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Parallelizability of Proper Actions, Global K-slices and Maximal Compact Subgroups

Herbert Abels

0.1. A locally compact topological space is by definition hausdorff. A pair (G, K) consisting of (1)a locally compact topological group G having a compact group of connected components and (2) a maximal compact subgroup K of G is called a *(G, K)-pair*. The results of the appendix (Theorem A.5, Corollary A.6) on (G, K)-pairs generalizing known results on Lie groups may be of independent interest.

Main Theorem. *Suppose we have a (G, K)-pair. Let X be a proper G-space* (Definition see 1.6). *If the orbit space $G\backslash X$ is paracompact then there is a G-mapping $f: X \to G/K$, where the action of G on G/K is induced by left translations.*

In the terminology of Palais' [32] a subset S of a proper G-space X is called a *global K-slice*, if there is a G-mapping $f: X \to G/K$ such that $S = f^{-1}(K)$. So the main theorem states the existence of a global K-slice.

The G-mapping $f: X \to G/K$ is actually the projection of a trivial fibre bundle with fibre $S := f^{-1}(K)$ and structure group K (1.2). Since the base space G/K is homeomorphic to a euclidean space $\mathbb{R}^n$, X is homeomorphic to $\mathbb{R}^n \times S$. We call n the *non-compact dimension of G* $n = nc - \dim(G)$, since n depends only on G.

There are two refinements of the main theorem: Suppose we have a (G, K)-pair. Then the problem of finding all G-actions on a topological space X admitting a global K-slice is equivalent to the two problems (1) of finding all subspaces S of X such that $\mathbb{R}^n \times S$ is homeomorphic to X and (2) to find all K-actions on such S (2.2).

The other refinement is the following: The K-space X is K-homeomorphic to $T \times S$, where T is a continuous K-module obtained as follows: Restrict the adjoint representation of G on the Lie algebra LG of G to K. Then T is the quotient K-module LG/LK. All these notions make sense not only for connected Lie groups but also for (G, K)-pairs (see Appendix).

As an application of the main theorem we show how it generalizes known results on ends of proper G-spaces for connected G (2.7f). We also see that there is a reasonable notion of non-compact dimension for arbitrary locally compact topological groups.

All results also hold in the differentiable category (differentiable always means C^∞-differentiable), e.g.:

Main Theorem (Differentiable Version). *Suppose G is a Lie group having a finite number of connected components. Let K be a maximal compact subgroup of G. Let X be a paracompact differentiable manifold and let G act differentiably and properly on X. Then there is a differentiable G-mapping $f : X \to G/K$.*

In particular S is a closed differentiable submanifold of codimension n.

0.2. Here are examples of locally compact topological groups G and proper G-spaces:

Example 1. Let X be a locally compact connected metric space. Let G be the group of all isometries of X endowed with the compact-open topology ([10]). More generally: Let X be a connected uniform space, fix a base $\mathfrak{B}$ of the uniformity and let G be the group of all homeomorphisms of X leaving every $U \in \mathfrak{B}$ fixed, i.e. for every $U \in \mathfrak{B}$ and every $g \in G$ we have: $(x, y) \in U$ implies $(g \cdot x, g \cdot y) \in U$. Endow G with the compact-open topology. Then X is a proper G-space ([3, Theorem 7 and Corollary, p. 606]. The fact that the action is proper is not stated there but follows as on p. 605).

Example 2. The natural action of the Lie group of all bijective (= differentiable) isometries of a Riemannian manifold ([19, IV, Theorem 2.5], [29, Chapter I, Theorem 4.7]).

Conversely: If a locally compact topological group G acts properly on a paracompact differentiable manifold X, there is a Riemannian metric ds^2 on X such that every $g \in G$ acts isometrically (cf. [30, p.9, Theorem 2]). The continuous homomorphism from G into the Lie group $I(X, ds^2)$ of all isometries of (X, ds^2) is injective iff the action of G on X is effective, i.e. $gx = x$ for every $x \in X$ implies $g = 1$. The *degree of symmetry* $N(X)$ of the paracompact differentiable manifold X is the maximum of the dimensions of $I(X, ds^2)$ for all possible Riemannian metrics ds^2 on X (cf. [25]). So the degree of symmetry $N(X)$ is the maximum of the dimensions of all Lie groups G acting properly and effectively on X. For similar statements concerning Example 1 s. [1, 30, 32].

Example 3. The locally compact topological group G acts properly upon itself by left translations, more generally on a coset space G/L where L is a compact subgroup.

Example 4. Let X be a completely regular hausdorff space. If X is the total space of a locally trivial principal G-fibre bundle, then G acts properly on X if the base space is regular and G is locally compact. If conversely X is a proper G-space and G acts freely on X, then $X \to G\backslash X$ is a locally trivial principal G-bundle ([32, Proposition 1.2.5, Theorem 4.1], [33, Théorème 1]).

Example 5. Let $(\mathbb{R}, X, t)$ be a dispersive dynamical system on a completely regular hausdorff space X. Then $X \to \mathbb{R}\backslash X$ is a locally trivial principal $\mathbb{R}$-bundle. So X is a proper $\mathbb{R}$-space iff $\mathbb{R}\backslash X$ is regular (cf. [18]).

If H is a closed subgroup of G and if X is a proper G-space, then the action restricted to H is proper.

The hypothesis of the main theorem that $G\backslash X$ be paracompact is unpleasant because hard to check. Let G be a locally compact topological group such that the group of connected components of G is compact. If X is a proper G-space, then $G\backslash X$ is paracompact in the following cases:

Case 1. X is locally Lindelöf and paracompact (proof as in [18, Corollary 14]: Let G_1 be the connected component of the neutral element in G). Note that $G_1\backslash X \to G\backslash X$ is a continuous closed surjective mapping and the inverse image of every point is compact. So $G_1\backslash X$ is paracompact iff $G\backslash X$ is paracompact [39, Problem 20G].

In particular:

Case 2. X is locally compact and paracompact.

More particularly:

Case 3. X is locally compact and metrizable.

Or

Case 4. X is a locally compact topological group and thus paracompact.

Hâjek [18] made a variation of the following

Conjecture. Let G be a connected locally compact topological group acting properly on a paracompact hausdorff space X. Then $G\backslash X$ is paracompact.

