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# Totally Geodesic Spheres in Compact Symmetric Spaces*

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## § 1. Introduction

Let $M$ be a compact, irreducible Riemannian globally symmetric space, $I_0(M)$ the largest connected group of isometries of $M$. As customary, let the Riemannian structure of $M$ be that induced by the negative of the Killing form of the Lie algebra of $I_0(M)$. Let $\kappa$ be the maximum of the sectional curvatures of $M$ whose values are then restricted to the interval $[0, \kappa]$. By a theorem of E. CARTAN, the maximum dimensional totally geodesic submanifolds of $M$ of constant curvature 0 are all conjugate under $I_0(M)$. In this paper we shall prove an analogous statement for the maximal curvature $\kappa$.

**Theorem 1.1.** *The space $M$ contains totally geodesic submanifolds of constant curvature $\kappa$. Any two such submanifolds of the same dimension are conjugate under $I_0(M)$. The maximal dimension of such submanifolds is $1+m(\bar{\delta})$ where $m(\bar{\delta})$ is the multiplicity of the highest restricted root $\bar{\delta}$* (see § 2). *Also, $\kappa = \|\bar{\delta}\|^2$, where $\| \;\|$ denotes length.*

*Remark.* Except for the case when $M$ is a real projective space the submanifolds above of dimension $1+m(\bar{\delta})$ are actually spheres.

During the proof of Theorem 1.1 we shall obtain the following result:

**Theorem 1.2.** *Assume that the space $M$ above is simply connected. Then the closed geodesics in $M$ of minimal length are permuted transitively by $I_0(M)$. The minimum length is $2\pi/\|\bar{\delta}\|$.*

For Grassmann manifolds a result like Theorem 1.2 is proved and applied by ELÍASSON [3] in a study of closed geodesics on manifolds homeomorphic to Grassmann manifolds. Certain totally geodesic spheres in Grassmann manifolds have been studied by WOLF [6], [7]. See also RAUCH [8].

*Notation.* We shall use the customary notation $\boldsymbol{Z}$, $\boldsymbol{R}$, and $\boldsymbol{C}$ for the integers, real numbers and complex numbers, respectively. Lie groups are denoted by capital Roman letters and their Lie algebras by the corresponding lower case German letters. The adjoint representation of a Lie group (resp. Lie algebra) is denoted Ad (resp. ad). If $N$ is a manifold, $p$ a point in $N$ then $N_p$ denotes the tangent space to $N$ at $p$.

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## § 2. Root space decompositions

Let $\mathfrak{u}$ be a compact semisimple Lie algebra over $\boldsymbol{R}$, $\theta$ an involutive automorphism of $\mathfrak{u}$. Let $\mathfrak{u}^c$ be the complexification of $\mathfrak{u}$, $\mathfrak{g}$ the real form of $\mathfrak{u}^c$ corresponding to $(\mathfrak{u}, \theta)$, that is $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$, where $\mathfrak{k}=\{T\in\mathfrak{u}|\theta T=T\}$ and $\mathfrak{p}=\{iX|X\in\mathfrak{u},\ \theta X=-X\}$. Then $\mathfrak{u}=\mathfrak{k}+\mathfrak{p}_*$, where $\mathfrak{p}_*=i\mathfrak{p}$. Let $\mathfrak{a}\subset\mathfrak{p}$ be a maximal abelian subspace, put $\mathfrak{a}_*=i\mathfrak{a}$ and extend $\mathfrak{a}_*$ to a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{u}$. Then the complexification $\mathfrak{t}^c\subset\mathfrak{u}^c$ is a Cartan subalgebra of $\mathfrak{u}^c$. Let $\Delta$ denote the corresponding system of nonzero roots, $\Delta_\mathfrak{p}$ the set of $\alpha\in\Delta$ which do not vanish identically on $\mathfrak{a}^c$ (the complexification of $\mathfrak{a}$ in $\mathfrak{u}^c$). Let $\Sigma$ denote the set of restrictions of elements of $\Delta_\mathfrak{p}$ to $\mathfrak{a}^c$; its elements are called restricted roots. Let $\mathfrak{t}_\mathfrak{k}=\mathfrak{k}\cap\mathfrak{t}$, $\mathfrak{t}_{\mathfrak{k}_*}=i\mathfrak{t}_\mathfrak{k}$. Then all $\alpha\in\Delta$ take real values on $\mathfrak{a}+\mathfrak{t}_{\mathfrak{k}_*}$. Select compatible orderings in the dual spaces of $\mathfrak{a}+\mathfrak{t}_{\mathfrak{k}_*}$ and $\mathfrak{a}$ and let $\Delta^+$ and $\Sigma^+$ denote the set of positive elements in $\Delta$ and $\Sigma$, respectively. If $f$ is a function on $\mathfrak{t}^c$ its restriction to $\mathfrak{a}^c$ is denoted $\bar{f}$. For each $\lambda\in\Sigma$ the number of $\alpha\in\Delta_\mathfrak{p}$ such that $\lambda=\bar{\alpha}$ is called the multiplicity of $\lambda$ and is denoted by $m(\lambda)$.

