introduce the following definition:

2.8. Definition[3]. A net $(x_i)_{i \in I}$ in G *F-o-converges to* $x \in G$ $\left(\text{denoted } F\text{-}o\text{-}\lim_{i \in I} x_i = x\right)$, if there is a decreasing net $(u_i)_{i \in I}$ in F such that $u_i \downarrow^{(F)} 0$ and $|x_i - x| \leqq u_i$ in G for all $i \in I$. A net $(x_i)_{i \in I}$ is *F-o-fundamental* if

$$F\text{-}o\text{-}\lim_{(i,j) \in I \times I} (x_i - x_j) = 0.$$

2.8a. Definition. Same as 2.8, with B in place of G, A in place of F and $x_i \dotplus x$, $x_i \dotplus x_j$ in place of $|x_i - x|$ and $|x_i - x_j|$ respectively.

[3]) Cf. [3, § 7].

If G is the l-group of all real functions on R and F the l-subgroup of all constant functions, then F-τ-convergence (which is defined in an obvious way) is equivalent to classical uniform convergence.

For all concepts of fundamentality introduced hitherto it is true that if $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$ are fundamental, then so are $x_i + y_j$, $(i, j) \in I \times J$ and generally all their combinations by means of the algebraic operations of G or B. Note also that limits are unique in the convergences of Defs. 2.1, 2.2 and 2.3. This is not always true of 2.8 and 2.8a:

2.9. Proposition. *F-o-limits of nets (resp. of sequences) are unique in G, if and only if F is regular (resp. σ-regular) in G.*

2.9a. Proposition. *A-o-limits of nets (resp. of sequences) are unique in B, if and only if A is regular (resp. σ-regular) in B.*

In fact, if $x_i \downarrow^{(F)} 0$ and $0 < a \leqq x_i$, $a \in G$, then F-o-$\lim x_i = a$ and F-o-$\lim x_i = 0$ at the same time.

We now restrict our study to sequence F-o-convergence and A-o-convergence. In view of Prop. 2.9 and 2.9a we assume that F and A are σ-regular in G and B respectively. It is easily proved that a sequence (x_n) in G is F-o-fundamental, if and only if there is a sequence (u_n) in F such that $u_n \downarrow 0$ and $|x_n - x_{n+r}| \leqq u_n$ for all n and r[4]). Similarly with A-o-fundamentality, where $x_n \dotplus x_{n+r}$ appears in place of $|x_n - x_{n+r}|$. If F-o-$\lim x_n = x$, then o-$\lim^{(G)} x_n = x$ (i.e. G-o-$\lim x_n = x$) and similarly with fundamentality. If o-$\lim^{(G)} x_n = x$ and (x_n) is F-o-fundamental, then F-o-$\lim x_n = x$. The analogues in B are also true.

Two sequences (x_n), (y_n) in G are *F-o-congruent*, if F-o-$\lim\, (x_n - y_n) = 0$. We denote: $(x_n) \underset{F}{\approx} (y_n)$. Analogously with B and A; here $\dotplus$ takes the place of $-$.

The key to most of the results of the next two sections is the following fundamental lemma:

2.10. Lemma. *A sequence (x_n) in G is F-o-fundamental, if and only if there are two sequences (a_n), (b_n) in G such that (a_n) is increasing, (b_n) is decreasing, $a_n \leqq x_n \leqq b_n$ for all $n \in N$ and F-o-$\lim\, (b_n - a_n) = 0$. In this case (a_n) and (b_n) are also F-o-fundamental and F-o-congruent with (x_n).*

Analogously with B and A.

Proof. Suppose $|x_i - x_{i+r}| \leqq u_i \downarrow^{(F)} 0$ in G for all i and r. Then $x_i - u_i \leqq x_n \leqq x_i + u_i$ for all $i \leqq n$, i.e. for $i = 1, 2, \ldots, n$. We define

$$a_n = \bigvee_{i=1}^{n} (x_i - u_i), \qquad b_n = \bigwedge_{i=1}^{n} (x_i + u_i)\,.$$

Clearly $x_n - u_n \leqq a_n \leqq x_n \leqq b_n \leqq x_n + u_n$, which implies $b_n - a_n \leqq (x_n + u_n) - (x_n - u_n) = 2u_n$, i.e. F-o-$\lim\, (b_n - a_n) = 0$. The rest of the proof is obvious.

In the case of a Boolean algebra $x_i \dotplus x_{i+r} \leqq u_i$ implies $x_i - u_i \leqq x_n \leqq x_i \vee u_i$ for $i = 1, 2, \ldots, n$. We define $a_n = \bigvee_{i=1}^{n} (x_i - u_i)$, $b_n = \bigwedge_{i=1}^{n} (x_i \vee u_i)$; then $b_n - a_n \leqq (x_n \vee u_n) - (x_n - u_n) = u_n$.

[4]) Compare the definition of o-regular sequence in [3, § 3 and § 7] and [2, p. 232, Ex. 1], where a weak version of our lemma 2.10 appears, under a severe and invalidating assumption.

Thus an F-o-fundamental sequence is of the same nature as an F-o-convergent sequence, except that it may lack a limit. Roughly speaking a fundamental sequence is essentially equivalent to a nest of intervals.

Two σ-regular l-subgroups F, F' of G are said to be *topologically equivalent in* G, if F-o-$\lim x_n = x$ implies F'-o-$\lim x_n = x$ in G (for sequences) and conversely. Analogously with A, A' in B.

2.11. Proposition. *F, F' are topologically equivalent in G, if and only if: $u_n \downarrow^{(F)} 0$ in F implies the existence of (v_n) in F' with $u_n \leqq v_n$ for all n and $v_n \downarrow 0$, and conversely. In such a case sequence F-o-fundamentality and F'-o-fundamentality are equivalent. Analogously with A, A' in B.*

2.12. Definition. *G is F-o-complete*, if every F-o-fundamental sequence in G is F-o-convergent in G. If G is G-o-complete, we say that it is *o-complete*. Analogously with B and A.

The following lemma is a sharpening of Lemma 3 in [3]:

2.13. Lemma. *If (x_n) in G is F-o-fundamental, then so is the sequence $x_1, x_1 \vee x_2, x_1 \vee x_2 \vee x_3, \ldots$ and its dual. Similarly with B and A.*

Proof. If $|x_n - x_{n+r}| \leqq u_n \downarrow^{(F)} 0$ and $y_n = x_1 \vee x_2 \vee \cdots \vee x_n$, then $|y_n - y_{n+r}| = |(x_1 \vee \cdots \vee x_n) \vee x_n - (x_1 \vee \cdots \vee x_n) \vee (x_{n+1} \vee \cdots \vee x_{n+r})| \leqq$ $\leqq |x_n - (x_{n+1} \vee \cdots \vee x_{n+r})| \leqq |x_{n+1} - x_n| \vee \cdots \vee |x_{n+r} - x_n| \leqq u_n$. The proof for Boolean algebras is similar.

§ 3. Closure

Assume again that G is a commutative l-group, F a σ-regular l-subgroup of G, B a Boolean algebra and A a σ-regular Boolean subalgebra of B. We begin with some definitions concerning G and F; the analogues for B and A run in a parallel and obvious way.

A subset K of G is *F-o-closed* in G, if $x_n \in K$, $n = 1, 2, \ldots$ and F-o-$\lim x_n = x$ $(x \in G)$ imply $x \in K$. The *F-o-closure* of a set L in G is the least F-o-closed set containing L. This defines a genuine closure operator which in turn defines a topology, but we shall not go into this topology now. The *first F-o-limit extension of L in G* is defined by:

$[L] = \{x \in G$: there is a sequence (x_n) in L with F-o-$\lim x_n = x\}$. In general $[L]$ is not F-o-closed.

Let ω_1 be the least uncountable ordinal. For each $\xi \leqq \omega_1$ we define the *F-o-limit extension of G of order ξ*, denoted by $L(\xi)$, inductively as follows: $L(1) = [L]$; if ξ has a predecessor $\xi - 1$, then $L(\xi) = [L(\xi - 1)]$; if $\xi < \omega_1$ is a limit ordinal, then $L(\xi) = \left[\bigcup_{\eta<\xi} L(\eta)\right]$. Finally $L(\omega_1) = \bigcup_{\xi<\omega_1} L(\xi)$. Using the fact that every sequence of ordinals less than ω_1 is bounded from above by an ordinal less than ω_1, we easily prove that if L is any subset of G, then the F-o-closure of L in G is $L(\omega_1)$. Similarly with B.

We can also define alternate upward and downward F-o-limit extensions. $L^1 = \{x \in G$: there is an increasing sequence (x_n) in L such that F-o-$\lim x_n = x\}$; L_1 is defined dually. Inductively $L^\xi = \left(\bigcup_{\eta<\xi} (L^\eta \cup L_\eta)\right)^1$, $L_\xi = \left(\bigcup_{\eta<\xi} (L^\eta \cup L_\eta)\right)_1$.

It can be shown that if G is F-o-complete and L is a sublattice of G, then $L^{\xi} \cup L_{\xi} \subseteqq L(\xi) \subseteqq L^{\xi+1} \cap L_{\xi+1}$ for every $\xi < \omega_1$. In particular $L(\omega_1) = L^{\omega_1} = L_{\omega_1}$. Similarly with B. The proof follows classical lines (cf. [5, Kap. IV]). It seems however that stronger hypotheses are required to prove the equality $L(\xi) = L^{\xi+1} \cap L_{\xi+1}$, which is valid in the cases of real functions and subsets of the real line. See [5, Thm. 34.2.6 as sharpened on p. 402, and Thm. 33.2.9]. Prop. 3.1 and 3.1a below, which are proved in [14], give a sufficient condition.

A partially ordered set P is said to satisfy *condition* Σ, if it satisfies the following condition and its dual:

If for each natural number i $(x_{ik})_{k \in N}$ is an increasing sequence in P such that $y_i = \bigvee\limits_{k=1}^{\infty}{}^{(P)} x_{ik}$ exists, if the sequence $(y_i)_{i \in N}$ is decreasing and if $\bigwedge\limits_{i=1}^{\infty}{}^{(P)} y_i = \bigwedge\limits_{i=1}^{\infty}{}^{(P)} \bigvee\limits_{k=1}^{\infty}{}^{(P)} x_{ik}$ exists, then the set

$$D = \{d \in P\colon \text{there is a choice-function } k(.) \text{ on } N \text{ with values in } N \text{ and such that } d \leqq x_{i,k(i)} \text{ for all } i \in N\}$$

is non-void and $\bigwedge\limits_{i=1}^{\infty}{}^{(P)} \bigvee\limits_{k=1}^{\infty}{}^{(P)} x_{ik} = \sup{}^{(P)} D$.

This is a weakening of the concept of "$\aleph_0$-regularity" introduced by K. MATTHES [9].