If this conjecture were true the "note" in Case 1 would enable us to replace the hypothesis "$G\backslash X$ paracompact" in the main theorem by the hypothesis "X paracompact". It is easy to see from the main theorem – or by elementary considerations – that under the hypotheses of the main theorem X is paracompact.

0.3. As far as I know only the following special cases of the main theorem are known. Example 4: "Reduction of the structure group of a principal G-bundle to the maximal compact subgroup" [36, 20]. Example 5: "Parallelization of dynamical systems" has been intensively studied: [4, 5, 6, 18, 31] and literature cited in [6, p. 55]. I thank Strantzalos for drawing my attention to these studies which were the starting point of the present paper (cf. [37]). Cf. also [17]: $K = \{1\}$, [28]: K normal, [9, Theorem 3.1].

0.4. The contents of the paper are: § 1 contains the proof of the main theorem. Some lemmas on the way can also be interpreted as corollaries of the main theorem, e.g. 1.3. § 2 contains the refinements and applications mentioned above. In order to keep the paper readable also for those interested in Lie groups only, the difficulties arising from considering non-Lie groups are dealt with in the appendix: We need a theorem on maximal compact subgroups for locally compact topological groups having a compact group of connected components analogous to the corresponding theorem on Lie groups. Once this theorem is established (A.5 and A.6) there are no extra difficulties arising from considering non-Lie groups.

Detailed proofs are given for the topological category, the differentiable case is treated only if it differs.

The ingredients of the proof of the main theorem are (1) Palais' theorem on the existence of local slices [32], (2) a study of the K-space G/K (Appendix A.5 and A.6).

§ 1

1.1. A *G-space* is a triple (G, X, φ) consisting of a topological group G, a topological space X and an action of G on X, i.e. a continuous mapping $\varphi: G\times X \to X$ such that $\varphi(g, \varphi(h, x)) = \varphi(g\cdot h, x)$ and $\varphi(1, x) = x$ for every $g, h \in G$ and $x \in X$ and 1 the neutral element of the group G. Since we usually consider only one action on a space X, we just speak of the G-space X and write $g\cdot x$ instead of $\varphi(g, x)$. A *G-mapping* of a G-space X into a G-space Y is a continuous mapping $f: X \to Y$ such that $f(g\cdot x) = g\cdot f(x)$ for every $x\in X$ and every $g \in G$. A G-mapping f is called a *G-homeomorphism* or a *G-isomorphism* if f is a homeomorphism. A subset A of a G-space is called *G-stable* if $g\cdot A = A$ for every $g \in G$.

Let X be a G-space. A G-orbit is a subset of X of the form $G\cdot x = \{g\cdot x; g\in G\}$. The *orbit space* $G\backslash X$ is the set of all G-orbits of X endowed with the finest topology making the natural mapping $X \to G\backslash X$ continuous.

If H is a closed subgroup of the topological group G, then G/H is the orbit space of G under the H-action $\varphi(h, g) = g\cdot h^{-1}$, i.e. the space of cosets $\{g\cdot H;\ g\in G\}$. We consider G/H has a G-space under the action $G\times G/H \to G/H$, $(g, x\cdot H) \mapsto g\cdot x\cdot H$.

1.2. Theorem. *Suppose we have a (G, K)-pair. Let $f: X\to G/H$ be a G-mapping from a G-space X to G/K. Then f is the projection in the trivial fibre bundle $f: X \to G/K$ with fibre $S := f^{-1}(K)$, base G/K, structure group K and associated trivial principal K-fibre bundle $G\to G/K$.*

For a more general version of this theorem see [32, 2.1.2]. It appears most convenient to state the properties of f in the language of fibre bundle theory.

Proof. The natural mapping $\pi: G\to G/K$ has a global section $t: G/K \to G$ (see Appendix Theorem 1.5). Let K act on $G\times S$ by $k\cdot(g, s) := (gk^{-1}, k\cdot s)$. Let $G\times_K S$ be the corresponding orbit space and denote the orbit of (g, s) by $[g, s]$. Let G act on $G \times_K S$ by $g \cdot [x, s] := [g \cdot x, s]$. The mapping $G \times_K S \to X$, $[g, s] \mapsto g \cdot s$ is a G-mapping and in fact a G-homeomorphism: The mapping $X \to G \times_K S, x \mapsto [t \circ f(x), (t \circ f(x))^{-1} \cdot x]$ is an inverse G-mapping.

The proof has the

1.3. Corollary. *$G \times_K S \to X$, $[g, s] \mapsto g \cdot s$, is a G-homeomorphism.*

1.4. Corollary. *Let Y be another G-space. Then the restriction mapping $\{G\text{-mappings } X \to Y\} \to \{K\text{-mappings } S \to Y\}$*

$$u \mapsto u \mid S$$

is a bijection.

This is obvious for $X = G \times_K S$. Then Corollary 1.3 implies Corollary 1.4.

By Corollary A.6 of the appendix there is a K-homeomorphism, say $\lambda : G/K \to T$, where T is a continuous K-module of dimension $n = nc - \dim(G)$. Therefore we can apply Weyl's trick: Let Y be a K-space and let $f : Y \to G/K$ be a continuous mapping. Then the mapping $\bar{f} : Y \to G/H$ defined by $\bar{f}(y) := \lambda^{-1} \int \lambda(k^{-1} \cdot f(k \cdot y))\, dk$ is a K-mapping. Here $\int \ldots dk$ denotes normalized Haar integral on K. If f is a K-mapping, we have $\bar{f} = f$. To prove the main theorem we need the following extension lemma:

1.5. Lemma. *Suppose we have a (G, K)-pair and a G-space X. Let U_i, $i = 1, 2$, be G-stable subsets of X and let $f_i : U_i \to G/K$ be G-mappings. If $S_2 := f_2^{-1}(K)$ is a normal space and if $U_1 \cap U_2$ is closed in $U_1 \cup U_2$, then there is a G-mapping $f : U_1 \cup U_2 \to G/K$ such that $f \mid U_1 = f_1$.*

Proof. Since $U_1 \cap U_2$ is closed in $U_1 \cup U_2$, a fortiori $S_2 \cap U_1$ is closed in S_2. Since G/K is homeomorphic to a euclidean space of finite dimension, by Tietze's extension theorem there is a continuous mapping $F : S_2 \to G/K$ such that $F \mid U_1 \cap S_2 = f_1 \mid U_1 \cap S_2$ ([39, Theorem 15.8]). By Weyl's trick there is a K-mapping $\bar{F} : S_2 \to G/K$ such that $\bar{F} \mid U_1 \cap S_2 = \bar{f}_1 \mid U_1 \cap S_2 = f_1 \mid U_1 \cap S_2$, since f_1 is a K-mapping. By Corollary 1.4 there is exactly one G-mapping $f' : U_2 \to G/K$ such that $f' \mid S_2 = \bar{F}$. Again by 1.4 we have $f' \mid U_1 \cap U_2 = f_1 \mid U_1 \cap U_2$. The mapping $f : U_1 \cup U_2 \to G/K$ such that $f \mid U_2 = f'$ and $f \mid U_1 = f_1$ is the desired G-mapping.