For each linear form $\lambda$ on $\mathfrak{a}^c$ put

$$\mathfrak{k}_\lambda=\{T\in\mathfrak{k}|(\operatorname{ad}H)^2T=\lambda(H)^2T\quad\text{for all}\quad H\in\mathfrak{a}_*\}$$
$$\mathfrak{p}_\lambda=\{X\in\mathfrak{p}_*|(\operatorname{ad}H)^2X=\lambda(H)^2X\quad\text{for all}\quad H\in\mathfrak{a}_*\}.$$

Then $\mathfrak{k}_\lambda=\mathfrak{k}_{-\lambda}$, $\mathfrak{p}_\lambda=\mathfrak{p}_{-\lambda}$ and $\mathfrak{k}_o$ and $\mathfrak{p}_o$ are the centralizers of $\mathfrak{a}_*$ in $\mathfrak{k}$ and $\mathfrak{p}_*$, respectively. The endomorphisms $(\operatorname{ad}H)^2$ $(H\in\mathfrak{a}_*)$ commute and are symmetric with respect to the Killing form $B$ of $\mathfrak{u}^c$. But the eigenvalues of $(\operatorname{ad}H)^2$ are $0$ and $\lambda(H)^2$ $(\lambda\in\Sigma)$ (cf. [4], p. 248). It follows that

$$\mathfrak{k}_\lambda=\mathfrak{p}_\lambda=\{0\}\quad\text{if}\quad\lambda\notin\Sigma\cup\{0\}$$

and that we have the direct decompositions

$$\mathfrak{k}=\mathfrak{k}_o+\sum_{\lambda\in\Sigma^+}\mathfrak{k}_\lambda,\tag{1}$$

$$\mathfrak{p}_*=\mathfrak{p}_o+\sum_{\lambda\in\Sigma^+}\mathfrak{p}_\lambda.\tag{2}$$

For each $\alpha\in\Delta$ select $X_\alpha\neq0$ in $\mathfrak{u}^c$ such that $[H, X_\alpha]=\alpha(H)X_\alpha$ for all $H\in\mathfrak{t}^c$. Extend $\theta$ to an automorphism of $\mathfrak{u}^c$, also denoted $\theta$, and let

$$\mathfrak{u}^+(\alpha)=\boldsymbol{C}(X_\alpha+\theta X_\alpha),\ \mathfrak{u}^-(\alpha)=\boldsymbol{C}(X_\alpha-\theta X_\alpha).$$

Then we have for $\lambda\in\Sigma$ (cf. Lemma 3.6, Ch. VI in [4])

$$\mathfrak{k}_\lambda+i\mathfrak{k}_\lambda=\sum_{\bar{\alpha}=\lambda}\mathfrak{u}^+(\alpha),\quad\mathfrak{p}_\lambda+i\mathfrak{p}_\lambda=\sum_{\bar{\alpha}=\lambda}\mathfrak{u}^-(\alpha).\tag{3}$$

Since $\lambda$ is real on $\mathfrak{a}$, the two right hand sides in (3) are invariant under the conjugation of $\mathfrak{u}^c$ with respect to $\mathfrak{g}$, so

$$\mathfrak{k}_\lambda=\mathfrak{k}\cap\sum_{\bar{\alpha}=\lambda}\mathfrak{u}^+(\alpha),\quad\mathfrak{p}_\lambda=\mathfrak{p}_*\cap\sum_{\bar{\alpha}=\lambda}\mathfrak{u}^-(\alpha)\tag{4}$$

and

$$\mathfrak{k}_o=\mathfrak{t}_\mathfrak{k}+\mathfrak{k}\cap\sum_{\bar{\alpha}=0}\mathfrak{u}^+(\alpha),\quad\mathfrak{p}_o=\mathfrak{a}_*.\tag{5}$$

Using the well-known commutation relations between the $X_\alpha$ the following two lemmas are obtained from (3), (4), and (5) without difficulty.

**Lemma 2.1.** *Let* $\lambda, \mu \in \Sigma \cup \{0\}$. *Then*

$$[\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \subset \mathfrak{p}_{\lambda+\mu} + \mathfrak{p}_{\lambda-\mu},$$
$$[\mathfrak{k}_\lambda, \mathfrak{k}_\mu] \subset \mathfrak{k}_{\lambda+\mu} + \mathfrak{k}_{\lambda-\mu},$$
$$[\mathfrak{p}_\lambda, \mathfrak{p}_\mu] \subset \mathfrak{k}_{\lambda+\mu} + \mathfrak{k}_{\lambda-\mu};$$