3.1. Proposition. *If G is sequentially v-complete*[5]) *and a direct union (cardinal product) $G = \bigtimes\limits_{\tau \in T} G^{\tau}$ of commutative l-groups, each of which satisfies condition Σ, if moreover $F = G$ and L is a sublattice of G, then $L(\xi) = L^{\xi+1} \cap L_{\xi+1}$ for every $\xi < \omega_1$.*

3.1a. Proposition. *If B is a Boolean σ-algebra satisfying condition Σ*[6]), *if $A = B$ and if L is a sublattice of B, then $L(\xi) = L^{\xi+1} \cap L_{\xi+1}$ for all $\xi < \omega_1$.*

We turn now to a particular case of importance. *In the sequel H always denotes an l-subgroup of G (not necessarily σ-regular in G) and C a Boolean subalgebra of B.*

3.2. Proposition. *If $H \supseteqq F$, then $H(1) = H^1 = H_1$ and H is regular in $H(1)$.*

Proof. The first half is an immediate consequence of Lemma 2.10 (applied to H and F). To prove the second half assume $S \subseteqq H$, $\inf^{(H)} S = 0$ and $a \in H(1)$, $a \leqq s$ for all $s \in S$. Since $a \in H^1$, there is a sequence (a_n) in H with $a = \bigvee\limits_{n=1}^{\infty}{}^{(G)} a_n$. Then $a_n \leqq s$ for all $n \in N$ and all $s \in S$, hence $a_n \leqq 0$, $a \leqq 0$.

3.2a. Proposition. *If $C \supseteqq A$, then $C(1) = C^1 = C_1$ and C is regular in $C(1)$.*

3.3. Lemma. *If $H \supseteqq F$ and if (x_n) is a decreasing sequence in $H(1)$, then there is a decreasing sequence (y_n) in H with $x_n \leqq y_n$ for all n and such that (x_n) and (y_n) have the same lower bounds in G. If, moreover, (x_n) is F-o-fundamental, then (y_n) can be chosen to be F-o-congruent with (x_n), hence F-o-fundamental.*

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[5]) See Definition 6.7 below.

[6]) If $B = \bigtimes\limits_{\tau \in T} B^{\tau}$, where each B^{τ} satisfies condition Σ, then B too satisfies condition Σ (a Boolean algebra is bounded).

Proof. By 3.2, for each $x_n \in H(1) = H_1$ there is a decreasing sequence $(b_{n,i})_{i \in N}$ in H with $F\text{-}o\text{-}\lim_i b_{n,i} = x_n$, say $b_{n,i} - x_n \leqq u_{n,i} \downarrow_i^{(F)} 0$. Define $y_n = b_{1,n} \wedge b_{2,n} \wedge \cdots \wedge b_{n,n}$ for every n. Clearly $y_n \in H$, (y_n) is decreasing and $x_n \leqq y_n$, since $x_n \leqq x_k \leqq b_{k,n}$ for all $k = 1, 2, \ldots, n$. If $a \leqq y_n$ for all n, then $a \leqq b_{k,n}$ for all k, n with $k \leqq n$; hence $a \leqq \bigwedge_n^{(G)} b_{k,n} = x_k$ for every k. This shows that (x_n) and (y_n) have the same lower bounds in G.

Assume now that (x_n) is F-o-fundamental, say $x_n - x_{n+r} \leqq v_n \downarrow^{(F)} 0$. Then $y_n - x_n = b_{1,n} \wedge b_{2,n} \wedge \cdots \wedge b_{n,n} - x_n = [b_{1,n} - x_n] \wedge [b_{2,n} - x_n] \wedge \wedge \cdots \wedge [b_{n,n} - x_n] = [(b_{1,n} - x_1) + (x_1 - x_n)] \wedge [(b_{2,n} - x_2) + (x_2 - x_n)] \wedge \wedge \cdots \wedge [(b_{n,n} - x_n) + (x_n - x_n)] \leqq [u_{1,n} + v_1] \wedge [u_{2,n} + v_2] \wedge \cdots \wedge \wedge [u_{n,n} + v_n] \equiv u_n^*$.

It is easily verified that $u_n^* \downarrow^{(F)} 0$.

3.3a. Lemma. *Same as 3.3. with B, A, C in place of G, F, H respectively.*

The proof follows the lines of 3.3. Thus for the A-o-fundamentality of (y_n) one shows : $y_n + y_{n+r} \leqq (u_{1,n} \vee v_1) \wedge (u_{2,n} \vee v_2) \wedge \cdots \wedge (u_{n,n} \vee v_n) \equiv u_n^* \downarrow^{(A)} 0$.

3.4. Theorem. *If $H \supseteq F$ and if H is σ-regular (resp. regular) in G, then $H(1)$ too is σ-regular (resp. regular) in G and topologically equivalent to H in G.*

Proof. Assume H is σ-regular in G and $x_n \downarrow^{H(1)} 0$, $x_n \in H(1)$. Let (y_n) be the sequence in H, the existence of which is asserted by Lemma 3.3. Then (x_n) and (y_n) have the same lower bounds in G, hence in $H(1)$ also. This implies $y_n \downarrow^{H(1)} 0$ and since $y_n \in H : y_n \downarrow^{(H)} 0$. By the σ-regularity of H in G $y_n \downarrow^{(G)} 0$. From this and the relation $0 \leqq x_n \leqq y_n$ we conclude $x_n \downarrow^{(G)} 0$. Thus $H(1)$ is σ-regular in G.

Assume now that H is regular in G and let $\bigwedge_{\delta \in \Delta}^{H(1)} x_\delta = 0$ $(x_\delta \in H(1))$. Each $x_\delta = \bigwedge_{n=1}^{\infty(G)} b_{\delta n}$, where $b_{\delta n} \in H$. Then $x_\delta = \bigwedge_n^{H(1)} b_{\delta n}$, $\bigwedge_\delta^{H(1)} \bigwedge_n^{H(1)} b_{\delta n} = 0$, $\bigwedge_{\delta, n}^{H(1)} b_{\delta n} = 0$, $\bigwedge_{\delta, n}^{(H)} b_{\delta n} = 0$, $\bigwedge_{\delta, n}^{(G)} b_{\delta n} = 0$. We easily conclude $\bigwedge_\delta^{(G)} x_\delta = 0$.

That H and $H(1)$ are topologically equivalent follows from 3.3 and 2.11.

3.4a. Theorem. *If $C \supseteq A$ and C is σ-regular (resp. regular) in B, then $C(1)$ too is σ-regular (resp. regular) in B and topologically equivalent to C in B.*

3.5. Theorem. *If $H \supseteq F$, then $H(1)$ is F-o-closed in G.*

Proof. It is sufficient to show that $H(1) = H(2)$. If $x \in H(2)$, then, by Proposition 3.2, there is a decreasing F-o-fundamental sequence (x_n) in $H(1)$ such that $x = \bigwedge_{n=1}^{\infty(G)} x_n$. By Lemma 3.3 there is another such sequence (y_n) in H, hence $x \in H_1 = H(1)$.

3.5a. Theorem. *If $C \supseteq A$, then $C(1)$ is A-o-closed in B.*

If however $H \not\supseteq F$, then $H(1)$ is in general not F-o-closed and the $H(\xi)$, $0 \leqq \xi \leqq \leqq \omega_1$ may be pairwise distinct. Similarly if $C \not\supseteq A$. Examples are given below.

§ 4. Completion

G is again a commutative l-group, F a σ-regular l-subgroup of G, B a Boolean algebra and A a σ-regular Boolean subalgebra of B.

Let $\mathfrak{G}_F$ be the commutative l-group of all F-o-fundamental sequences in G; (the operations are defined thus: $(x_n) + (y_n) = (x_n + y_n)$, $-(x_n) = (-x_n)$, $(x_n) \vee (y_n) = (x_n \vee y_n)$). G is represented in $\mathfrak{G}_F$ by the constant sequences and is regular in $\mathfrak{G}_F$.

Let G_F be the quotient l-group of $\mathfrak{G}_F$ modulo the l-ideal of all F-o-null sequences, i.e. the l-group of all classes into which $\mathfrak{G}_F$ is partitioned by the congruence relation $\underset{F}{\approx}$ (F-o-congruence). The class to which a sequence (x_n) belongs is denoted by $[(x_n)]$; we further set $\bar{x} = [(x, x, \ldots)]$. By Lemma 2.10 each class $[(x_n)]$ can be represented in both the forms $[(a_n)]$ and $[(b_n)]$ where (a_n) is increasing and (b_n) decreasing. It is not difficult to see that $[(x_n)] \leqq [(y_n)]$ in G_F, if and only if $a_n \leqq d_n$ in G for all n, where (a_n) is an increasing sequence in $[(x_n)]$ and (d_n) a decreasing sequence in $[(y_n)]$. Using this fact one can easily prove that G_F is a regular extension of G; the embedding of G in G_F is given by $G \ni x \rightarrow \bar{x} \in G_F$ [7]).

An exactly similar construction we can make with B to get an extension B_A. The analogous assertions are true in this case too.

4.1. Proposition. *If (x_n) is an F-o-fundamental sequence in G, then F-o-lim $\bar{x}_n = [(x_i)]$ in G* [7]).

Proof. Let (a_n), (b_n) be the two sequences of Lemma 2.10. Clearly $\bar{a}_n \leqq \bar{x}_n \leqq \bar{b}_n$; we have only to prove that $\bigvee_{n=1}^{\infty}{}^{(G_F)} \bar{a}_n = \bigwedge_{n=1}^{\infty}{}^{(G_F)} \bar{b}_n = [(x_i)]$.

Let $[(y_i)]$ be an upper bound of $\bar{a}_n$, $n = 1, 2, \ldots$; (y_i) can be chosen to be decreasing. Then $y_i \geqq a_i$ for all i, hence $[(y_i)] \geqq [(a_i)] = [(x_i)]$. The dual is established similarly. This shows that $o\text{-lim}^{(G_F)} \bar{x}_n = [(x_i)]$ and since (x_n) is F-o-fundamental we conclude F-o-lim $\bar{x}_n = [(x_i)]$.

4.1 a. Proposition. *Same as* 4.1, *with B in place of G and A in place of F.*

G_F is therefore the F-o-closure of G in G_F and hence, by Thms. 3.4 and 3.4a:

4.2. Theorem. *G and G_F are topologically equivalent in G_F. B and B_A are topologically equivalent in B_A.*

In the particular case $F = G$ we see that G_G as an extension of G is "topologically invariant" over G, i.e. the restriction (relativization) to G of sequence o-convergence in G_G is equivalent to sequence o-convergence in G. This disproves a conjecture of C. J. EVERETT, who expressed the view that "some condition akin to regularity (a concept introduced by KANTOROVITCH [14], which has nothing to do with the term as used in the present paper) seems lacking" for this to be true (see [3, p. 114]). He gave a sufficient condition [3, Thm. 5].

4.3. Theorem. *G_F is F-o-complete. B_A is A-o-complete.*

Proof. Let (X_n) be an F-o-fundamental sequence in G_F and (Y_n) a decreasing sequence F-o-congruent with it (Lemma 2.10). By Lemma 3.3 there is a decreasing F-o-fundamental sequence (y_n) in G which satisfies $Y_n \leqq y_n$ for all n and has the same lower bounds with (Y_n) in G. It is easily verified that $[(y_i)] = \bigwedge_{n=1}^{\infty}{}^{(G_F)} Y_n$ hence $[(y_i)] = o\text{-lim}^{(G_F)} Y_n = o\text{-lim}^{(G_F)} X_n$. Since (X_n) is F-o-fundamental $[(y_i)] = F$-o-lim X_n. Similarly with B.

[7]) Cf. [3, § 4, Thm. 4].