1.6. Following Palais [32, Definition 1.2.2] we call a *G-space X proper* if (1) G is a locally compact topological group, (2) X is a completely regular hausdorff space and (3) every point of X has a nhood V such that for every point of X there is a nhood U with the property that $\{g \in G;\, g \cdot U \cap V \neq \emptyset\}$ has compact closure in G.

Note that this definition differs from Bourbaki's [8, Chapter III, § 4]: In case (1) and (2) is satisfied Bourbaki's definition is equivalent to the condition (3_B): For any two points x and y of X there are nhoods

U and V of x and y resp. such that $\{g \in G;\, g \cdot U \cap V \neq \emptyset\}$ has compact closure in G. In the language of topological dynamics the systems ($=\mathbb{R}$-spaces) satisfying (3_B) are called dispersive (s. [6]). The two notions of proper G-spaces coincide if X is locally compact, more generally: if the orbit space is regular [32]. In general: Palais-proper implies Bourbaki-proper. The converse is false as is shown by a very interesting example due to Bebutov [6, IV, 1.5.5 and 18, p. 79]. For a further discussion of these and similar notions see [32], for dynamical systems [6, IV; 5; 31].

The following statements are easily checked: Let X be a proper G-space (always in the sense of Palais). The isotropy group G_x of every point $x \in X$ is compact and the proper G-space G/G_x is homeomorphic to the orbit $G \cdot x$ of x. The main result of [32] is

1.7. Theorem. *If G is a Lie group and X is a proper G-space, then for every point $x \in X$ there is a G-stable nhood U and a G-mapping $f: U \to G/G_x$.*

1.8. Corollary. *Suppose we have a (G, K)-pair. Then every point of a proper G-space has a G-stable nhood U with a G-mapping $f: U \to G/K$.*

Proof. By a theorem of Gluškov's cited in the appendix G contains a compact normal subgroup B such that G/B is a Lie group. Since the compact normal subgroup B is contained in any maximal compact subgroup (A.5 (iii)) it is enough to show that every point of the proper G/B-space $B\backslash X$ has a G/B-stable nhood U with a G/B-mapping $f: U \to G/B/K/B \cong G/K$. So we may assume that G is a Lie group having a finite number of connected components. Let x be a point of X. Since the isotropy group G_x is compact there is an element $g \in G$ such that $g \cdot G_x \cdot g^{-1} \subset K$. The mapping $G \to G/K$, $y \mapsto y \cdot g^{-1} \cdot K$ is constant on the cosets $y \cdot G_x$, thus induces a G-mapping $G/G_x \to G/K$. Compose this G-mapping with the G-mapping $f: U \to G/G_x$ of Palais' theorem to obtain the required G-mapping.

1.9. Main Theorem. *Suppose we have a (G, K)-pair. Let X be a proper G-space. If $G\backslash X$ is paracompact then there is a G-mapping $X \to G/K$.*

Proof. Let $\pi: X \to G\backslash X =: Y$ be the natural mapping. By Corollary 1.8 and since Y is regular there is an open cover $\mathfrak{U}$ of X such that every $U \in \mathfrak{U}$ is G-stable and there is a G-mapping from the closure $\bar{U}$ to G/K. If Y is paracompact there is an open σ-discrete refinement of the cover $\pi(\mathfrak{U}) = \{\pi(U);\, U \in \mathfrak{U}\}$ of Y ([27, Chapter V, Theorem 28]), i.e. there is a sequence $\mathfrak{A}_n$, $n \in \mathbb{N}$, of families $\mathfrak{A}_n$ of open subsets of Y such that $\bigcup_{n=1}^{\infty} \mathfrak{A}_n$ is a cover of Y which refines $\pi(\mathfrak{U})$ and every family $\mathfrak{A}_n$ is discrete, i.e. for every point $y \in Y$ there is a nhood V of y such that $V \cap A \neq \emptyset$

for at most one element $A \in \mathfrak{A}_n$. In particular $Y_n := \bigcup_{A \in \mathfrak{A}_n} \bar{A}$ is closed in Y. Since every $A \in \mathfrak{A}_n$ is contained in some $\pi(U)$, $U \in \mathfrak{U}$, there is a G-mapping $f_A : \pi^{-1}(\bar{A}) \to G/K$. For every $n \in \mathbb{N}$ the G-mappings f_A, $A \in \mathfrak{A}_n$, compose to a G-mapping $f_n : X_n \to G/K$ where $X_n := \pi^{-1}(Y_n) = \bigcup_{\bar{A} \in \mathfrak{A}_n} \pi^{-1}(\bar{A})$ because $\mathfrak{A}_n$ is discrete.

We claim: There is a sequence of G-mappings $F_j : X_1 \cup \cdots \cup X_j \to G/K$, $j \in \mathbb{N}$, such that $F_j | X_1 \cup \cdots \cup X_{j-1} = F_{j-1}$ for $j > 1$. We set $F_1 := f_1$ and define F_n inductively by Lemma 1.5: X_n, $X_1 \cup \cdots \cup X_{n-1}$, f_n, F_{n-1} play the role of U_2, U_1, f_2, f_1 of that lemma. It remains to prove that $S := f_n^{-1}(K)$ is a normal space. The mapping $\pi | S : S \to Y_n$ is surjective continuous closed and the inverse image of any point is compact, because the inverse image of any set $\pi(M)$, $M \subset S$, is $K \cdot M$. The closed subspace Y_n of Y is paracompact, thus S is paracompact [39, Exercise 20G], a fortiori normal. So by Lemma 1.5 there is a G-mapping $F_n : X_1 \cup \cdots \cup X_n \to G/K$ such that $F_n | X_1 \cup \cdots \cup X_{n-1} = F_{n-1}$.