For each linear function $\xi$ on $\mathfrak{t}^c$ let $H_\xi \in \mathfrak{t}^c$ be determined by $B(H_\xi, H) = \xi(H)$ $(H \in \mathfrak{t}^c)$ and if $\eta$ is a linear function on $\mathfrak{a}^c$ let $A_\eta \in \mathfrak{a}^c$ be determined by $B(A_\eta, H) = \eta(H)$ for all $H \in \mathfrak{a}^c$. Put $\langle \xi_1, \xi_2 \rangle = B(H_{\xi_1}, H_{\xi_2})$, $\langle \eta_1, \eta_2 \rangle = B(A_{\eta_1}, A_{\eta_2})$ and $|\xi| = |H_\xi| = B(H_\xi, H_\xi)^{1/2}$ if $H_\xi \in i\mathfrak{t}$, $\|\eta\| = \|A_\eta\| = B(A_\eta, A_\eta)^{1/2}$ if $A_\eta \in \mathfrak{a}$. If $\lambda \in \Sigma$ let $\mathfrak{a}_\lambda$ denote the subspace $\boldsymbol{R} i A_\lambda$ of $\mathfrak{a}_*$. If $\alpha \in \Delta$ then $H_\alpha - A_{\bar{\alpha}} \in \mathfrak{t}_{\mathfrak{k}_*}$.

**Lemma 2.2.** *Let* $\lambda \in \Sigma$. *Then*

$$[\mathfrak{k}_\lambda, \mathfrak{p}_\lambda] \subset \mathfrak{p}_{2\lambda} + \mathfrak{a}_\lambda .$$

Let $W_\mathfrak{t}$ and $W_\mathfrak{a}$ be the Weyl groups of $(\mathfrak{u}, \mathfrak{t})$ and $(\mathfrak{u}, \mathfrak{a}_*)$, respectively. They act on $\mathfrak{t}^c$ and $\mathfrak{a}^c$ as well as on their duals ($s H_\xi = H_{s\xi}$ etc.) The sets $\Delta$ and $\Sigma$ have been ordered. Let $C \subset \mathfrak{a} + \mathfrak{t}_{\mathfrak{k}_*}$ be the Weyl chamber where all $\alpha \in \Delta^+$ take positive values and let $C_\mathfrak{p} \subset \mathfrak{a}$ be the Weyl chamber where all $\lambda \in \Sigma^+$ take positive values. By the compatibility of the orderings, $C_\mathfrak{p} \subset \mathrm{Cl}(C)$, Cl denoting closure.

**Lemma 2.3.** *Let* $\delta \in \Delta^+$ *denote the highest root. Then* $H_\delta \in \mathrm{Cl}(C)$, $A_{\bar{\delta}} \in \mathrm{Cl}(C_\mathfrak{p})$ *and* $A_{\bar{\delta}} \neq 0$.

*Proof.* Since $W_\mathfrak{t}$ permutes $\Delta$ and $W_\mathfrak{a}$ permutes $\Sigma$, the two first statements are contained in the general statement that the closed Weyl chamber consists of those elements which dominate their transforms under the Weyl group. Finally $\delta \geqq \alpha$ for all $\alpha \in \Delta$ so, by the compatibility of the orderings, $\bar{\delta} \geqq \lambda$ for all $\lambda \in \Sigma$; hence $\bar{\delta} \neq 0$.

**Lemma 2.4.** *Let* $\alpha \in \Delta_\mathfrak{p}$. *Then*

$$\langle \alpha, \alpha \rangle = m \langle \bar{\alpha}, \bar{\alpha} \rangle, \quad \textit{where} \quad m = 1, 2 \text{ or } 4 .$$

This is well known (cf. ARAKI [1], p. 6—7) but we give a direct proof. The linear form $\alpha + \theta\alpha$ is 0 on $\mathfrak{a}$; it is not in $\Delta$ because then $[X_\alpha, \theta X_\alpha]$ would lie in $\mathfrak{k} + i\mathfrak{k}$, whereas clearly $[X_\alpha, \theta X_\alpha] \subset \mathfrak{p} + i\mathfrak{p}$. If $\alpha + \theta\alpha = 0$ then $m = 1$. If $\alpha + \theta\alpha \neq 0$ then $\alpha + \theta\alpha$ is not a root so the integer $2\langle \theta\alpha, \alpha \rangle / \langle \alpha, \alpha \rangle$ is $\geqq 0$, whence the angle between $H_\alpha$ and $H_{\theta\alpha}$ is 0, $\pi/3$ or $\pi/2$. But

$$A_{\bar{\alpha}} = \frac{1}{2}(H_\alpha - H_{\theta\alpha}) \tag{6}$$

so the lemma follows.

## § 3. Closed geodesics of minimal length. Proof of Theorem 1.2

Preserving the notation of § 2, we assume now that $\mathfrak{u}$ is simple. Let $U$ be the simply connected Lie group with Lie algebra $\mathfrak{u}$, "extend" $\theta$ to an automorphism of $U$ and let $K$ be the group of fixed points of $\theta$. As is well known, $K$ is connected. The negative of the Killing form $B$ of $\mathfrak{u}^c$ induces a Riemannian structure on $U/K$ and on $U$. Let $o$ denote the point $\{K\}$ in $U/K$ and let Exp denote the Exponential mapping of $\mathfrak{p}_*$ onto $U/K$.