In virtue of Theorems 3.4 and 3.4a the preceding result can be sharpened:

4.4. Theorem. *If F' is the F-o-closure of F in G_F, then G_F is F'-o-complete. If A' is the A-o-closure of A in B_A, then B_A is A'-o-complete.*

In fact F and F' are topologically equivalent in G_F by Thms. 3.4. and 3.5. In particular if $F = G$, then $F' = G_G$, hence:

4.5. Corollary. *G_G is o-complete. B_B is o-complete.*

The first half disproves the opinion of C. J. EVERETT that "it is not possible to complete G by the Cantor process in a single step except in special instances" [3, § 4]. The diagonality of G, a condition which Everett imposes on G in order to secure completion in one step, is unnecessary. In the general case he accepts as a solution to the completion problem the extension G^* mentioned in § 2, which is however much "larger" than G_G. B. BANASCHEWSKI [1, Satz 14] has essentially shown that G^* is the completion of G relative to natural convergence of nets in general. We shall return to this point later (see Thm. 6.8 below). Theorem 4.4 shows that the apparent successive extensions in [3, § 7] actually terminate with the first step.

4.6. Theorem. *If a commutative l-group E is an extension of G which is σ-regular over F and F-o-complete, then G_F is isomorphic with the F-o-closure of G in E, under an isomorphism that maps each element of G onto itself. Analogously with B.*

Thus G_F is the *minimal* F-o-complete extension of G. If G is simply (linearly) ordered, then so is G_G.

One might suspect that if G is Archimedean, then G_G is conditionally σ-saturated. If G is simply ordered this is true, since G is then a subgroup of R ([2, Chap. XIV, Thm. 15]). In the general case it is not:

4.7. Proposition. *A necessary and sufficient condition that G_G be conditionally σ-saturated is the following: If (x_n) is an increasing bounded sequence in G, then there is a decreasing sequence (y_n) in G with $x_n \leqq y_n$ for all n and $(y_n - x_n) \downarrow^{(G)} 0$.*

Roughly: If $[A, B]$ is a cut in G which can be approached from below by an ordinary sequence, then it can also be approached from above by an ordinary sequence. It is easily seen that the condition is equivalent to its dual.

4.7a. Proposition. *A necessary and sufficient condition that B_B be a Boolean σ-algebra is the following: If (x_n) is an increasing sequence in B, then there is a decreasing sequence (y_n) in B with $x_n \leqq y_n$ for all n and $(y_n - x_n) \downarrow^{(B)} 0$.*

We shall now give examples that do not satisfy these conditions. We begin with a Boolean algebra: Let S be an uncountable set, B the Boolean algebra $\{A \subseteq S$: either A or $A^C = S - A$ is finite$\}$ and (x_n) a sequence of pairwise distinct elements of S. Define $X_n = \{x_1, x_2, \ldots, x_n\}$. Then (X_n) is increasing but does not satisfy the above condition.

For examples of commutative l-groups define:

G (resp. G') $= \{f : f$ is a bounded real function on $D = [0, 1] \times [0, 1]$ such that there is a continuous function f^* on D with $f(x, y) = f^*(x, y)$ for all but a finite (resp. countable) number of points $(x, y) \in D\}$.

The MacNeille conditional saturation of both G and G' is the l-group $\hat{G}$ of all bounded functions on D. Define $\hat{g} \in \hat{G}$ by $\hat{g}(x, y) = 0$ if $x = 0$ and 1 if $x \neq 0$. Also $f_n(x, y) = nx$ if $0 \leqq x \leqq 1/n$ and 1 if $1/n < x \leqq 1$. Then $\hat{g} = \bigvee_{n=1}^{\infty}{}^{(\hat{G})} f_n$ ($f_n \in G$) but it is not difficult to prove that there is no sequence (h_n) in G or G' with $\hat{g} = \bigwedge_{n=1}^{\infty}{}^{(\hat{G})} h_n$.

Defining $\varphi_n(x, y) = nxe^{1-nx} \leqq e$ on D we see that $\varphi_n \in G$, $\lim \varphi_n(x, y) = 0$, hence $\hat{G}$-o-$\lim \varphi_n = 0$; however it is not true that G-o-$\lim \varphi_n = 0$. Thus the restriction to G of the o-convergence in $\hat{G}$ is in general weaker (more convergent sequences) than the o-convergence in G; in other words (Thm. 2.5) natural convergence in G is weaker than o-convergence. The derived topologies are also non-equivalent.

§ 5. Relativized convergence

The last mentioned example shows the possibility of defining new concepts of sequence convergence in a commutative l-group G or a Boolean algebra B by considering the restriction (relativization) to them of the o-convergence of a (not necessarily σ-regular) extension. In the next section we shall describe natural convergence of sequences as a convergence of this kind.

Let E be a commutative l-group which is an extension of G. In introducing E-o-convergence in G we can assume without loss of generality that E is o-complete. For if it is not we can extend it to E_E, which is o-complete (Corollary 4.5) and introduces the same sequence convergence and fundamentality in G (Thm. 4.2). We can then complete G by taking successive E-o-limit extensions $G(\xi)$, $0 \leqq \xi \leqq \omega_1$ until we arrive at $G(\omega_1)$ which is E-o-closed in E hence E-o-complete. We can also proceed by the Cantor method, construct the first extension $G\{1\}$, embed it in E and repeat ω_1 times. The resulting completions are isomorphic. Similar considerations apply to Boolean algebras.

If G is the l-group of all continuous real functions on $[0, 1]$ and $E = M$ (the l-group of all bounded functions on $[0, 1]$), then the corresponding extensions $G(\xi)$, $0 \leqq \xi \leqq \omega_1$ constitute the well-known Baire classes of bounded functions (cf. [5, Kap. IV], [13, Kap. XV]). In the case of Boolean algebras we get the Borel classes if we consider the Boolean algebra B of all finite unions of (open, closed or half-open) subintervals of $[0, 1]$ and the extension algebra D consisting of all subsets of $[0, 1]$ (cf. [5, Kap. IV], [8, p. 1190]).

§ 6. Natural convergence

In the case of Boolean algebras (but not Boolean rings in general) natural convergence, as we defined it, is equivalent to the intrinsic convergence, defined and studied extensively in the case of sequences by H. Löwig in [8]. In the present section we shall study natural convergence in commutative l-groups. Since most of the results more or less have their parallels in Boolean algebras we shall not make detailed references to [8]. We assume that G is a commutative l-group.

A directed net (x_i) in G is *eventually bounded from above* (resp. *from below*) *in* G, if it has at least one superelement (resp. subelement) in G. It is *eventually bounded*, if it has both a superelement and a subelement. For ordinary sequences "bounded" and "eventually bounded" are equivalent.

6.1. Proposition. *A net* (x_i) *in* G *is naturally fundamental, if and only if it is eventually bounded in* G *and*

$$\text{(1)} \qquad \inf{}^{(G)}\{u - v : u \in U \quad \textit{and} \quad v \in V\} = 0\,,$$

where U *is the set of superelements and* V *the set of subelements of* (x_i) *in* G.

Proof. If $\nu\text{-}\lim_{i,j} |x_i - x_j| = 0$ and W is the set of superelements of $|x_i - x_j|$, $(i, j) \in I \times I$ in G, then $\inf^{(G)} W = 0$ and for each $w \in W$ there is $j(w) \in I$ such that $|x_i - x_{j(w)}| \leqq w$ for all $i \geqq j(w)$, i.e. $x_{j(w)} - w \leqq x_i \leqq x_{j(w)} + w$ for all $i \geqq j(w)$. Hence $x_{j(w)} - w \in V$, $x_{j(w)} + w \in U$ for every $w \in W$; obviously $\inf^{(G)}\{(x_{j(w)} + w) - (x_{j(w)} - w) : w \in W\} = \inf^{(G)}\{2w : w \in W\} = 0$, which implies (1).

The converse is an immediate consequence of the fact that each $u - v$ $(u \in U,\ v \in V)$ is a superelement of $(|x_i - x_j|)$.

It follows that in an Archimedean l-group an increasing net is naturally fundamental if and only if it is bounded from above, and dually.

6.2. Proposition. *If* E *is a regular extension of* G[8]*),* (x_i) *a net in* G *and* $x \in G$, *then* $\nu\text{-}\lim^{(G)} x_i = x$ *implies* $\nu\text{-}\lim^{(E)} x_i = x$.

Proof. If U is the set of superelements and V the set of subelements of (x_i) in G, then $x = \inf^{(G)} U$ implies $x = \inf^{(E)} U$ and dually. Apply now Prop. 2.4 ((ii)).

6.3. Lemma. *If* $a \in G$, $(x_i)_{i \in I}$ *is a net in* G *which is eventually bounded from above and if* U *is the set of superelements of* (x_i) *in* G, *then assertion* (i) *below implies* (ii). *If moreover* G *is Archimedean, then* (ii) *also implies* (i).

(i) $\bigvee^{(G)}_{j \geqq i} (x_j \wedge a) = a$ for every $i \in I$.

(ii) $a \leqq u$ for all $u \in U$.

The dual lemma is also true.

Proof. If (i) is true and $u \in U$, then there is an $i_0 \in I$ such that $x_j \leqq u$ for all $j \geqq i_0$, hence $x_j \wedge a \leqq u$ for all $j \geqq i_0$ and finally $a = \bigvee_{j \geqq i_0} (x_j \wedge a) \leqq u$.

Conversely, assume that G is Archimedean and that (ii) is satisfied. Fix $i \in I$. Obviously $x_j \wedge a \leqq a$ for all $j \geqq i$. Let b be any upper bound of $(x_j \wedge a)_{j \geqq i}$. We shall prove:

(A) If $u \in U$, then $u + b - a \in U$.

In fact let u be in U and k an index such that $x_j \leqq u$ for all $j \geqq k$. Assume moreover that $k \geqq i$. Then for every $j \geqq k$ we have: $x_j = x_j \wedge a + x_j - x_j \wedge a \leqq b + 0 \vee (x_j - a) \leqq b + 0 \vee (u - a) = b + u - a$ by (ii). Hence $u + b - a \in U$.

Now choose $u_0 \in U$. Applying (A) repeatedly we deduce:

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[8]) Here and in the sequel "regular extension" in such a context means "commutative l-group which is a regular extension."

(B) $u_0 + n(b-a) \in U$ for every $n \in N$
hence by (ii) $a \leqq u_0 + n(b-a)$, i.e. $n(a-b) \leqq u_0 - a$. Since G is Archimedean $a - b \leqq 0$, $a \leqq b$. It follows that $a = \bigvee_{j \geqq i} (x_j \wedge a)$.

Note that the assumption that G is Archimedean is essential for the implication (ii) $\Rightarrow$ (i). In fact let M be the l-group of all bounded real functions on $[0, 1]$. Define $\sigma_n(x) = n^2 x(1-x^2)^n$, $x \in [0, 1]$, $n = 1, 2, \ldots$. (The sequence (σ_n) will be used freely in the sequel without explicit reference to its definition here.) The sequence (σ_n) is not bounded from above in M (if $x_n = 1/\sqrt{n+1}$, then $\lim_n \sigma_n(x_n) = +\infty$). Let J be the ordered group of all integers and define $G = J \circ M$ (lexicographic or ordinal product; see [2, p. 9]). An element $(k, f) \in G$ is a superelement of the sequence $(0, \sigma_n)$, $n = 1, 2, \ldots$ if and only if $k \geqq 1$. Hence any element $a = (m, g)$ with $m \leqq 0$ satisfies condition (ii) of the lemma, but for (i) to be satisfied it is necessary and sufficient that a be $\leqq (0, 0)$.