The mapping $f : X \to G/K$ such that $f | X_n = F_n$ is a G-mapping: The mapping f is continuous at every point $x \in X$ because $\pi(x)$ is an inner point of some $A \in \bigcup_{n=1}^{\infty} \mathfrak{A}_n$ and thus x is an inner point of some X_n.

1.10. The differentiable case. Differentiable always means C^∞-differentiable. Let G be a Lie group. A *differentiable G-space* X is a differentiable manifold together with a differentiable action $G \times X \to X$. A *differentiable G-mapping* $f : X \to Y$ from a differentiable G-space X to a differentiable G-space Y is a mapping which is both a G-mapping and differentiable.

Theorem. *Let G be a Lie group having a finite number of connected components. Let K be a maximal compact subgroup. Let X be a paracompact differentiable manifold with a proper differentiable G-action. Then there is a differentiable G-mapping $X \to G/K$.*

We just indicate how the proof differs from the proof of the topological version. We can replace throughout the whole paragraph every topological notion and statement by the corresponding differentiable one, expect where the Tietze extension theorem is used: Lemma 1.5 and hence in the proof of Theorem 1.9. But with a little care the differentiable case can be handled by the same method:

Let A be a subset of a differentiable manifold X. We call a mapping f from A to a differentiable manifold Y differentiable if there is an open nhood U of A and a differentiable mapping $F : U \to Y$ such that $F | A = f$. A partition of unity argument proves the following extension

Lemma. *Let A be a closed subset of a paracompact differentiable manifold X and let $f: A \to \mathbb{R}$ be a differentiable function. Then there is a differentiable function $F: X \to \mathbb{R}$ such that $F|A = f$.*

The following definition is useful. A mapping $f: A \to Y$ of some G-stable subset A of a differentiable G-space X to a differentiable G-space Y is called a *differentiable G-mapping* if there is an open G-stable nhood U of A in X and a differentiable G-mapping $F: U \to Y$ such that $F|A = f$. Note that a differentiable G-mapping is both differentiable and a G-mapping. I do not know whether every mapping that is both a G-mapping and differentiable is actually a differentiable G-mapping. The following lemma replaces Lemma 1.5.

1.11. Lemma. *Let G be a Lie group with a finite number of connected components, let K be a maximal compact subgroup of G and let X be a paracompact differentiable G-space. Let U_i, $i = 1, 2$, be G-stable closed subsets of X and let $f_i: U_i \to G/H$ be differentiable G-mappings. Then there is a differentiable G-mapping $f: U_1 \cup U_2 \to G/K$ such that $f|U_1 = f_1$.*

Proof. Let V_i be open G-stable nhoods of U_i in X and let $g_i: V_i \to G/K$ be differentiable G-mappings such that $g_i|U_i = f_i$. The set $T_i := g_i^{-1}(K)$ is a closed submanifold of V_i. Since $G\backslash X$ is paracompact (see § 0.2, Case 2) hence normal there is a closed G-stable subset W_1 of V_1 such that $U_1 \subset \mathring{W}_1$. Since G/K is diffeomorphic to a euclidean space there is – by the extension lemma above – a differentiable mapping $F: X \to G/K$ such that $F|W_1 = g_1|W_1$. By Weyl's trick (differentiable version) we construct the differentiable K-mapping $\bar{F}: X \to G/K$. By Corollary 1.4 (differentiable version) there is a unique differentiable G-mapping $f': V_2 \to G/K$ such that $f'|T_2 = \bar{F}|T_2$. The G-mapping $g_1|W_1 \cap V_2$ coincides on $W_1 \cap V_2 \cap T_2 = W_1 \cap T_2$ with $g_1|W_1 \cap T_2 = F|W_1 \cap T_2 = \bar{F}|W_1 \cap T_2$. By the uniqueness part of Corollary 1.4 (not necessarily differentiable version) we have $g_1|W_1 \cap V_2 = f'|W_1 \cap V_2$. So there is a well defined differentiable G-mapping $f: \mathring{W}_1 \cup V_2 \to G/K$ on the open G-stable submanifold $\mathring{W}_1 \cup V_2$ of X such that $f|\mathring{W}_1 = g_1|\mathring{W}_1$, $f|V_2 = f'$. The restriction of f to $U_1 \cup U_2$ has the desired properties.

Now the proof of the main theorem – differentiable version – runs just like the proof of the topological version except that we start with a cover $\mathfrak{U}$ of X by G-stable open submanifolds U of X such that there is a differentiable G-mapping from $\bar{U} \to G/K$. Such a cover exists by the differentiable version of Palais' theorem [32, Proposition 2.2.2]. We also need the following fact: If $\mathfrak{A}$ is a discrete family of subsets of $G\backslash X$ and $f_A: \pi^{-1}(\bar{A}) \to G/K$ is a differentiable G-mapping for every $A \in \mathfrak{A}$, then the composite mapping $f: U\pi^{-1}(\bar{A}) \to G/K$ is a differentiable G-mapping. This is implied by the following "very strong normality condition" of the paracompact space $G\backslash X$ [27, Chapter V, Lemma 3.1]: There is a discrete family $\{V(A); A \in \mathfrak{A}\}$ of nhoods $V(A)$ of the sets $A \in \mathfrak{A}$.

§ 2

This paragraph contains refinements and applications of the main Theorems 1.9 and 1.10. Everything in this paragraph also holds in the differentiable category.

A first refinement of the main theorem describes the K-space X.

2.1. Theorem. *Suppose we have a (G, K)-pair. Let X be a G-space admitting a global K-slice. Then there is a K-homeomorphism $\varphi : X \to T \times S$ such that*

(1) *T is the continuous K-module of dimension $n = nc - \dim(G)$ described in the Appendix* A.6.

(2) *The action of K on $T \times S$ is the product action.*

(3) *$\varphi(s) = (0, s)$ for $s \in S$.*

(4) *$\varphi(G \cdot s) = T \times K \cdot s$ for $s \in S$.*

Proof. This is just an application of Theorems A.5 and A.6 of the appendix. We use the notations of these theorems. By Corollary 1.3 X is G-homeomorphic to $G \times_K S$. Let the inverse mapping of the multiplication $E \times K \to G$ be denoted $x \mapsto (e(x), k(x))$. The mapping $e : G \to E$ is a K-mapping if we let K act on G and E by inner automorphisms (A.5 (i)). The mapping $G \times_K S \to E \times S$, $[g, s] \mapsto (e(g), k(g) \cdot s)$ is a K-homeomorphism. The fact that the K-space E is K-isomorphic to the continuous K-module $L(G)/L(K)$ of dimension $n = nc - \dim(G)$ implies the theorem.