Let $\gamma(t)$ $(-\infty < t < \infty)$ be a geodesic in a Riemannian manifold. The geodesic is called *closed* if there exists a number $L > 0$ such that $\gamma(t+L) = \gamma(t)$ for all $t$. The geodesic is said to be *simply closed* if in addition $\gamma(t_1) \neq \gamma(t_2)$ for $0 < t_1 < t_2 \leqq L$. If $|t|$ is the arc parameter, $L$ is then called the length of the simply closed geodesic.

If a geodesic in the symmetric space $U/K$ intersects itself then since it is an orbit of a one-parameter group of $U$ it is a simply closed geodesic (see e.g. [4], p. 355).

**Lemma 3.1.** *The shortest, periodic one-parameter subgroups in a simple, simply connected, compact Lie group $U$ have length $4\pi/|\delta|$ and they are all conjugate in $U$.*

Let $C_* = iC \subset \mathfrak{t}$ and consider the polyhedron $P_* \subset C_*$ given by

$$P_* = \{H \subset \mathfrak{t} \mid (2\pi i)^{-1}\alpha(H) > 0 (\alpha \in \Delta^+),\ (2\pi i)^{-1}\delta(H) < 1\}.$$

Since $U$ is simply connected, the unit lattice $\mathfrak{t}_e = \{H \in \mathfrak{t} \mid \exp H = e\}$ satisfies the well-known relation

$$(1) \qquad \mathfrak{t}_e \cap \mathrm{Cl}(P_*) = \{0\}$$

([4], p. 264). For $\alpha \in \Delta$ let $\mathfrak{t}_\alpha = \{H \in \mathfrak{t} \mid \alpha(H) \in 2\pi i \boldsymbol{Z}\}$ and let $T_\alpha$ be the centralizer of $\exp \mathfrak{t}_\alpha$ in $U$. In view of Lemma 4.5, Ch. VII in [4], $T_\alpha$ contains an element $u$ such that the restriction of $\mathrm{Ad}(u)$ to $\mathfrak{t}$ is the reflection $s_\alpha$ in the hyperplane $\alpha = 0$. It follows that

$$(2) \qquad \exp(2H(\alpha)) = e,$$

where $H(\alpha) = 2\pi i H_\alpha/|\alpha|^2$ is the projection of 0 onto the hyperplane $\alpha = 2\pi i$ in $\mathfrak{t}$. Each $H \in \mathfrak{t}_e$ satisfies $\beta(H) \in 2\pi i \boldsymbol{Z}$ for all $\beta \in \Delta$ so it is clear from (1), (2) and Lemma 2.3 that the one-parameter subgroup $t \to \exp(2tH(\delta))$ is periodic and has length $2|H(\delta)|$. Now let $t \to \exp tX$ be any periodic one-parameter subgroup of $U$ of length $\leqq 2|H(\delta)|$, the parameter being fixed such that $\exp t\ X \neq e$ for $0 < t < 1$ and $\exp X = e$. Then for some $u \in U$, $H_1 = \mathrm{Ad}(u)X$ lies in $\mathrm{Cl}(C_*)$ and since $\exp H_1 = e$, $\delta(H_1) = n 2\pi i$ where $n \geqq 0$ in $\boldsymbol{Z}$. Since $H_1 \neq 0$ we have $n \neq 0$. Also $n \neq 1$ by (1). But the point $2H(\delta)$ is the only point which minimizes the distance from 0 to the union of the hyperplanes $\delta = n 2\pi i (n \geqq 2)$ in $\mathfrak{t}$. Thus $H_1 = 2H(\delta)$ and the lemma is proved.

We can now prove Theorem 1.2 for the space $U/K$. First observe that if $H \in \mathfrak{a}_*$ then $\mathrm{Exp} H = o$ if and only if $\exp 2H = e$. Let $A(\bar{\alpha}) = \pi i A_{\bar{\alpha}}/\|\bar{\alpha}\|^2$, $(\alpha \in \Delta_\mathfrak{p})$, and let us verify the relation

$$(3) \qquad \mathrm{Exp}(2A(\bar{\alpha})) = o.$$

Let $|\alpha|^2 = m\|\bar{\alpha}\|^2$, so by Lemma 2.4, $m = 1, 2, 4$. In the first case, $H_\alpha = A_{\bar{\alpha}}$ and (3) reduces to (2). In the cases $m = 2, 4$ we use $iH_\alpha - iA_{\bar{\alpha}} \in \mathfrak{k}$, so by (2),

$$\exp(2\pi i A_{\bar{\alpha}}/\|\bar{\alpha}\|^2) \in \exp(2\pi i H_\alpha/\|\bar{\alpha}\|^2) K = K,$$

proving (3). In particular $4A(\bar{\delta}) \in \mathfrak{t}_e$, and since $\delta(4A(\delta)) = 4\pi i$ and $4A(\bar{\delta}) \in \mathrm{Cl}(C_*)$ we have by (1), $4tA(\bar{\delta}) \notin \mathfrak{t}_e$ for $0 < t < 1$. Consequently the geodesic

$$(4) \qquad t \to \mathrm{Exp}(2tA(\bar{\delta}))$$

is simply closed and has length $2\|A(\bar{\delta})\| = 2\pi/\|\bar{\delta}\|$.