6.4. Theorem. *If G is Archimedean, E is a regular extension of G, (x_i) is a net in G which is eventually bounded in G and $x \in G$, then $\nu\text{-}\lim^{(G)} x_i = x$ and $\nu\text{-}\lim^{(E)} x_i = x$ are equivalent.*

Proof. In view of Prop. 6.2 we need only show that $\nu\text{-}\lim^{(E)} x_i = x$ implies $\nu\text{-}\lim^{(G)} x_i = x$. Let U be the set of superelements and V the set of subelements of (x_i) in G. We shall prove $x = \inf^{(G)} U$ and dually. Every $u \in U$ is a superelement of (x_i) in E also, hence $x \leqq u$ for all $u \in U$. If $a \in G$ is any lower bound of U, then by Lemma 6.3 $\bigvee^{(G)}_{j \geqq i} (x_j \wedge a) = a$ for every i, which implies $\bigvee^{(E)}_{j \geqq i} (x_j \wedge a) = a$ for every i.

Let U' be the set of superelements of (x_i) in E. The last assertion implies, by 6.3 again, that $a \leqq u'$ for all $u' \in U'$, i.e. $a \leqq \inf^{(E)} U' = x$. This establishes $x = \inf^{(G)} U$.

The requirement that (x_i) be eventually bounded in G is essential. Let E be the l-group $R^{[0,1]}$ of all real functions on $[0, 1]$ and $G = M$; then $\nu\text{-}\lim^{(E)} \sigma_n = 0$, although (σ_n) is unbounded in M. The Archimedity of G is also essential: take $G = J \circ M$ and $E = J \circ R^{[0,1]}$ and consider the sequence $(0, \sigma_n)$, $n = 1, 2, \ldots$.

Recall that G^* denotes the Everett extension of G.

6.5. Proposition. *If (x_i) is a net in G and $x \in G$, then $\nu\text{-}\lim^{(G)} x_i = x$ if and only if $\nu\text{-}\lim^{(G^*)} x_i = x$.*

Proof. Suppose $\nu\text{-}\lim^{(G^*)} x_i = x$. Every superelement u^* of (x_i) in G^* is of the form $u^* = \inf^{(G^*)} U(u^*)$, $U(u^*) \subseteqq G$, where obviously the elements of $U(u^*)$ are superelements of (x_i) in G. Setting $U = \bigcup_{u^*} U(u^*)$ we have $\inf^{(G^*)} U = x$, hence $\inf^{(G)} U = x$.

Theorems 6.6 and 6.8 below were proved by BANASCHEWSKI [1, Sätze 13, 14].

6.6. Theorem. (BANASCHEWSKI). *Every naturally fundamental net in G converges naturally in G^*.*

This follows from Prop. 6.1, since V, U determine (they do not constitute) a cut in G which by Prop. 2.6 belongs to G^* and is obviously the natural limit of (x_i) in G^*.

6.7. Definition. A commutative l-group E is said to be *ν-complete* (resp. *sequentially ν-complete)*, if every naturally fundamental net (resp. sequence) in E converges naturally in E.

6.8. Theorem. (BANASCHEWSKI). *G is ν-complete, if and only if $G = G^*$.*

Proof. If G is ν-complete and $[A, B]$ is a cut in G with $\inf^{(G)}\{b - a : b \in B, a \in A\} = 0$, then A, considered as an increasing "net", is naturally fundamental (by Prop. 6.1), hence converges naturally in G. Its limit $[A, B]$ in G^* must therefore belong to G. Thus $G^* \subseteq G$, $G^* = G$. The converse follows from Thm. 6.6.

6.9. Theorem. *If E is a ν-complete regular extension of G, then E contains a regular l-subgroup isomorphic with G^*, under an isomorphism which maps each element of G onto itself.*

Proof. If x^* is an element of G^*, there is a net (x_i) in G, naturally fundamental relative to G and such that $\nu\text{-}\lim^{(G^*)} x_i = x$. Since, by Prop. 6.2, the net (x_i) is naturally fundamental relative to E also, there is $y^* \in E$ such that $\nu\text{-}\lim^{(E)} x_i = y^*$. The mapping $G^* \ni x^* \to y^* \in E$ is the desired isomorphism.

To show that G^* (more precisely its image) is regular in E assume $\bigwedge_{i \in I}^{(G^*)} x_i^* = 0$. For each $i \in I$ there is a set $A_i \subseteq G$ such that $x_i^* = \inf^{(G^*)} A_i$. Thus $0 = \bigwedge_{i \in I}^{(G^*)} \inf^{(G^*)} A_i = \inf^{(G^*)} A = \inf^{(G)} A$, where $A = \bigcup_{i \in I} A_i$. Then $0 = \inf^{(E)} A$ which implies $0 = \bigwedge_{i \in I}^{(E)} x_i^*$.

Thus if G is an arbitrary commutative l-group, its *minimal* ν-completion is G^*. We return now to ordinary sequences to determine the sequential ν-completion of G. In the proof of the next theorem we use the fact that if (x_n) is a naturally fundamental sequence, then so are $x_1, x_1 \vee x_2, x_1 \vee x_2 \vee x_3, \ldots$ and its dual. The easy proof follows the lines of 2.13.

6.10. Proposition. *If G is sequentially ν-complete, then a sequence (x_n) is naturally fundamental in G if and only if it is o-convergent.*

Proof. If (x_n) is naturally fundamental in G, then so is $x_n, x_n \vee x_{n+1}, x_n \vee x_{n+1} \vee x_{n+2}, \ldots$ and its dual, for each n; hence $y_n = \bigvee_{i \geqq n}^{(G)} x_i$ and $z_n = \bigwedge_{i \geqq n}^{(G)} x_i$ exist for each $n = 1, 2, \ldots$. If u is any superelement and v any subelement of (x_n) in G, then $v \leqq z_n \leqq x_n \leqq y_n \leqq u$ eventually; therefore $y_n - z_n \downarrow 0$, which shows that (y_n), (z_n) are G-o-fundamental (hence naturally fundamental), G-o-congruent and define a limit x in G which is the o-limit of (x_n).

6.11. Corollary. *A sequence (x_n) in a commutative l-group G is naturally fundamental relative to G if and only if it is o-convergent in G^*.*

This follows from 6.5.

6.12. Corollary. *For ordinary sequences natural convergence and fundamentality in G is the restriction to G of o-convergence and o-fundamentality in G^*.*

To complete G relative to naturally fundamental sequences we proceed as in § 5 and construct $G(\omega_1)$ in G^*. This is sequentially ν-complete, topologically invariant and regular over G as well as regular in G^*. This is seen from the following propositions:

6.13. Proposition. *If H is an l-subgroup of G^* containing G, then G is regular in H and H is regular in G^*.*

6.14. Proposition. *If H is an l-subgroup of G^* containing G and if $x_i \in G$, $x \in G$, then $\nu\text{-}\lim^{(G)} x_i = x$, $\nu\text{-}\lim^{(H)} x_i = x$ and $\nu\text{-}\lim^{(G^*)} x_i = x$ are equivalent.*

They follow from the fact that $H^* = G^*$ and from Prop. 6.5. Now 6.14 and 6.10 imply:

6.15. Corollary. *A sequence (x_n) in G is naturally fundamental relative to G, if and only if it is o-convergent in $G(\omega_1)$.*

6.16. Theorem. *If E is a sequentially ν-complete regular extension of G, then $G(\omega_1)$ is isomorphic with a regular l-subgroup of E, containing G, under an isomorphism that maps each element of G onto itself.*

Proof. G is regular in E^* and by Thm. 6.9 G^* too can be regularly embedded in E^*. Then $G(\omega_1)$, being regular in G^*, is regular in E^* also. However $G(\omega_1)$ is contained in E, since E is sequentially ν-complete, and is therefore regular in E.

A direct proof is given in [14]. Thus $G(\omega_1)$ is the *minimal* sequential ν-completion of G. The importance of sequential completion relative to natural convergence is seen from the following:

6.17. Theorem. *An l-group is conditionally σ-saturated, if and only if it is Archimedean and sequentially ν-complete* (in either case it is commutative).

Proof. Assume G is conditionally σ-saturated. Then it is Archimedean (see [2, Chap. XIV, Thm. 17]), hence $G^* = \hat{G}$. If (x_n) is a naturally fundamental sequence in G, then there is $\hat{x} \in \hat{G}$ with

$$\hat{x} = \bigwedge_{n=1}^{\infty}{}^{(\hat{G})} \bigvee_{i \geqq n}{}^{(\hat{G})} x_i = \bigvee_{n=1}^{\infty}{}^{(\hat{G})} \bigwedge_{i \geqq n}{}^{(\hat{G})} x_i .$$

Since G is conditionally σ-saturated and regular in $\hat{G}$, we infer that $\hat{x} \in G$ and

$$\hat{x} = \bigwedge_{n=1}^{\infty}{}^{(G)} \bigvee_{i \geqq n}{}^{(G)} x_i = \bigvee_{n=1}^{\infty}{}^{(G)} \bigwedge_{i \geqq n}{}^{(G)} x_i .$$

Hence $G\text{-}o\text{-}\lim x_n = \hat{x}$, $\nu\text{-}\lim^{(G)} x_n = \hat{x}$.

Conversely assume G is Archimedean and sequentially ν-complete and let (x_n) be an increasing bounded sequence in G, say $x_n \leqq b$. Then $\hat{s} = \bigvee_{n=1}^{\infty}{}^{(\hat{G})} x_n$ exists in $\hat{G}$ and $\hat{G}\text{-}o\text{-}\lim x_n = \hat{s}$. By 6.11 (x_n) is naturally fundamental in G, hence naturally convergent in G, say $\nu\text{-}\lim^{(G)} x_n = x$. But then $\hat{G}\text{-}o\text{-}\lim x_n = x$, hence $x = \hat{s}$, i.e. $x = \bigvee_{n=1}^{\infty}{}^{(\hat{G})} x_n$ $x = \bigvee_{n=1}^{\infty}{}^{(G)} x_n$.

6.18. Corollary. *If G is Archimedean, then $G(\omega_1)$ is the minimal conditionally σ-saturated regular extension of G.*

Before closing this section we remark that one can develop a theory analogous to that of join-extensions introduced by LÖWIG for Boolean rings [8]. It is immediately seen that in the case of l-groups a join-extension is necessarily a "meet-and-join" extension and the parallel of Löwig's Theorem 66 [8] is trivial (compare loc. cit. Thm. 67). However we must be content with accepting G^* as a satisfactory "saturation" of G. In fact G^* "fills" as many "gaps" in G as we can hope to fill. If G is not Archimedean, the remaining gaps are of a deeper nature and are due to the non-Archimedity of G. They can only be filled at the cost of reducing the extension algebra to a semigroup. If, for instance, $[A, B]$ is a cut in G not belonging to G^*, i.e. not satisfying $\bigwedge^{(G)}_{b \in B, a \in A} (b - a) = 0$, then there is $x_0 \in G$ with $0 < x_0 \leqq b - a$ for all $b \in B$, $a \in A$. This implies that for every $a \in A$ $a + x_0 \leqq b$ for all $b \in B$, hence $a + x_0 \in A$, since $[A, B]$ is a cut. If E is any extension of G to a commutative l-group and $B' = \{b' \in E : b'$ is an upper bound of $A\}$, $A' = \{a' \in E : a'$ is a lower bound of $B'\}$, then $A \subsetneqq A'$, $B \subsetneqq B'$ but $x_0 \leqq b' - a'$ for all $b' \in B'$, $a' \in A'$. ($a + x_0 \in A$ implies $a + x_0 \leqq b'$ for all $b' \in B'$, hence $a \leqq b' - x_0$ for all $a \in A$, $b' - x_0 \in B'$, $a' \leqq b' - x_0$, $x_0 \leqq b' - a'$). In particular $\sup^{(E)} A$ cannot exist.