Theorem 2.1 implies for the orbit space: $G \backslash X$ is homeomorphic to $K \backslash S$ and $K \backslash S \hookrightarrow K \backslash X$ is a strong deformation retraction.

A second refinement of the main theorems is to reduce the problem of finding all proper G-actions on certain spaces X to the two problems of finding all subspaces S of X such that $\mathbb{R}^n \times S$ is homeomorphic to X and of finding all actions of K on such S.

Suppose we have a (G, K)-pair. Let $\mathfrak{E}(G, X)$ be the set of all G-homeomorphism classes of G-actions on X admitting a global K-slice. If X is completely regular hausdorff, a G-action admitting a global K-slice is obviously proper. The converse is true by our main theorem if $G \backslash X$ is paracompact. If K is a compact topological group, then $\mathfrak{E}(K, X)$ is just the set of all K-homeomorphism classes of K-actions on X, because X is a global K-slice.

2.2. Theorem. *Suppose we have a (G, K)-pair. Let $n = nc - \dim(G)$. Then for every topological space X there is a bijection*

$$\Psi : \mathfrak{E}(G, X) \to U \mathfrak{E}(K, S)$$

where the union is taken over topological spaces S such that $\mathbb{R}^n \times S$ is homeomorphic to X, one such space out of every class of homeomorphic spaces.

Note that the theorem implies that $\mathfrak{E}(G, X)$ depends only on the maximal compact subgroup of G and on n, not on the algebraic relations between K and the non-compact part of G. So $\mathfrak{E}(G, X) \cong \mathfrak{E}(K \times \mathbb{R}^n, X)$, where $K \times \mathbb{R}^n$ is the direct product of topological groups.

Proof. The mapping Ψ is defined as follows. Let φ be an action of G on X that admits a global K-kernel, i.e. there is a G-mapping $f: X \to G/K$. Define $\Psi(\varphi) =$ K-homeomorphism class of $f^{-1}(K)$. The point is to show that Ψ depends on the G-homeomorphism class of φ only. We first define the inverse mapping Φ (cf. [32, p. 306, Theorem]). Pick a homeomorphism $t: E \to \mathbb{R}^n$, where E is a subset of G such that the multiplication $E \times K \to G$ is a homeomorphism. We denote the inverse homeomorphism $G \to E \times K$ by $g \mapsto (e(g), k(g))$. Let S be a K-space and let $q: \mathbb{R}^n \times S \to X$ be a homeomorphism. Then we have a composite homeomorphism h

$$G \times_K S \to E \times S \xrightarrow{t \times 1_S} \mathbb{R}^n \times S \xrightarrow{q} X.$$

$$[g, s] \mapsto (e(g), k(g) \cdot s)$$

Now $G \times_K S$ is a G-space under the action $g \cdot [x, s] = [g \cdot x, s]$. There is exactly one G-action on X such that h is a G-homeomorphism. The G-homeomorphism class thus defined obviously does not depend on the homeomorphisms chosen. Let $\Phi(S)$ be the G-homeomorphism class of this G-action. There is a G-mapping $f: X \to G/K$, $f(h([g, s])) = g \cdot K$, such that $f^{-1}(K) = q(S)$ is K-homeomorphic to S. If S_1 and S_2 are K-homeomorphic spaces such that $\mathbb{R}^n \times S_1$ and $\mathbb{R}^n \times S_2$ are homeomorphic to X, then $\Phi(S_1) = \Phi(S_2)$. So Φ is well defined on $U\mathfrak{E}(K, S)$ and takes values in $\mathfrak{E}(G, X)$.

Concerning Ψ we have

2.3. Lemma. *Suppose we have a (G, K)-pair. Let X be a G-space admitting two G-mappings $f_1, f_2: X \to G/K$. Then there is a G-autohomeomorphism h of X such that $f_1 \circ h = f_2$.*

This lemma implies the theorem: Ψ depends only on the G-homeomorphism class of the G-action on X and Ψ is obviously the inverse mapping of Φ.

Proof of 3.2. Again we make use of the fact that the restriction of the natural mapping $\pi: G \to G/K$ to E is a K-mapping, if we let K act on G/K by left translations and on E by inner automorphisms. So $\tilde{f}_i: X \to E$, $\tilde{f}_i := (\pi | E)^{-1} \circ f_i$ are K-mappings. Thus $p_i: X \to S_i := f_i^{-1}(K)$, $p_i(x) := (\tilde{f}_i(x))^{-1} \cdot x$ are K-retractions preserving orbits.

So we have two K-mappings $p_1 | S_2: S_2 \to S_1$ and $p_2 | S_1: S_1 \to S_2$ which are unfortunately not inverse of each other. For instance $p_1 \circ p_2(s) = (l \circ \tilde{f}_2(s)) \cdot s$ for $s \in S_1$ where $l: E \to K$ is defined by the condition $g^{-1} \in E \cdot l(g)$, so $l(g) = k(g^{-1})$ in our earlier notation. Since

E is a K-space, l is a K-mapping. The K-mapping $j: S_1 \to S_1$, $j(s) := (l \circ \tilde{f}_2(s))^{-1} \cdot s$, is the inverse of $p_1 \circ p_2 | S_1$. Similarly we conclude that $p_2 \circ p_1 : S_2 \to S_2$ is a K-homeomorphism. So $p_1 | S_2 : S_2 \to S_1$ and $p_2 | S_1 : S_1 \to S_2$ are K-homeomorphisms and $p_2 \circ j : S_1 \to S_2$ is an inverse K-homeomorphism of $p_1 | S_2 : S_2 \to S_1$. By Corollary 1.4 these two K-mappings induce mutually inverse G-mappings $X \to X$. Let $h: X \to X$ be the G-homeomorphism such that $h | S_2 = p_1 | S_2$. Since the two G-mappings $f_1 \circ h$ and f_2 from X to G/K coincide on the global K-slice S_2, they are equal, again by Corollary 1.4.

As a further application we show how the results of the present paper generalize known facts on ends of proper G-spaces for connected G [2, 11–13, 22, 26, 35, 38, 40]. We need some preparatory material 2.4–2.6.