Now let $t \to \mathrm{Exp}\, tX$ be any simply closed geodesic of length $\leqq 2\pi/\|\bar{\delta}\|$ such that $\mathrm{Exp}\, X = o$, $\mathrm{Exp}\, t_1 X \neq \mathrm{Exp}\, t_2 X$ for $0 < t_1 < t_2 \leqq 1$. Select $k \in K$ such that $H = \mathrm{Ad}_U(k) X \in \mathrm{Cl}(C_{\mathfrak{p}_*})$ where $C_{\mathfrak{p}_*} = iC_{\mathfrak{p}}$. Then $2H \in \mathfrak{t}_e$ so

$$\delta(H) = m\pi i,$$

where $m \geqq 0$ in $\boldsymbol{Z}$. But $H \neq 0$ so $m \neq 0$ and $m \neq 1$ because of (1). But the point $2A(\bar{\delta})$ is the only point which minimizes the distance from 0 to the union of the hyperplanes $\bar{\delta} = m\pi i (m \geqq 2)$ in $\mathfrak{a}_*$. Thus $H = 2A(\bar{\delta})$ and Theorem 1.2 is proved for the space $U/K$.

In order to complete the proof of Theorem 1.2 we must consider the group $U$ as a symmetric space $U \times U/U^*$ ($U^* =$ diagonal in $U \times U$) with Riemannian structure $Q^*$ defined by the Killing form $B^*$ of $\mathfrak{u} \times \mathfrak{u}$, the identification being made via the mapping $\tau: (u_1, u_2) U^* \to u_1 u_2^{-1}$ $(u_1, u_2 \in U)$. The tangent space to $U \times U/U^*$ at $o$ is according to the usual identification given by the orthogonal complement of the Lie algebra of $U^*$ in $\mathfrak{u} \times \mathfrak{u}$, hence equals the space $\{(X, -X) | X \in \mathfrak{u}\}$. Since

$$(5) \qquad B^*((X, -X), (Y, -Y)) = 2B(X, Y) \qquad X, Y \in \mathfrak{u}$$

and since $d\tau_o(X, -X) = 2X$ it is clear that for each tangent vector $Z$ to $U \times U/U^*$,

$$(6) \qquad 2Q^*(Z, Z) = Q(d\tau(Z), d\tau(Z)).$$

Here $Q$ is the Riemannian structure on $U$ defined by $B$. The space $\{(H, -H) | H \in \mathfrak{t}\}$ is a maximal abelian subspace of the tangent space above and is contained in the Cartan subalgebra $\mathfrak{t} \times \mathfrak{t}$ of $\mathfrak{u} \times \mathfrak{u}$. The roots of $(\mathfrak{u} \times \mathfrak{u})^c$ with respect to $(\mathfrak{t} \times \mathfrak{t})^c$ are given by the linear forms

$$(H_1, H_2) \to \alpha(H_1), \; (H_1, H_2) \to \alpha(H_2) \quad (H_1, H_2 \in \mathfrak{t}^c)$$

as $\alpha$ runs through the roots of $\mathfrak{u}^c$ with respect to $\mathfrak{t}^c$. The highest restricted root, say $\tilde{\delta}$, is given by $\tilde{\delta}(H, -H) = \delta(H)$ $(H \in \mathfrak{t})$ and by (5) we find $A_{\tilde{\delta}} = (\frac{1}{2} H_\delta, -\frac{1}{2} H_\delta)$ and $2\|\tilde{\delta}\|^2 = |\delta|^2$. In the Riemannian structure $Q^*$ the one-parameter subgroups from Lemma 3.1 have length $2\pi\sqrt{2}/|\delta|$ which equals $2\pi/\|\tilde{\delta}\|$. This concludes the proof of Theorem 1.2.

We note some simple consequences of the proof. First we have from (2) and (3),

**Corollary 3.2.** *Let $\alpha \in \Delta$. Then $|\alpha| \leqq |\delta|$ and $\|\bar{\alpha}\| \leqq \|\bar{\delta}\|$.*

**Corollary 3.3.** *Suppose $\alpha \in \Delta$ and $\bar{\alpha} \neq \pm \bar{\delta}$. Then*

$$2 \frac{\langle \bar{\alpha}, \bar{\delta} \rangle}{\langle \bar{\delta}, \bar{\delta} \rangle} = 0 \quad \text{or} \quad \pm 1$$

*so the value is 0 if $2\bar{\alpha} \in \Sigma$.*

In fact, $\delta(A(\bar{\delta}))=\pi i$ and by (3), $\alpha(A(\bar{\delta}))=\frac{1}{2}\pi i n$ where $n\in\mathbf{Z}$. But $\delta-\alpha$ is a positive integral linear combination of the simple roots in $\Delta^+$, so $n=0$, $\pm 1$, $\pm 2$. But if $n=\pm 2$ we deduce from Cor. 3.2 that $\bar{\alpha}=\pm\bar{\delta}$.