§ 7. L-convergence

Every o-fundamental or naturally fundamental net in a commutative l-group is eventually bounded; sequences in particular are bounded. However the sequence of functions (σ_n) in M considered earlier converges to 0 in M relative to pointwise convergence, without being bounded. It is therefore natural to have convergent or fundamental nets which are not eventually bounded. The purpose of the present section is to study a weakening of natural convergence allowing for such nets.

In [11] H. Nakano introduced the following definition of convergence in a conditionally σ-saturated vector lattice (see also [12, § 5 and note on p. 314]): A sequence (x_n) is said to be individually convergent to x, if for every pair of elements a, b $o\text{-}\lim (a \vee x_n) \wedge b = (a \vee x) \wedge b$. The next definition is a modification of this and coincides with it in the particular case of Nakano. Compare also [8].

Let G be a commutative l-group.

7.1. Definition. A net (x_i) in G *L-converges to $x \in G$ relative to G* (denoted $\operatorname{Lim}^{(G)}_{i \in I} x_i = x$), if for each pair of elements a^*, b^* in G $\nu\text{-}\lim^{(G)} (a^* \vee x_i) \wedge b^* = (a^* \vee x) \wedge b^*$.

Obviously for an eventually bounded net (x_i) $\operatorname{Lim} x_i = x$ and $\nu\text{-}\lim x_i = x$ are equivalent. If a net (x_i) L-converges to x, then every subnet of (x_i) L-converges to x.

7.2. Proposition. *$\operatorname{Lim}^{(G)} x_i = x$ if and only if for every $b \geqq 0$ in G $\nu\text{-}\lim^{(G)} |x_i - x| \wedge b = 0$.*

Proof. Suppose $\operatorname{Lim} x_i = x$, i.e. $\nu\text{-}\lim (a^* \vee x_i) \wedge b^* = (a^* \vee x) \wedge b^*$ for every a^*, b^*, and let b be any positive element. Choosing $a^* = x$, $b^* = b + x$ we see that $\nu\text{-}\lim (x \vee x_i) \wedge (x + b) = x$ from which we infer

$$\nu\text{-}\lim [0 \vee (x_i - x)] \wedge b = 0 . \tag{1}$$

Next, choosing $a^* = x - b$, $b^* = x$ we have $\nu\text{-}\lim [(x - b) \vee x_i] \wedge x = x$, hence $\nu\text{-}\lim (-[(x - b) \vee x_i] \wedge x) = -x$, therefore

$$\nu\text{-}\lim [0 \vee (x - x_i)] \wedge b = 0 . \tag{2}$$

(1) and (2) now imply:

$$\nu\text{-lim}\{[0 \vee (x_i - x)] \wedge b\} \vee \{[0 \vee (x - x_i)] \wedge b\} = 0$$

$$\nu\text{-lim}\,[0 \vee (x_i - x) \vee (x - x_i)] \wedge b = 0$$

$$\nu\text{-lim}\,|x_i - x| \wedge b = 0.$$

The converse follows from the fact that

$$|(a^* \vee x_i) \wedge b^* - (a^* \vee x) \wedge b^*| \leqq |x_i - x| \wedge (b^* - a^* \wedge b^*)\,.$$

7.3. Lemma. *In an l-group, if x, y and b are $\geqq 0$, then*

$$(x + y) \wedge b \leqq x \wedge b + y \wedge b\,.$$

7.4. Theorem. *If $\operatorname{Lim}_{i\in I} x_i = x$ and $\operatorname{Lim}_{j\in J} y_j = y$ in G, then $\operatorname{Lim}_{i\in I}(-x_i) = -x$, $\operatorname{Lim}_{(i,j)\in I\times J}(x_i + y_j) = x + y$, $\operatorname{Lim}_{(i,j)\in I\times J}(x_i \vee y_j) = x \vee y$ and dually.*

This follows from 7.2 and 7.3. For instance
$|x_i \vee y_j - x \vee y| \leqq |x_i \vee y_j - x \vee y_j| + |x \vee y_j - x \vee y| \leqq |x_i - x| + |y_j - y|$,
hence $|x_i \vee y_j - x \vee y| \wedge b \leqq |x_i - x| \wedge b + |y_j - y| \wedge b$.

Compare the next theorem (and its proof) with [8, Thm. 50, p. 1158].

7.5. Theorem. *If G is Archimedean, E is a regular extension of G, (x_i) is a net in G and $x \in G$, then $\operatorname{Lim}^{(E)} x_i = x$ implies $\operatorname{Lim}^{(G)} x_i = x$. If, moreover, E too is Archimedean, then $\operatorname{Lim}^{(G)} x_i = x$ and $\operatorname{Lim}^{(E)} x_i = x$ are equivalent.*

Proof. By 7.2 it is sufficient to prove the theorem for $x_i \geqq 0$, $x = 0$. If $\operatorname{Lim}^{(E)} x_i = 0$, then $\nu\text{-lim}^{(E)} x_i \wedge b = 0$ for all $b \geqq 0$, $b \in E$; in particular this is true for $b \in G$. Since the net $(x_i \wedge b)$ is bounded in G, this implies by 6.4 $\nu\text{-lim}^{(G)} x_i \wedge b = 0$ for all $b \geqq 0$, $b \in G$, i.e. $\operatorname{Lim}^{(G)} x_i = 0$.

Conversely, assume E is Archimedean and $\operatorname{Lim}^{(G)} x_i = x$. Let b' be any positive element of E. To show that $\nu\text{-lim}^{(E)} x_i \wedge b' = 0$ we need only show $\inf^{(E)} U' = 0$, where U' is the set of superelements of $(x_i \wedge b')$ in E.

Let $z \in E$ be such that $z \leqq u'$ for all $u' \in U'$. Then $z \leqq b'$ and by Lemma 6.3:

(3) $$z = \bigvee_{i \geqq i_0}^{(E)} (x_i \wedge b' \wedge z) = \bigvee_{i \geqq i_0}^{(E)} (x_i \wedge z) \quad \text{for every } i_0\,.$$

Now let a be any positive element of G. From (3) we get

(4) $$z \wedge a = \bigvee_{i \geqq i_0}^{(E)} ((x_i \wedge a) \wedge z \wedge a) \quad \text{for every } i_0\,.$$

However $\nu\text{-lim}^{(G)} x_i \wedge a = 0$ implies $\nu\text{-lim}^{(E)} x_i \wedge a = 0$, by Thm. 6.4, and this combined with (4) and Lemma 6.3 shows that $z \wedge a \leqq 0$.

Since this is true for every positive element a of G we have in particular $z \wedge x_i \leqq 0$ for all i. By (3) $z \leqq 0$, which establishes the equality $\inf^{(E)} U' = 0$.

The hypotheses concerning Archimedity are essential in the above theorem. Thus if $G = M$, $E = J \circ M$ we have $\operatorname{Lim}^{(G)} \sigma_n = 0$ but not $\operatorname{Lim}^{(E)}(0, \sigma_n) = (0, 0)$. If $G = J \circ M$, $E = J \circ R^{[0,1]}$, then $\operatorname{Lim}^{(E)}(0, \sigma_n) = (0, 0)$ but not $\operatorname{Lim}^{(G)}(0, \sigma_n) = (0, 0)$. The difficulties with these two examples are eliminated if we employ another concept of convergence: $\lim x_i = x$, if x is the only element satisfying $x = \bigvee_{i \geqq i_0} (x_i \wedge x) = \bigwedge_{i \geqq i_0} (x_i \vee x)$ for every i_0. This is weaker than L-convergence and coincides with it in an Archimedean l-group. The analogue of 7.4 is true for this concept of convergence too. Proofs of these facts and others concerning this convergence will be incorporated in another paper.

It follows from the above considerations that L-convergence is most natural in an Archimedean l-group. However we shall state theorems and make constructions for the general case, whenever possible.

7.6. Proposition. *If* (x_i) *is a net in* G *and* $x \in G$, *then* $\mathrm{Lim}^{(G)} x_i = x$ *if and only if* $\mathrm{Lim}^{(G^*)} x_i = x$.

Proof. Suppose $\mathrm{Lim}^{(G)} x_i = x$ and let b^* be any positive element of G^*. Then there is $b \in G$ with $b^* \leqq b$. Now $\nu\text{-}\lim^{(G)} |x_i - x| \wedge b = 0$ implies $\nu\text{-}\lim^{(G^*)} |x_i - x| \wedge b = 0$, hence $\nu\text{-}\lim^{(G^*)} |x_i - x| \wedge b^* = 0$.

Conversely, if $\nu\text{-}\lim^{(G^*)} |x_i - x| \wedge b^* = 0$ for every positive $b^* \in G^*$, then in particular $\nu\text{-}\lim^{(G^*)} |x_i - x| \wedge b = 0$ for every positive $b \in G$, and hence $\nu\text{-}\lim^{(G)} |x_i - x| \wedge b = 0$ (Prop. 6.5).

7.7. Theorem. *If* G *is the direct union (cartesian product)* $G = \mathop{\times}\limits_{\tau \in T} G^\tau$ *of commutative* l*-groups* $(G^\tau)_{\tau \in T}$ *and if* $x_i \in G$, $x \in G$, *then* $\mathop{\mathrm{Lim}}\limits_i{}^{(G)} x_i = x$ *if and only if* $\mathop{\mathrm{Lim}}\limits_i{}^{(G^\tau)} x_i^\tau = x^\tau$ *for every* $\tau \in T$ *(here* y^τ *denotes the* τ-th *coordinate of the element* $y \in G$*).*

This follows without difficulty from the fact that for each $b \geqq 0$, $b \in G$ the net $(|x_i - x| \wedge b)$ is bounded in G. In fact if u is a superelement of $(|x_i - x| \wedge b)$ in G, then for each $\tau \in T$ u^τ is a superelement of $(|x_i^\tau - x^\tau| \wedge b^\tau)$ in G^τ and conversely if s is a superelement of $(|x_i^{\tau_0} - x^{\tau_0}| \wedge b^{\tau_0})$ in G^{τ_0}, then there is a superelement u of $(|x_i - x| \wedge b)$ in G such that $s = u^{\tau_0}$ (choose $u^\tau = b^\tau$ for all $\tau \neq \tau_0$).

The analogous proposition for natural convergence is true for ordinary sequences (which can be proved to be bounded) but fails to generalize to nets. (If T is infinite, a net (x_i) in G may have no superelement, even though $\nu\text{-}\lim^{(G^\tau)} x_i^\tau = x^\tau$ for every $\tau \in T$.)

The significance of the above theorem is illustrated in the following examples: Consider M and $R^{[0,1]}$. In $R^{[0,1]}$ L-convergence (but not natural convergence) is equivalent to pointwise convergence; however, for ordinary sequences L-convergence, natural convergence and pointwise convergence coincide. In M L-convergence is again pointwise convergence (by 7.5) but natural convergence is not even for sequences equivalent to it. Observe that both M and $R^{[0,1]}$ are conditionally saturated.

§ 8. Completion

A net $(x_i)_{i \in I}$ in G is said to be *L-fundamental relative to* G, if $\mathop{\mathrm{Lim}^{(G)}}\limits_{(i,j) \in I \times I} (x_i - x_j) = 0$.

8.1. Definition. A commutative l-group G is said to be *L-complete* (resp. *sequentially L-complete)*, if every L-fundamental net (resp. sequence) is L-convergent in G.