2.4. Theorem. *Let the non-compact locally compact connected topological group G act properly on the locally compact topological space X. Then the embedding of X into its one-point-compactification $X \cup \{\infty\}$ is homotopic to the constant mapping $X \to \infty$.*

Proof. [38]. In the notations of Theorem A.5 of the appendix: E is homeomorphic to some $\mathbb{R}^n$, $n > 0$. So there is a path $\gamma: [0, 1] \to G \cup \{\infty\}$ in the one-point-compactification of G such that $\gamma(0) = \infty$ and $\gamma(1) =$ identity element of G. The desired homotopy $F: X \times [0, 1] \to X \cup \{\infty\}$ is then given by

$$F(x, t) = \begin{cases} \gamma(t) \cdot x & \text{for} \quad \gamma(t) \neq \infty \\ \infty & \text{for} \quad \gamma(t) = \infty . \end{cases}$$

F is continuous because the action is proper.

We need the notion of cohomology at infinity [16]. By cohomology we mean Čech cohomology with values in a fixed ring A. Let X be a locally compact paracompact topological space. Let $\mathfrak{T}$ be the system of all closed subsets F of X whose complement is relatively compact. $\mathfrak{T}$ is a projective system ordered by inclusion.

2.5. Definition. The cohomology at infinity $H^n_\infty(X)$ is the direct limit $\lim\limits_{F \in \mathfrak{T}} H^n(F)$ of the inductive system $H^n(F)$, $F \in \mathfrak{T}$.

There is a long exact sequence [16]

$$\cdots \to H^n_c(X) \to H^n(X) \to H^n_\infty(X) \to H^{n+1}_c(X) \to H^{n+1}(X) \to \cdots$$

involving cohomology groups $H^n(X)$ and cohomology groups $H^n_c(X)$ with compact support. In the case of Theorem 2.4 this exact sequence falls apart since $H^n_c(X) \to H^n(X)$ is the zero-mapping.

The first group $H^0_\infty(X)$ is closely related to the number of ends of X. Freudenthal's end point compactification $\hat{X}$ of the locally compact space X is defined by the following properties: (1) $\hat{X}\setminus X$ is totally disconnected or equivalently 0-dimensional. (2) If Y is any compactification of X such that $Y\setminus X$ is 0-dimensional, then the embedding $X \to Y$ has a unique continuous extension $\hat{X} \to Y$. The cardinality of $\hat{X}\setminus X$ is called the number of ends of X and denoted $e(X)$.

2.6. Theorem. [16, 34, 38]. *Let X be a connected locally compact paracompact topological space. Then $H^0_\infty(X)$ is isomorphic to the ring of continuous functions from $\hat{X}\setminus X$ to the discrete ring A.*

In particular: If A is a field, then $e(X) = \dim_A H^0_\infty(X)$ in case one of the two numbers is finite.

We are now ready to state

2.7. Theorem. *Let G be a locally compact topological group having a compact group of connected components. Let $n = nc - \dim(G)$. Let X be a locally compact paracompact proper G-space. Then*

(a) *$H^m_c(X) = 0$ for $m < n$.*

(b) *$H^n_c(X) \neq 0$ iff $G\backslash X$ has a compact open subset $\neq \emptyset$. In particular, if X is connected: $H^n_c(X) \neq 0$ iff $G\backslash X$ is compact.*

(c) *If $G\backslash X$ is compact: $H^k_c(X) \cong H^{k-n}(X)$ for every $k \in \mathbb{Z}$.*

2.8. Corollary. *If G is non-compact and X is connected, then $e(X) \leqq 2$. If $e(X) = 2$, then $G\backslash X$ is compact and G is a semidirect product of a maximal compact subgroup K and a normal subgroup E isomorphic to $\mathbb{R}$. The group K contains an open subgroup L of index $\leqq 2$ such that $E \cdot L$ is a direct product of E and L. The group L is normal in G.*

Proof of Theorem 2.7. By the main theorem there is a G-mapping $f: X \to G/K$. Then X is homeomorphic to $\mathbb{R}^n \times S$, where $S := f^{-1}(K)$. So $H^*_c(X) \cong H^*_c(S) \otimes H^*_c(\mathbb{R}^n; \mathbb{Z})$ which implies (a). Since $G\backslash X$ is homeomorphic to $K\backslash S$, $G\backslash X$ is compact iff S is compact. $H^n_c(X) = H^0_c(S)$ is non zero iff S has a compact open subset $\neq \emptyset$. Since $G\backslash X$ is homeomorphic to $K\backslash S$, $G\backslash X$ has a compact open subset $\neq \emptyset$ iff S does, which implies (b). Finally if S is compact we have $H^k_c(X) \cong H^{k-n}_c(S) \cong H^{k-n}(S) \cong H^{k-n}(X)$ for every $k \in \mathbb{Z}$, because $\mathbb{R}^n$ is contractible.

Proof of Corollary 2.8. By Theorem 2.4 we have short exact sequences

$$0 \to H^k(X) \to H^k_\infty(X) \to H^{k+1}_c(X) \to 0 .$$

Take a field A as ring of coefficients for the cohomology. Since X is connected we have $\dim_A H^1_c(X) = \dim_A H^0_\infty(X) - 1 = e(X) - 1$ by Theorem 2.6 in case one of these numbers is finite.

Since G is non-compact we have $n = nc - \dim(G) > 0$. Theorem 2.7 implies: If $H^1_c(X) \neq 0$ then $n = 1$ (a), $G\backslash X$ is compact (b) and $H^1_c(X) \cong H^0(X)$ (c). So $e(X) = 1 + \dim_A H^1_c(X) \leqq 2$ and $e(X) = 2$ iff $H^1_c(X) \neq 0$, in which case $n = 1$. We now apply Theorem A.5 of the appendix for

$n=1$. In the notations given there E is a one-parameter subgroup of G (iv), normal (i) and G is the semidirect product of E and a maximal compact subgroup K of G (ii). Now consider the continuous homomorphism $r: K \to \mathrm{Aut}(E)$, $k \to I_k | E$, where I_k is the inner automorphism of G, $I_k(g) = k \cdot g \cdot k^{-1}$. The image of r is a compact group of automorphisms of E, so contained in $\{\pm 1_E\}$. So the kernel L of r is an open subgroup of K of index $\leqq 2$. By construction $E \cdot L$ is a direct product of E and L. The normalizer of L contains K and E, so L is a normal subgroup of G.