Let $M$ be as in Theorem 1.2. For each $p\in M$ let $A_p$ denote the set of midpoints of closed geodesics of minimal length passing through $p$. We call $A_p$ the *midpoint locus* (associated with $p$).

**Corollary 3.4.** *For each $p\in M$ the midpoint locus*[1] *$A_p$ is a totally geodesic submanifold of $M$ and is an orbit of the isotropy subgroup of $I_o(M)$ at $p$.*

In fact let $q\in A_p$. Then the geodesic symmetry of $M$ with respect to $q$ leaves $p$ fixed and consequently maps $A_p$ into itself. In addition to Theorem 1.2 we now need the following more general lemma whose proof is contained in the proof of Prop. 5.1 in [5].

**Lemma 3.5.** *Let $Q$ be a Riemannian globally symmetric space, $N$ a submanifold of $Q$ such that for each $n\in N$, $N$ is invariant under the geodesic symmetry of $Q$ with respect to $n$. Then $N$, with the Riemannian structure induced by $Q$, is totally geodesic in $Q$.*

We conclude this section with a further description of the midpoint locus and the space of closed geodesics of minimal length.

For $\varepsilon=0, 1/2, 1$, put

$$\mathfrak{k}(\varepsilon)=\sum_{\langle\lambda,\bar{\delta}\rangle=\varepsilon\langle\bar{\delta},\bar{\delta}\rangle}\mathfrak{k}_\lambda;\ \mathfrak{p}(\varepsilon)=\sum_{\langle\lambda,\bar{\delta}\rangle=\varepsilon\langle\bar{\delta},\bar{\delta}\rangle}\mathfrak{p}_\lambda .$$

Then by Cor. 3.3, $\mathfrak{k}(1)=\mathfrak{k}_{\bar{\delta}}$, $\mathfrak{p}(1)=\mathfrak{p}_{\bar{\delta}}$ and

$$\mathfrak{k}=\mathfrak{k}_o+\mathfrak{k}(0)+\mathfrak{k}\left(\frac{1}{2}\right)+\mathfrak{k}(1),\quad \mathfrak{p}_*=\mathfrak{p}_o+\mathfrak{p}(0)+\mathfrak{p}\left(\frac{1}{2}\right)+\mathfrak{p}(1).$$

**Proposition 3.6.** *Let $S$ and $S'$, respectively, denote the centralizers of* $\exp 2A(\bar{\delta})$ *and* $A(\bar{\delta})$ *in $K$. Their respective Lie algebras are given by*

$$\mathfrak{s}=\mathfrak{k}_o+\mathfrak{k}(0)+\mathfrak{k}_{\bar{\delta}},\quad \mathfrak{s}'=\mathfrak{k}_o+\mathfrak{k}(0). \tag{7}$$

*In the space $U/K$ the midpoint locus $A_o$ can be identified with $K/S$ and the space of closed geodesics of minimal length is identified with $U/S'$. Let $o'=\mathrm{Exp}A(\bar{\delta})$; then*

$$A_{o'}=\mathrm{Exp}\,\mathfrak{p}\left(\frac{1}{2}\right).$$

*Proof.* The relations (7) are immediate. Using $\theta$ one finds that an element $k\in K$ commutes with $\exp 2A(\bar{\delta})$ if and only if $k$ leaves the point $\mathrm{Exp}A(\bar{\delta})$ fixed. Hence $A_o=K/S$; the identification of $U/S'$ is also immediate. Finally

$$A_{o'}=\exp(-A(\bar{\delta}))K\exp A(\bar{\delta})\cdot o .$$

If $T\in\mathfrak{k}$ it is clear that

$$\mathrm{Ad}_U(\exp(-A(\bar{\delta})))T\equiv-\sinh(\mathrm{ad}(A(\bar{\delta})))T \pmod{\mathfrak{k}}$$

[1] This leads to a natural definition of the Radon transform for $M$, generalizing the one given in [5].

so the curve

$$t \to \exp(-A(\bar{\delta}))\exp tT\exp(A(\bar{\delta}))\cdot o$$

has tangent vector $-\sinh(\operatorname{ad}A(\bar{\delta}))T$ at $t=0$. But

$$\sinh(\operatorname{ad}A(\bar{\delta}))\mathfrak{k} = \mathfrak{p}\left(\frac{1}{2}\right).$$

which finishes the proof.

## § 4. Totally geodesic spheres. Proof of Theorem 1.1

In order to compute the maximal curvature $\kappa$ for the space $U/K$ from § 3 it suffices to consider plane sections $\subset \mathfrak{p}_*$ spanned by orthonormal vectors $H$, $X$ where $H \in C_{\mathfrak{p}_*}$. The corresponding curvature is then given by a formula of Cartan (cf. [4], p. 205) as

(1) $$B((\operatorname{ad}H)^2 X, X).$$

Decomposing $X$ by (2) § 2,

$$X = X_o + \sum_{\lambda \in \Sigma^+} X_\lambda$$

we have

$$B((\operatorname{ad}H)^2 X, X) = \sum_{\lambda \in \Sigma^+} \lambda(H)^2 B(X_\lambda, X_\lambda) \leqq \delta(H)^2$$

so $\kappa \leqq \|\bar{\delta}\|^2$.