In the case of a Boolean ring B the Boolean ring $\mathfrak{B}$ of all normal ideals of B is the minimal regular extension of B to a saturated Boolean ring (cf. [8, Thm. 68]). At the same time $\mathfrak{B}$ can be easily shown to be the minimal regular extension of B to a Boolean ring which is complete relative to Löwig's intrinsic convergence (applied to nets in general).

However the situation with commutative l-groups is different. The l-group M is conditionally saturated but not sequentially L-complete. Thus L-completion goes beyond order saturation; this reveals the importance of obtaining the L-completion of an arbitrary commutative l-group.

One can construct a sequential L-completion of G by the Cantor process in ω_1 steps, following the lines of [8]. There are points in [8] where LÖWIG shifts things from B to $\mathfrak{B}$ to facilitate proofs (see for instance the proof that if (x_n) is fundamental in B, then $\lim x_n = [(x_i)]$ in the "first fundamental extension" of B [8, Thm. 133]) but it is possible to give direct, although more elaborate, proofs. We shall not, however, follow this line. Instead, we shall obtain an L-completion $\tilde{G}$ of G; a sequential L-completion (isomorphic with the one obtained by the Cantor method) can then be constructed by limit extensions within $\tilde{G}$.

Our method of L-completion applied to the l-group of all bounded real functions on a set X yields (to within isomorphism) the l-group of all real functions on X. Taking M as our prototype we observe that every positive real function on $[0, 1]$ (i.e. every positive element of $R^{[0,1]}$) can be approached in the sense of L-convergence by an increasing L-fundamental net (in fact sequence) of positive elements of M. As with the classical Cantor process, the idea is to represent positive elements $\tilde{x}$ of the sought after extension $\tilde{G}$ by such nets in G. We shall then have $\tilde{x} = \bigvee^{(\tilde{G})}_{i \in I} x_i$. Obviously $\tilde{x}$ can be represented by many such nets. To avoid the trouble of taking equivalence classes we choose the net of *all* positive $x \in G$ with $x \leqq \tilde{x}$. The property $x \leqq \tilde{x}$ can easily be characterized in terms of G and (x_i): $x \leqq \tilde{x} = \bigvee^{(\tilde{G})}_{i\in I} x_i$ if and only if $x = x \wedge \tilde{x} = \bigvee^{(\tilde{G})}_{i\in I} (x \wedge x_i) = \bigvee^{(G)}_{i\in I} (x \wedge x_i)$. We thus arrive at the definition of *normal pyramid* below (8.3).

Taking into account Prop. 7.6, as well as the fact that L-convergence is weaker than natural convergence, we assume, without loss of generality, that $G = G^*$, i.e. that G is ν-complete. The assumption is not essential but greatly facilitates proofs. We repeat that the most interesting case is that of an Archimedean l-group G.

Two nets $(x_i)_{i\in I}$ and $(y_j)_{j\in I}$ are said to be *L-congruent* in G, if $\operatorname{Lim}^{(G)}_{(i,j)\in I\times J} (x_i - y_j) = 0$.

8.2. Proposition. *A net $(x_i)_{i\in I}$ in G is L-fundamental, if and only if there is a net $(y_j)_{j\in J}$ L-congruent with it.*

In fact if say $|x_i - y_j| \wedge b \leqq u$ for all $i \geqq i_0$, $j \geqq j_0$, then $|x_i - x_{i'}| \wedge b \leqq$ $\leqq |x_i - y_{j_0}| \wedge b + |y_{j_0} - x_{i'}| \wedge b \leqq 2u$ for all $i, i' \geqq i_0$.

Let $P = G^+ = \{x \in G : x \geqq 0\}$. If S is a subset of P which is directed upwards, then the identity mapping of S onto S makes S into an increasing net. This net is L-fundamental in G if and only if for each $b \in P$ there is a set $U \subseteq P$

satisfying: (i) $\inf^{(G)} U = 0$ and (ii) for every $u \in U$ there is an $s_0 \in S$ such that $(s - s_0) \wedge b \leqq u$ for all $s \geqq s_0$, $s \in S$. In this case we shall say that *the set S is L-fundamental.*

8.3. Definition. A non-void subset S of P is said to be a *pyramid*, if it is directed upwards and L-fundamental. A *normal pyramid* is a pyramid S which contains every element $x \in P$ satisfying $x = \bigvee^{(G)}_{s \in S} (x \wedge s)$. If S is any pyramid we define $S^- = \{x \in P : x = \bigvee^{(G)}_{s \in S} (x \wedge s)\}$.

A normal pyramid S is an ideal in P in the sense that $x \in S$, $y \in S$ imply $x \vee y \in S$ and $0 \leqq x \leqq y$, $y \in S$ imply $x \in S$. S^- is the least normal pyramid containing S and is L-congruent with S.

8.4. Proposition. *If S and T are pyramids, then so are the sets $\{s + t : s \in S, t \in T\}$, $\{s \vee t : s \in S, t \in T\}$ and $\{s \wedge t : s \in S, t \in T\}$. If S and T are normal pyramids, then $\{s \wedge t : s \in S, t \in T\}$ too is a normal pyramid and coincides with the set-theoretic intersection $S \cap T$.*

Proof. Put $Q = \{s + t : s \in S, t \in T\}$. Let

$$(1) \qquad \begin{cases} (s - s_0) \wedge b \leqq u & \text{for all} \quad s \geqq s_0, s \in S \\ (t - t_0) \wedge b \leqq v & \text{for all} \quad t \geqq t_0, t \in T, \end{cases}$$

and set $q_0 = s_0 + t_0$. If $q \in Q$, say $q = s_1 + t_1$, and $q \geqq q_0$, choose $s_2 \in S$, $t_2 \in T$ such that $s_2 \geqq s_0, s_1$ and $t_2 \geqq t_0, t_1$. Then $(q - q_0) \wedge b \leqq [(s_2 + t_2) - (s_0 + t_0)] \wedge b \leqq$ $\leqq (s_2 - s_0) \wedge b + (t_2 - t_0) \wedge b \leqq u + v$. Thus Q is L-fundamental. Similarly with the rest sets.

8.5. Proposition. *Two normal pyramids S and T are identical if and only if they are L-congruent.*

Proof. Assume S, T are L-congruent and let $x \in S$. Suppose $|s - t| \wedge x \leqq u$ for all $s \geqq s_0$, $s \in S$ and all $t \geqq t_0$, $t \in T$. We can assume $s_0 \geqq x$; then for every $t \geqq t_0 : x - x \wedge t = |x \wedge s_0 - x \wedge t| \wedge x \leqq |s_0 - t| \wedge x \leqq u$. We easily conclude $\bigwedge_{t \in T} (x - x \wedge t) = 0$, i.e. $x = \bigvee_{t \in T} (x \wedge t)$, hence $x \in T$. Conversely $x \in T$ implies $x \in S$.

8.6. Definition. *Let $\tilde{P}$ be the set of all normal pyramids in P. If S, $T \in \tilde{P}$ we define:*

$$S + T = \{s + t : s \in S, t \in T\}^-$$

$$S \vee T = \{s \vee t : s \in S, t \in T\}^-$$

$$S \wedge T = S \cap T = \{s \wedge t : s \in S, t \in T\}$$

$$S \leqq T \quad \text{if and only if } S \subseteq T\,.$$

8.7. Proposition. *$\tilde{P}$ is a commutative semigroup under $+$, with the pyramid $\{0\}$ as zero element. The cancellation law holds in $\tilde{P}$ and*

$$(2) \qquad S + T = \{0\} \quad \textit{implies} \quad S = \{0\} \quad \textit{and} \quad T = \{0\}\,.$$

Proof of the cancellation law. If $S + T = S + T'$, i.e. $\{s + t : s \in S, t \in T\}^- = \{s + t' : s \in S, t' \in T'\}^-$, then $\{s + t : s \in S, t \in T\}$ and $\{s + t' : s \in S, t' \in T'\}$ are L-congruent; say $|\lambda - \lambda'| \wedge b \leqq u$ for all $\lambda = s + t \geqq \lambda_0 = s_0 + t_0$ and all $\lambda' = s' + t' \geqq \lambda'_0 = s'_0 + t'_0$. Then for all $t \geqq t_0$, $t \in T$ and all $t' \geqq t'_0$, $t' \in T'$

we have: $|t - t'| \wedge b = |(s_0 \vee s_0' + t) - (s_0 \vee s_0' + t')| \wedge b \leqq u$, since $s_0 \vee s_0' + t \geqq$ $\geqq s_0 + t_0$ and $s_0 \vee s_0' + t' \geqq s_0' + t_0'$. We infer that T and T' are L-congruent and, by 8.5, $T = T'$.

8.8. Proposition. *$S \leqq T$ in $\tilde{P}$, if and only if there is $Q \in \tilde{P}$ such that $S + Q = T$. This Q is unique.*

Proof. Suppose $S \leqq T$. For each $t \in T$ the net $(t \wedge s)_{s \in S}$ is L-fundamental; being bounded it is naturally fundamental and since G is ν-complete $d_t = \bigvee^{(G)}_{s \in S} (t \wedge s)$ exists. Then

$$(3) \qquad \bigwedge_{s \in S} (d_t - t \wedge s) = 0, \quad \text{i.e.} \quad \bigvee_{s \in S} (t \wedge s - d_t) = 0 \quad \text{for all} \quad t \in T\,.$$

If $t_1, t_2 \in T$, then $\nu\text{-}\lim_s t_1 \wedge s = d_{t_1}$ and $\nu\text{-}\lim_s t_2 \wedge s = d_{t_2}$, hence $\nu\text{-}\lim |t_1 \wedge s - t_2 \wedge s| = |d_{t_1} - d_{t_2}|$. However $|t_1 \wedge s - t_2 \wedge s| \leqq |t_1 - t_2|$ for all s, which implies:

$$(4) \qquad |d_{t_1} - d_{t_2}| \leqq |t_1 - t_2|\,.$$

In particular if $t_1 \geqq t_2$, then $t_1 - t_2 \geqq d_{t_1} - d_{t_2}$, i.e. $t_1 - d_{t_1} \geqq t_2 - d_{t_2}$. Since T is directed upwards we infer from this that the set $A = \{t - d_t : t \in T\}$ is also directed upwards. A is L-fundamental too, for by (4):

$$|(t_1 - d_{t_1}) - (t_2 - d_{t_2})| \leqq |t_1 - t_2| + |d_{t_1} - d_{t_2}| \leqq 2\,|t_1 - t_2|\,.$$

Thus A is a pyramid and $Q = A^-$ is a normal pyramid. It remains to be shown that $S + Q = T$.

Assume $t_0 \in T$. To show $t_0 \in S + Q$ it is sufficient, by 8.6, to show:

$$(5) \qquad t_0 = \bigwedge_{s \in S,\, q \in Q} [t_0 \wedge (s + q)]\,.$$

If $\varphi \geqq t_0 \wedge (s + q)$ for all s, q then in particular $\varphi \geqq t_0 \wedge (s + t_0 - d_{t_0})$ for all $s \in S$, hence $\varphi \geqq 0 \wedge (s - d_{t_0}) + t_0 \geqq 0 \wedge (t_0 \wedge s - d_{t_0}) + t_0$. Taking supremum over $s \in S$, we infer from (3) that $\varphi \geqq 0 \wedge 0 + t_0 = t_0$, which establishes (5). Thus $T \subseteq S + Q$.