2.9. Let G be an arbitrary locally compact topological group, let G_1 be its connected component of 1. Then

$$nc-\dim G_1 = \min\{m \in \mathbb{Z};\, H_c^m(G) \neq 0\} \tag{2.9.1}$$

by 2.7 (b) applied to the proper G_1-space G defined by right translations, say. If G/G_1 is compact we have

$$nc-\dim G = \min\{m \in \mathbb{Z};\, H_c^m(G) \neq 0\} \tag{2.9.2}$$

by 2.7 (a) and (b) applied to the proper G-space G. So we define for an arbitrary locally compact topological group G:

$$nc-\dim G := nc-\dim G_1 = \min\{m \in \mathbb{Z};\, H_c^m(G) \neq 0\}$$

and this definition is by (2.9.2) in keeping with our previous definition.

2.10. Corollary. *If H is a closed subgroup of the locally compact topological group G we have:*

$$nc-\dim H \leqq nc-\dim G$$

and

$$nc-\dim H = nc-\dim G$$

iff G_1/H_1 is compact.

Proof. Look at the proper H_1-space G and apply 2.7.

Mostow [31a] has shown: If H and G are connected Lie groups, L and K their maximal compact subgroups, H a closed subgroup of G and $nc-\dim G = d + nc-\dim H$, then G/H is a fiber bundle over K/L with typical fiber $\mathbb{R}^d$. In case $d=1$, Borel [7, Theorem 2] has shown that this bundle is trivial, which is not true in general. The number of ends of G/H is 0, 2, 1 according to the cases $d=0$, 1, $\geqq 2$. For ends of homogeneous spaces cf. also [24].

Appendix

We will prove a refined version of the theorem of Malcev-Iwasawa on maximal compact subgroups for locally compact topological groups having a compact group of connected components. The corresponding theorem on Lie groups can be found in [21, Chapter XV, Theorem 3.1]. Using this theorem, the solution of Hilbert's fifth problem (A.2), a local splitting theorem (A.1) and the notion of Lie algebra of a topological group we get the desired result (A.5 and A.6).

A.1. Theorem *(Gluškov* [14, Theorem A]*). Let G be a locally compact topological group, U a nhood of the neutral element 1 of G. Then there is a nhood $V \subset U$ of 1 in G that splits as a direct product of a compact group and a connected local Lie group.*

A.2. Theorem *(Gluškov* [14, Theorem 8]*). Let G be a locally compact topological group having a compact group of connected components. Then every nhood of 1 in G contains a compact normal subgroup B such that G/B is a Lie group.*

A.3. Let G be a topological group. We will define a sort of Lie algebra of G (see e.g. [15, 23]): Let $L(G)$ be the set of all continuous homomorphisms from the additive group $\mathbb{R}$ to G endowed with the compact-open topology, which is the same as the topology of uniform convergence on compact sets [8]. $L(\cdot)$ is a functor from topological groups to topological spaces. In particular we have an adjoint representation of G on $L(G)$ defined by

$$(\mathrm{Ad}(x)\,X)(s) = x \cdot X(s) \cdot x^{-1} \quad \text{for} \quad x \in G,\ X \in L(G),\ s \in \mathbb{R}.$$

The mapping $G \times L(G) \to L(G)$ defined by the adjoint representation is continuous at $(1, X)$ for every $X \in L(G)$, as is easily checked by looking at the topology of uniform convergence on compact sets, and hence everywhere, since $\mathrm{Ad}(x): L(G) \to L(G)$ is continuous.

Lemma. *Let G and H be topological groups. A continuous homomorphism $f: G \to H$ which is a local isomorphism induces a homeomorphism $L(f): L(G) \to L(H)$.*

Proof. Any local homomorphism from $\mathbb{R}$ to any topological group, defined on a connected nhood of 0 in $\mathbb{R}$ extends uniquely to a continuous homomorphism on all of $\mathbb{R}$. So $L(f)$ is bijective. It remains to prove that $L(f)$ is open. Let N be an open nhood of 1 in G such that $f|N$ is injective and $f^{-1}: f(N) \to N$ is continuous and multiplicative. Let $W_G(K, U)[X] = \{Y \in L(G);\ Y(s) \in X(s) \cdot U$ for every $s \in K\}$ be a nhood of X in $L(G)$, K compact $\subset \mathbb{R}$, U a nhood of 1 in G. We have to show that its image under $L(f)$ is a nhood of $L(f)(X)$. We may assume that K is a closed interval $[-r, r] \subset \mathbb{R}$. Let n be a positive integer such that $X\left(\frac{K}{n}\right) \subset N$. Let V_1 be a nhood of 1 in G such that $V_1^n \subset U$ and let V be a nhood of 1 in G such that $X\left(\frac{K}{n}\right) \cdot V \subset N$ and $V \cdot X(s) \subset X(s) \cdot V_1$ for every $s \in K$. Then $W_G\left(\frac{K}{n}, V\right)[X] \subset W_G(K, U)[X]$ and $L(f)\, W_G\left(\frac{K}{n}, V\right)$ $[X] = W_H\left(\frac{K}{n}, f(V)\right)[f \circ X]$.

A.4. Theorem *Let G be a locally compact topological group. Let $L(G)$ be the set of all continuous homomorphisms $\mathbb{R} \to G$, endowed with the compact-open topology. If $X, Y \in L(G)$ then the sequence of points $\left(X\left(\frac{s}{n}\right) \cdot Y\left(\frac{s}{n}\right)\right)^n \in G$ converges to a point which we denote $(X+Y)(s)$. Then $X + Y : \mathbb{R} \to G$ is a continuous homomorphism. $L(G)$ with this addition and the multiplication by scalars defined by the formula*

$$(r \cdot X)(s) := X(r \cdot s)$$

turns $L(G)$ into a complete locally convex topological $\mathbb{R}$ vector space.

Proof. The theorem is true for Lie groups and for compact groups (see [23]), hence for the direct product of a compact group K and a Lie group H. By Gluškov's Theorem A.1 and since any local Lie group is locally isomorphic to a simply connected Lie group, there is a continuous homomorphism $K \times H \to G$ of a direct product as above to G, which is a local isomorphism and hence induces a homeomorphism $L(H \times H) \to L(G)$. This proves the theorem. For a different proof s. [15].

Remark. By the same argument one can also define a continuous Lie algebra structure on $L(G)$, such that a Campbell-Hausdorff-theorem holds using [23]. But the G-module structure is enough for our purposes.