Now since $2\bar{\delta}$ is not a restricted root it is easily seen from Lemmas 2.1 and 2.2 that the subspace $\mathfrak{a}_{\bar{\delta}} + \mathfrak{p}_{\bar{\delta}}$ of $\mathfrak{p}_*$ is a Lie triple system. Let $S$ and $\mathfrak{s}$ be as in Prop. 3.6. If $\lambda \in \Sigma^+$ such that $\langle \lambda, \bar{\delta} \rangle = 0$ then by Cor. 3.3 neither $\bar{\delta} + \lambda$ nor $\bar{\delta} - \lambda$ is a restricted root. Hence by (7) § 3 and Lemmas 2.1 and 2.2,

(2) $$[\mathfrak{s}, \mathfrak{a}_{\bar{\delta}}] = \mathfrak{p}_{\bar{\delta}}, \quad [\mathfrak{s}, \mathfrak{p}_{\bar{\delta}}] = \mathfrak{a}_{\bar{\delta}} + \mathfrak{p}_{\bar{\delta}}.$$

Since $\mathfrak{a}_{\bar{\delta}}$ is a maximal abelian subspace of $\mathfrak{a}_{\bar{\delta}} + \mathfrak{p}_{\bar{\delta}}$, the globally symmetric space $M_\delta = \operatorname{Exp}(\mathfrak{a}_{\bar{\delta}} + \mathfrak{p}_{\bar{\delta}})$ has rank one. If $H$, $X$ are orthonormal vectors, $H \in \mathfrak{a}_{\bar{\delta}}$, $X \in \mathfrak{p}_{\bar{\delta}}$, the corresponding sectional curvature of $M_\delta$ is by (1) equal to $\|\bar{\delta}\|^2$. It follows that $\kappa = \|\bar{\delta}\|^2$ and that $M_\delta$ has constant curvature. Let $S_o$ denote the identity component of $S$. Since $[\mathfrak{s}, \mathfrak{a}_{\bar{\delta}}] = \mathfrak{p}_{\bar{\delta}}$ it is clear that the orbit $\operatorname{Ad}_U(S_o)\cdot A(\bar{\delta})$ is the sphere in $\mathfrak{a}_{\bar{\delta}} + \mathfrak{p}_{\bar{\delta}}$ with center 0, passing through $A(\bar{\delta})$. Since all $s \in S_o$ leave $\operatorname{Exp}A(\bar{\delta})$ fixed all geodesics in $M_\delta$ through $o$ pass through $o' = \operatorname{Exp}A(\bar{\delta})$. Hence $M_\delta - o'$ is the diffeomorphic image of a ball ([4], p. 356) so $M_\delta$ is simply connected, thus a sphere.

Now since the geodesic symmetry $s_{0'}$ of $U/K$ leaves $o$ fixed it is clear that any geodesic segment of minimum length joining $o$ and $o'$ is a part of a closed geodesic of length $\leqq 2\pi/\|\bar{\delta}\|$. By Theorem 1.2 it is a simply closed geodesic of length $2\pi/\|\bar{\delta}\|$. The closed geodesics of this length, starting at $o$, and passing through $o'$ are permuted transitively by $S$, hence form finitely many distinct spheres $\Sigma_i$ of dimension $1 + m(\bar{\delta})$, namely the images of $M_\delta$ under $S$. Let $\sigma_i$ denote the unit sphere in the tangent space $(\Sigma_i)_o$. Each $\sigma_i$ is an orbit of

$\mathrm{Ad}_U(S_o)$ so any two different $\sigma_i$ are disjoint. Now let $\Sigma$ be any totally geodesic sphere in $U/K$ of curvature $\|\bar{\delta}\|^2$. Using a motion $u \in U$ we may assume that $\Sigma$ passes through $o$ and $o'$. Then the unit sphere $\sigma$ in the tangent space $(\Sigma)_o$ is contained in the union of the $\sigma_i$. But $\sigma$ is connected so it is contained in a single $\sigma_i$. Hence there exists an element $s \in S$ such that $s \cdot \Sigma \subset M_\delta$.