Conversely, if $s_0 \in S$, $q_0 \in Q$, then

$$(6) \qquad \begin{aligned} s_0 + q_0 &= s_0 + \bigvee_{t \in T} q_0 \wedge (t - d_t) = \bigvee_{t \in T} [s_0 + q_0 \wedge (t - d_t)] \\ &= \bigvee_{t \in T} [(s_0 + q_0) \wedge (s_0 + t - d_t)]\,. \end{aligned}$$

But $s_0 + t - d_t = s_0 \vee t + s_0 \wedge t - d_t \leqq s_0 \vee t$ (since $t \wedge s_0 - d_t \leqq 0$ by (3)) and $s_0 \vee t \in T$ (since $S \subseteq T$). T being an ideal in P, we infer $s_0 + t - d_t \in T$ for all $t \in T$ and (6) shows that $s_0 + q_0 = \bigvee_{t' \in T} [(s_0 + q_0) \wedge t']$, i.e. $s_0 + q_0 \in T$. Thus $\{s + q : s \in S,\ q \in Q\} \subseteq T$, hence $S + Q = \{s + q : s \in S,\ q \in Q\}^- \subseteq T$.

That $S + Q = T$ implies $S \leqq T$ is obvious and uniqueness of Q follows from the cancellation law.

Summing up the consequences of 8.8: If $S, T \in \tilde{P}$, then $S \vee T$ is the least common "multiple" and $S \wedge T$ the greatest common „divisor" of S and T in $\tilde{P}$ relative to the operation $+$. At the same time $S \vee T$ and $S \wedge T$ are respectively the join and meet of S, T relative to the partial ordering $\leqq$.

For each $x \in P$ the set $\{y \in P : y \leqq x\}$ is a normal pyramid in P, which we denote by $\tilde{x}$.

8.9. Proposition. *If* S *is a pyramid, then* $S^- = \bigvee_{s \in S}^{(\tilde{P})} \tilde{s} = \sup^{(\tilde{P})} \{\tilde{s} : s \in S\}$.

8.10. Proposition. *The mapping* $P \ni x \to \tilde{x} \in \tilde{P}$ *is an embedding of* P *in* $\tilde{P}$ *which preserves sums, differences, the ordering relation and all existing joins and meets.*

We now extend the semigroup $\tilde{P}$ to a commutative l-group $\tilde{G}$ by considering formal differences $S - T$ of elements of $\tilde{P}$. Cf. [2, Chap. XIV, § 3, pp. 217–218]. The set of positive elements of $\tilde{G}$ is $\tilde{P}$ with the original ordering, join operation and meet operation. Prop. 8.10 implies:

8.11. Theorem. $\tilde{G}$ *is a commutative l-group which is a regular extension of* G.

If S is a normal pyramid we shall sometimes find it helpful to distinguish between S as a subset of P and S as an element of $\tilde{P}$. Elements of $\tilde{P}$ will be denoted by boldface letters $\mathbf{S}, \mathbf{T}, \ldots$, while $S, T, \ldots$ will be retained for the corresponding sets. The elements $\tilde{x}, \tilde{y}, \ldots$ will be identified with $x, y, \ldots$.

8.12. Lemma. *If* $0 \leqq \mathbf{S} \leqq x$ *where* $x \in G$, *then* $\mathbf{S}$ *is an element of* G *(more precisely there is* $y \in G$ *such that* $S = \{s \in G : s \leqq y\}$*).*

Proof. The pyramid S is L-fundamental and bounded in G. It is therefore naturally fundamental, hence naturally convergent in G.

8.13. Proposition. *If* (x_i) *is eventually bounded in* G *and* $x \in G$, *then* $\nu\text{-}\lim^{(G)} x_i = x$ *if and only if* $\nu\text{-}\lim^{(\tilde{G})} x_i = x$.

Proof. Suppose $|x_i - x| \leqq c$ for all $i \geqq i_0$ $(c \in G)$. Suppose further $\nu\text{-}\lim^{(\tilde{G})} x_i = x$ and set $\mathfrak{A} = \{\mathbf{S} : \mathbf{S}$ is a superelement of $(|x_i - x|)$ in $\tilde{G}$ such that $\mathbf{S} \leqq c\}$. Then $\mathfrak{A} \subseteq G$ by Lemma 8.12 and $\inf^{(G)} \mathfrak{A} = 0$.

8.14. Theorem. *If* (x_i) *is a net in* G *and* $x \in G$, *then* $\mathrm{Lim}^{(G)} x_i = x$ *if and only if* $\mathrm{Lim}^{(\tilde{G})} x_i = x$.

Proof. Suppose $\mathrm{Lim}^{(G)} x_i = x$. We shall show that for every $\mathbf{S} \geqq 0$ in $\tilde{G}$ $\nu\text{-}\lim^{(\tilde{G})} |x_i - x| \wedge \mathbf{S} = 0$. Fix $\mathbf{S}$ and define $\mathfrak{A} = \{\mathbf{S} - s + u \in \tilde{G} : s \in S$ and u is a superelement of $(|x_i - x| \wedge s)_i$ in $G\}$.

Then $\inf^{(\tilde{G})} \mathfrak{A} = 0$, since $\mathbf{X} \leqq \mathbf{S} - s + u$ for all s and all u $(\mathbf{X} \in \tilde{G})$ implies $\mathbf{X} \leqq \mathbf{S} - s$ for all s, hence $\mathbf{X} \leqq 0$ by 8.9. Moreover every element of $\mathfrak{A}$ is a superelement of $(|x_i - x| \wedge \mathbf{S})$ in $\tilde{G}$, since given s and u $|x_i - x| \wedge \mathbf{S} = |x_i - x| \wedge (s + \mathbf{S} - s) \leqq |x_i - x| \wedge s + |x_i - x| \wedge (\mathbf{S} - s) \leqq |x_i - x| \wedge s + \mathbf{S} - s \leqq u + \mathbf{S} - s$ eventually. Thus $\nu\text{-}\lim^{(\tilde{G})} |x_i - x| \wedge \mathbf{S} = 0$.

Conversely, suppose $\mathrm{Lim}^{(\tilde{G})} x_i = x$. Then for every positive $\mathbf{S}$ in $\tilde{G}$ $\nu\text{-}\lim^{(\tilde{G})} |x_i - x| \wedge \mathbf{S} = 0$; in particular $\nu\text{-}\lim^{(\tilde{G})} |x_i - x| \wedge b = 0$ for every positive $b \in G$. By Prop. 8.13 $\nu\text{-}\lim^{(G)} |x_i - x| \wedge b = 0$ and the proof is complete.

We are now in a position to prove that every L-fundamental net in $\tilde{G}$ is L-convergent in $\tilde{G}$.

8.15. Lemma. *If* $(\mathbf{S}_i)_{i \in I}$ *is an increasing net in* $\tilde{P}$ *which is L-fundamental relative to* $\tilde{G}$, *then* $\bigvee_{i \in I}^{(\tilde{G})} \mathbf{S}_i$ *exists.*

Proof. We shall show that $R = \bigcup_{i \in I} S_i$ is a pyramid and that R^- is the required supremum.

Since $(\mathbf{S}_i)$ is L-fundamental $\nu\text{-}\lim_{i,j}^{(\tilde{G})} |\mathbf{S}_i - \mathbf{S}_j| \wedge \mathbf{B} = 0$ for every $\mathbf{B} \in \tilde{P}$, in particular for every $b \in P$. Fix b and for each superelement $\mathbf{A} \in \tilde{P}$ of

$(|\mathbf{S}_i - \mathbf{S}_j| \wedge b)_{i,j}$ choose $i(\mathbf{A}) \in I$ so that $|\mathbf{S}_i - \mathbf{S}_j| \wedge b \leqq \mathbf{A}$ for all $i, j \geqq i(\mathbf{A})$. Define

$\Sigma = \{2(\mathbf{A} + \mathbf{S}_{i(\mathbf{A})} - s) : \mathbf{A}$ is a superelement of $(|\mathbf{S}_i - \mathbf{S}_j| \wedge b)_{i,j}$ and $s \in S_{i(\mathbf{A})}\}$.

Clearly $\inf^{(\tilde{G})} \Sigma = 0$. We shall prove that every element of Σ is a superelement of $|s - s'| \wedge b$, s, $s' \in R\uparrow$. In fact let $2(\mathbf{A} + \mathbf{S}_{i(\mathbf{A})} - s_0) \in \Sigma$. Then for every $s \in R$ with $s \geqq s_0$ there is an index $i(s) \geqq i(\mathbf{A})$ such that $s \in S_{i(s)}$ ($\mathbf{S}_i$, $i \in I$ is increasing), or equivalently $s \leqq \mathbf{S}_{i(s)}$; hence $|s - s_0| \wedge b = (s - s_0) \wedge b \leqq$ $\leqq (\mathbf{S}_{i(s)} - s_0) \wedge b \leqq (\mathbf{S}_{i(s)} - \mathbf{S}_{i(\mathbf{A})}) \wedge b + (\mathbf{S}_{i(\mathbf{A})} - s_0) \wedge b \leqq \mathbf{A} + \mathbf{S}_{i(\mathbf{A})} - s_0$.

We infer $\nu\text{-}\lim^{(\tilde{G})}_{(s,s')\in R\times R} |s - s'| \wedge b = 0$. By Prop. 8.13 $\nu\text{-}\lim^{(G)}_{(s,s')\in R\times R} |s - s'| \wedge b = 0$ and thus $R\uparrow$ is L-fundamental relative to G. Defining $S = R^- = \left(\bigcup_{i\in I} S_i\right)^-$ we see, by 8.9, that $\mathbf{S} = \bigvee^{(\tilde{G})}_{s\in R} s = \bigvee^{(\tilde{G})}_{i\in I} \bigvee^{(\tilde{G})}_{s\in S_i} s = \bigvee^{(\tilde{G})}_{i\in I} \mathbf{S}_i$.

8.16. Corollary. *$\tilde{G}$ is ν-complete and $\tilde{\tilde{G}} = \tilde{G}$.*

8.17. Lemma. *In a ν-complete commutative l-group, if (x_i) is an L-fundamental net of positive elements and V is the set of positive subelements of (x_i), then (x_i) and $V\uparrow$ are L-congruent.* (This lemma is actually true in every commutative l-group.)

Proof. For each $i \in I$ the net $(x_i \wedge x_k)_{k\in I}$ is bounded and L-fundamental, hence naturally fundamental, therefore naturally convergent, say:

(7) $$\nu\text{-}\lim_k x_i \wedge x_k = y_i .$$

Now $\nu\text{-}\lim_k |x_i \wedge x_k - x_j \wedge x_k| = |y_i - y_j|$ and since $|x_i \wedge x_k - x_j \wedge x_k| \leqq |x_i - x_j|$ for all k we infer

(8) $$|y_i - y_j| \leqq |x_i - x_j| .$$

Let $V_i = \{v : v$ is a positive subelement of $(x_i \wedge x_k)_k\}$. It is easy to prove that

(9) $$V = \bigcup_{i\in I} V_i$$

(10) $$y_i = \sup V_i \quad \text{(by (7))} .$$

Assertion. Every $v \in V$ is frequently in $(V_i)_{i\in I}$, i.e. for every $i \in I$ there is $p \geqq i$ such that $v \in V_p$.

In fact let $v \in V$ and $i \in I$. Then $v \in V_j$ for some $j \in I$ by (9), i.e. there is $k(v) \in I$ such that $v \leqq x_j \wedge x_k$ for all $k \geqq k(v)$, hence $v \leqq x_k$ for all $k \geqq k(v)$. Choosing $p \geqq i, k(v)$ we have $v \leqq x_k$ for all $k \geqq p$, hence $v \leqq x_p \wedge x_k$ for all $k \geqq p$, i.e. $v \in V_p$.