Next we observe that

$$\mathfrak{s} \supset [\mathfrak{a}_{\bar{\delta}} + \mathfrak{p}_{\bar{\delta}},\ \mathfrak{a}_{\bar{\delta}} + \mathfrak{p}_{\bar{\delta}}]\,.$$

The right hand side of this relation is a Lie algebra $\mathfrak{k}'$ contained in $\mathfrak{k}$ and $\mathrm{ad}_{\mathfrak{u}}(\mathfrak{k}')$ restricted to $\mathfrak{a}_{\bar{\delta}} + \mathfrak{p}_{\bar{\delta}}$ is the Lie algebra of the special orthogonal group of the tangent space $(M_\delta)_o$. It follows that this group is contained in the restriction of $\mathrm{Ad}_U(S_o)$ to $(M_\delta)_o$. Thus any two totally geodesic spheres of the same dimension, contained in $M_\delta$, are conjugate under a member of $U$; to finish the proof of Theorem 1.1 for the space $U/K$ it remains to verify that any totally geodesic submanifold $N$ of $U/K$ of constant curvature $\|\bar{\delta}\|^2$ is a sphere. But if $N$ were not a sphere it is clear, passing to the universal covering of $N$, that $N$ would contain a simply closed geodesic of length $<2\pi/\|\bar{\delta}\|^2$ which is impossible.

If $M$ is any compact Riemannian globally symmetric space such that $I_o(M)$ is simple then Theorem 1.1 can be applied to the universal covering space of $M$ and the theorem follows for $M$.

Now let us establish the remark following Theorem 1.1 in the case when $I_o(M)$ is simple. Let $Z$ denote the center of $U$ and put

$$K_Z = \{u \mid u^{-1}\theta(u) \in Z\}\,.$$

Then $U/K_Z$ is a Riemannian globally symmetric space and any such space, associated with $(\mathfrak{u}, \theta)$, actually covers $U/K_Z$ ([4], Theorem 8.1, Ch. VII). It is appropriate to call $U/K_Z$ the *adjoint space* of the orthogonal symmetric Lie algebra $(\mathfrak{u}, \theta)$ because if its construction (which does not require $\mathfrak{u}$ to be simple) is carried out for the group case $\mathfrak{u} = \mathfrak{v} \times \mathfrak{v}$, $\theta$ interchanging the two factors, then one obtains the adjoint group of $\mathfrak{v}$. Under the covering map $\varphi: uK \to uK_Z$ of $U/K$ onto $U/K_Z$ the sphere $\mathrm{Exp}(\mathfrak{a}_{\bar{\delta}} + \mathfrak{p}_{\bar{\delta}})$ is mapped onto a sphere if and only if $\varphi(\mathrm{Exp}\,A(\bar{\delta})) \neq o$. But $\varphi(\mathrm{Exp}\,A(\bar{\delta})) = o$ if and only if $\exp A(\bar{\delta}) \in K_Z$, which is equivalent to $\exp 2A(\bar{\delta}) \in Z$, which in turn implies $K = S$. But if $K = S$ then by Prop. 3.6, $\mathfrak{p}\left(\frac{1}{2}\right) = 0$ so

$$\mathfrak{p}_* = \mathfrak{a}_* = \mathfrak{p}_{\bar{\delta}} + \sum_{\langle \lambda, \bar{\delta} \rangle = 0} \mathfrak{p}_\lambda\,.$$

Thus $\Sigma^+$ has the property that all $\lambda \in \Sigma^+ - \{\bar{\delta}\}$ are perpendicular to $\bar{\delta}$. This means $\Sigma^+ = \{\bar{\delta}\}$ so $U/K$ is a sphere and consequently, $U/K_Z$ is a real projective space.

We have now established Theorem 1.1 and the subsequent remark for the case when $I_o(M)$ is simple. The remaining case when $M$ is a group (cf. [4], Prop. 1.2, p. 327) is handled quite similarly. We only indicate the necessary changes.

Suppose $U$ has Riemannian structure given by $-B$. Then if $X$, $Y$ are orthonormal vectors in $\mathfrak{u}$ the corresponding sectional curvature of $U$ is

$$\frac{1}{4}B((\operatorname{ad}X)^2 Y, Y),$$

and the maximum sectional curvature is readily found to be $\frac{1}{4}|\delta|^2$. However, with the conventions in Theorem 1.1 we must view $U$ as $U \times U/U^*$, whereby the Riemannian structure is multiplied by $\frac{1}{2}$ (cf. (6) § 3). Accordingly all sectional curvatures are multiplied by 2 so $\kappa = 2|\delta|^2/4$ which, as we have seen, equals $\|\tilde{\delta}\|^2$. Since the multiplicity $m(\tilde{\delta})$ now equals 2 the totally geodesic spheres in $U$ of curvature $\|\tilde{\delta}\|^2$ have dimension $\leqq 3$. In the adjoint group $U/Z$ the totally geodesic submanifolds of constant curvature $\|\tilde{\delta}\|^2$ and dimension 3 are spheres unless the point $\exp H(\delta)$ lies in $Z$. As is well known, $\exp H(\delta) \in Z$ if and only if $\alpha(H(\delta)) \in 2\pi i \boldsymbol{Z}$ for all $\alpha \in \Delta$ (see e.g. [4], p. 268). But in view of Cor. 3.2, $\alpha(H(\delta)) \in 2\pi i \boldsymbol{Z}$ implies that all roots except $\pm\delta$ are orthogonal to $\delta$, which implies $U = \boldsymbol{SU}(2)$ so $U/Z$ is a real projective space.

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