We now proceed to show that

(11) $$\operatorname*{Lim}_{i\in I, v\in V} |x_i - v| = 0 .$$

Fix $b \geqq 0$. Let A be the set of superelements of $(|x_i - x_j| \wedge b)_{i,j}$ and for each $a \in A$ choose $i(a)$ such that

(12) $$|x_i - x_j| \wedge b \leqq a \quad \text{for all} \quad i, j \geqq i(a) .$$

Then, by (8):

(13) $$|y_i - y_j| \wedge b \leqq a \quad \text{for all} \quad i, j \geqq i(a) .$$

Recall that $y_{i(a)} = \sup V_{i(a)}$ by (10) and define $\Omega = \{3a + y_{i(a)} - v : a \in A$ and $v \in V_{i(a)}\}$. Obviously $\inf \Omega = \bigwedge_{a \in A} \bigwedge_{v \in V_{i(a)}} (3a + y_{i(a)} - v) = 0$. We shall show that every element of Ω is a superelement of $(|x_i - v| \wedge b)_{i,v}$. More precisely:

If $3a + y_{i(a)} - v_0 \in \Omega$, then for all $i \geqq i(a)$ and all $v \geqq v_0$ $(v \in V)$

$$|x_i - v| \wedge b \leqq 3a + y_{i(a)} - v_0 \,. \tag{14}$$

Let $i \geqq i(a)$ and $v \geqq v_0$. By the Assertion proved earlier there is some $p \geqq i$ such that $v \in V_p$, i.e. $v \leqq y_p$. Then

$$|x_i - v| \leqq |x_i - y_i| + |y_i - y_p| + y_p - v \leqq |x_i - y_i| + |y_i - y_p| + y_p - v_0 \leqq$$
$$\leqq |x_i - y_i| + |y_i - y_p| + |y_p - y_{i(a)}| + y_{i(a)} - v_0$$

since $v_0 \in V_{i(a)}$. Applying Lemma 7.3 and using (12), (13) we infer:

$$|x_i - v| \wedge b \leqq |x_i - y_i| \wedge b + a + a + y_{i(a)} - v_0 \,. \tag{15}$$

However $|x_i - y_i| \wedge b = v\text{-}\lim_k |x_i - x_i \wedge x_k| \wedge b$ by (7) and since $|x_i - x_i \wedge x_k| \wedge b = |x_i \wedge x_i - x_i \wedge x_k| \wedge b \leqq |x_i - x_k| \wedge b \leqq a$ eventually (for $k \geqq i(a)$), we have $|x_i - y_i| \wedge b \leqq a$ and (15) yields (14), which in turn implies (11). The proof is complete.

8.18. Proposition. *Every L-fundamental net (x_i) in G is L-convergent in $\tilde{G}$.*

Proof. It is sufficient to prove the theorem for nets of positive elements, for if (z_i) is an arbitrary L-fundamental net then the nets $(z_i^+) = (z_i \vee 0)$ and $(z_i^-) = (-z_i \vee 0)$ are L-fundamental $(|z_i^+ - z_j^+| \leqq |z_i - z_j|)$ and $\operatorname{Lim}_i^{(\tilde{G})} z_i^+ - \operatorname{Lim}_i^{(\tilde{G})} z_i^- = \operatorname{Lim}_i^{(\tilde{G})} z_i$.

If (x_i) is an L-fundamental net of positive elements in G and V is the set of its positive subelements, then $V\uparrow$ is L-fundamental relative to G, by Proposition 8.2 and the preceding lemma. Hence V is a pyramid. $S = V^-$ is a normal pyramid and $\mathbf{S} = \sup^{(\tilde{G})} V = \operatorname{Lim}_{v \in V}^{(\tilde{G})} v = \operatorname{Lim}_{i \in I}^{(\tilde{G})} x_i$ by the lemma again.

8.19. Theorem. *$\tilde{G}$ is L-complete.*

In fact every L-fundamental net in $\tilde{G}$ is L-convergent in $\tilde{\tilde{G}} = \tilde{G}$.

8.20. Lemma. *If G is conditionally saturated, T is a pyramid in P and A a subset of T which is directed upwards, then A too is a pyramid.*

Proof. For each $t \in T$ define $d_t = \bigvee_{a \in A}^{(G)} (t \wedge a) \leqq t$. By the regularity of G in $\tilde{G}$ $d_t = \bigvee_{a \in A}^{(\tilde{G})} (t \wedge a)$. The net $(d_t)_{t \in T}$ is directed upwards and L-fundamental, since $|d_{t_1} - d_{t_2}| \leqq |t_1 - t_2|$. Hence $B = \{d_t : t \in T\}$ is a pyramid; setting $S = B^-$ we have

$$\mathbf{S} = \bigvee_{t \in T}^{(\tilde{G})} d_t = \bigvee_{t \in T}^{(\tilde{G})} \bigvee_{a \in A}^{(\tilde{G})} (t \wedge a) = \bigvee_{a \in A}^{(\tilde{G})} \bigvee_{t \in T}^{(\tilde{G})} (t \wedge a) = \bigvee_{a \in A}^{(\tilde{G})} a \,,$$

since $a \in A \subseteq T$ implies $a = \bigvee_{t \in T}^{(\tilde{G})} (t \wedge a)$. Thus $A\uparrow$ is L-convergent in $\tilde{G}$, hence L-fundamental in $\tilde{G}$. By Thm. 8.14 it is L-fundamental in G too.

8.21. Theorem. *If G is conditionally saturated, then so is $\tilde{G}$.*

Proof. If $\mathbf{S}_i \uparrow \leqq \mathbf{T}$ in $\tilde{G}$ for all $i \in I$, then the set $A = \bigcup_{i \in I} S_i$ is obviously directed upwards and $A \subseteq T$. By the preceding lemma $A\uparrow$ is a pyramid; setting $S = A^-$ we have $\mathbf{S} = \sup^{(\tilde{G})} A = \bigvee_{i \in I}^{(\tilde{G})} \bigvee_{s \in S_i}^{(\tilde{G})} s = \bigvee_{i \in I}^{(\tilde{G})} \mathbf{S}_i$.

The construction of $\tilde{G}$ was made under the assumption that G is ν-complete. If G is not ν-complete we define $\tilde{G} = \widetilde{G^*}$. Combining Prop. 7.6 and Thm. 8.14 we see that the latter theorem is valid without the tacit assumption $G = G^*$. Theorem 8.21 now reads: If G is Archimedean, then $\tilde{G}$ is conditionally saturated. Every positive element of $\tilde{G}$ is a join of positive elements of G. It follows that every element of $\tilde{G}$ is the Limit of some net in G. Hence:

8.22. Proposition. *G is dense in $\tilde{G}$ relative to L-convergence of nets.*

8.23. Corollary. *G is L-complete, if and only if $G = \tilde{G}$.*

Thus $\tilde{G}$ can serve as a "minimal" L-complete extension of G, in the sense that it is generated by G with respect to L-convergence. In this connection notice that in the next theorem the assumption of Archimedity is essential (consider $G = M$, $E = J \circ M$).

8.24. Theorem. *If G is Archimedean and E is an L-complete Archimedean and regular extension of G, then $\tilde{G}$ is isomorphic with a regular l-subgroup of E containing G, under an isomorphism that maps each element of G onto itself.*

Proof. G^* can be regularly embedded in E by Thm. 6.9. If S is a normal pyramid in G^*, then $S\uparrow$ is L-fundamental in E, by Thm. 7.5, hence $\bigvee_{s \in S}^{(E)} s$ exists. Mapping $\mathrm{S} \to \bigvee_{s \in S}^{(E)} s$ and then identifying S with its image in E we get $\mathrm{S} = \bigvee_{s \in S}^{(E)} s$. This embedding can obviously be extended to non-positive elements of $\tilde{G}$.

Now let $\bigwedge_{i \in I}^{(\tilde{G})} \mathrm{S}_i = 0$ and $e \leqq \mathrm{S}_i$ for all $i \in I$ $(e \in E)$. Choosing an $i_0 \in I$ we have:

$$e = e \wedge \mathrm{S}_{i_0} = e \wedge \bigvee_{s \in S_{i_0}}^{(E)} s = \bigvee_{s \in S_{i_0}}^{(E)} (e \wedge s) \,. \tag{16}$$

However $e \wedge s \leqq \mathrm{S}_i \wedge s$ for all $i \in I$ and by 8.12 $\mathrm{S}_i \wedge s \in G^*$. Since $\bigwedge_{i \in I}^{(\tilde{G})} (\mathrm{S}_i \wedge s) = \bigwedge_{i \in I}^{(G^*)} (\mathrm{S}_i \wedge s) = 0$ and G^* is regular in E we infer $e \wedge s = 0$ for every $s \in S_{i_0}$ and by (16) $e \leqq 0$.

8.25. Theorem. *A sequence (x_n) in G is L-fundamental in G, if and only if it is o-convergent in $\tilde{G}$.*

In fact in $\tilde{G}$ an L-fundamental sequence is bounded (if (x_n) is L-fundamental, so is $x_1, x_1 \vee x_2, x_1 \vee x_2 \vee x_3, \ldots$ hence $\bigvee_{n=1}^{\infty (\tilde{G})} x_n$ exists) and therefore naturally fundamental; the theorem then follows from 6.10. Thus sequence L-convergence in G is the restriction to G of sequence o-convergence in $\tilde{G}$ and we can construct a sequential L-completion of G by repeated extensions in $\tilde{G}$ (see § 5); we arrive at a commutative l-group $G[\omega_1]$ which is regular over G as well as regular in $\tilde{G}$. It is also topologically invariant over G:

8.26. Proposition. *If H is an l-subgroup of $\tilde{G}$ containing G, then G is regular in H and H is regular in $\tilde{G}$. For (x_i) and x in G $\mathrm{Lim}^{(G)} x_i = x$, $\mathrm{Lim}^{(H)} x_i = x$ and $\mathrm{Lim}^{(\tilde{G})} x_i = x$ are equivalent.*

Proof. Suppose $\mathrm{S}_i \downarrow^{(H)} 0$, $\mathrm{S}_i \in H$. If $0 \leqq \mathrm{T} \leqq \mathrm{S}_i$ for all $i \in I$ $(\mathrm{T} \in \tilde{G})$ and if $t \in T$, then $t \leqq \mathrm{S}_i$ for all $i \in I$. Since $t \in G \subseteq H$ we infer $t = 0$. Thus $T = \{0\}$, $\mathrm{S}_i \downarrow^{(\tilde{G})} 0$.

That $\mathrm{Lim}^{(G)} x_i = x$ implies $\mathrm{Lim}^{(H)} x_i = x$ and $\mathrm{Lim}^{(H)} x_i = x$ implies $\mathrm{Lim}^{(\tilde{G})} x_i = x$ is proved much as the first half of Thm. 8.14.

8.27. Theorem. *If G is Archimedean and E is a regular extension of G which is also Archimedean and sequentially L-complete, then $G[\omega_1]$ is isomorphic with the E-o-closure of G in E which is regular in E, under an isomorphism that maps each element of G onto itself.*

The embeddability of $G[\omega_1]$ in E is a consequence of Thm. 7.5. The regularity of $G[\omega_1]$ in E is proved by the same argument that served to establish Thm. 6.16.

8.28. Theorem. *If G is the l-group of all bounded real functions on some set X, then $G[\omega_1]$ and $\tilde{G}$ are isomorphic with the l-group of all real functions on X.*

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