There is a continuous mapping, classically called the exponential

$$\exp : L(G) \to G, \qquad \exp(X) = X(1).$$

A.5. Theorem. *Let G be a locally compact topological group whose group of connected components is compact. Then there is a maximal compact subgroup K of G and a subset E of G such that*

(i) *$x \cdot E \cdot x^{-1} = E$ for every element $x \in K$,*

(ii) *the multiplication $E \times K \to G$ is a homeomorphism,*

(iii) *for every compact subgroup L of G there is an element $e \in E$ such that $e \cdot L \cdot e^{-1} \subset K$,*

(iv) *consider $L(G)$ as a K-module by restricting the adjoint representation of G to K. Then there is a finite set of finite dimensional K-submodules $S_1, \ldots, S_k$ of $L(G)$ such that their sum $S_1 + \cdots + S_k$ is direct, the mapping*

$$\sigma : S_1 \oplus \cdots \oplus S_k \to G$$

$$\sigma(x_1 + \cdots + x_k) = \exp(x_1) \cdot \cdots \cdot \exp(x_k)$$

is a homeomorphism onto E and for each $E_i := \exp(S_i)$ we have $x \cdot E_i \cdot x^{-1} = E_i$ for every $x \in K$.

A.6. Corollary. *The finite dimensional K-submodule* $T := S_1 \oplus \cdots \oplus S_k$ *of* $L(G)$ *is a complementary submodule of* $L(K)$ *in* $L(G)$, *i.e.* $L(K) \oplus T = L(G)$. *We have K-isomorphisms*

$$T \xrightarrow{\sigma} E \xrightarrow{\pi|E} G/K.$$

Here K acts on $E \subset G$ *by inner automorphisms and on* G/H *by left translations of cosets. The mapping* π *is the natural mapping* $G \to G/K$.

By (iii) for any two maximal compact subgroups K and L of G we have $\dim G/K = \dim G/L$. We call it the *non-compact dimension of* G:

$$nc-\dim(G) = \dim G/K = \dim_{\mathbb{R}} L(G)/L(K).$$

Proof. For a Lie group G the space of continuous homomorphisms $L(G)$ can be identified with the Lie algebra of left invariant vector fields on G. For a Lie group having a finite number of connected components the theorem and its corollary are known [21, Chapter XV, Theorem 3.1]. That the S_i are actually K-modules and σ is a K-mapping is seen from the following facts: The exponential mapping is a local homeomorphism; $\exp \mathrm{ad}(x)(X) = x \cdot \exp(X) \cdot x^{-1}$ for every $x \in G$, $X \in L(G)$. Now let G be a locally compact topological group such that the group of connected components is compact. There is a compact normal subgroup B of G such that G/B is a Lie group. G/B has only a finite number of connected components. Let $p: G \to G/B$ be the natural homomorphism. Let W be a nhood of 1 in G/B containing no subgroup except $\{1\}$. Then $p^{-1}(W)$ contains a nhood of 1 in G that splits as a direct product of a compact group A and a local Lie group. So there is a direct product of A and a Lie group H and a continuous homomorphism $A \times H \xrightarrow{f} G$ which is a local isomorphism and such that $f \mid A = 1_A$. Now $p(A)$ is a subgroup of W, so A is contained in the kernel of p. The continuous homomorphism $p \circ f \mid H : H \to G/B$ of Lie groups is continuous and open, hence induces a surjection $L(H) \to L(G/B)$. By means of the isomorphism $L(f): L(A) \times L(H) \to L(G)$ the two vector spaces are identified. For every element x of the open subgroup $f(A \times H) \subset G$ we have $\mathrm{Ad}(x) L(H) \subset L(H)$. Since the group of connected components of G is compact, $f(A \times H)$ is of finite index in G. So there is a finite dimensional G-submodule M of $L(G)$ such that $L(p): M \to L(G/B)$ is surjective.

Now let K, E_i, S_i, T be the objects of the theorem and the corollary for the Lie group G/B. Then $p^{-1}(K) =: L$ is a maximal compact subgroup of G and any compact subgroup of G is conjugate to a subgroup of L. Let $\tilde{S}_i$ be the inverse image of S_i under $L(p) | M : M \to L(G/B)$. Then $\tilde{S}_i$ is an L-module containing the L-module $\ker(L(p) | M)$. Since every finite dimensional continuous L-module is semisimple, there is an

L-module $T_i \subset \tilde{S}_i$ such that $L(p) | T_i$ is a vector space isomorphism onto S_i. Let $F_i := \exp_G(T_i)$. The commutative diagram

$$\begin{array}{ccc} T_i & \xrightarrow{\exp_G} & F_i \\ {\scriptstyle L(p)}\downarrow\wr & & \downarrow{\scriptstyle p} \\ S_i & \underset{\sim}{\xrightarrow{\exp_{G/B}}} & E_i \end{array}$$

shows that $\exp_G : T_i \to F_i$ is a homeomorphism. The formula $x \cdot \exp(X) \cdot x^{-1} = \exp \mathrm{Ad}(x)(X)$ shows that $x \cdot F_i \cdot x^{-1} = F_i$ for $x \in L$. The sum $T_1 + \cdots + T_k$ is direct since any non trivial relation $t_1 + \cdots + t_k = 0$ would imply the non trivial relation $L(p)\, t_1 + \cdots + L(p)\, t_k = 0$. An analogous diagram as above shows that $T_1 \oplus \cdots \oplus T_k \xrightarrow{\tau} F_1 \cdot \ldots \cdot F_k$, $\tau(t_1 + \cdots + t_k) = \exp_G(t_1) \cdot \ldots \cdot \exp_G(t_k)$ is a homeomorphism.

Let $R := \oplus T_i \subset L(G)$. The homomorphism $L(p) | R : R \to T$ is a vector space isomorphism. We have a continuous projection mapping $L(G) \to R$ rendering the diagram

$$\begin{array}{ccc} L(G) & \xrightarrow{L(p)} & L(G/B) = L(K) \oplus T \\ \downarrow & & \downarrow{\scriptstyle \mathrm{prox}} \\ R & \underset{\sim}{\xrightarrow{L(p)}} & T \end{array}$$

commutative. This projection mapping is an L-module homomorphism. Its kernel $L(p)^{-1}(L(K))$ is just $L(L)$ injected into $L(G)$.

The multiplication $F \times L \to G$ is continuous. Let g be the following composite continuous mapping $g : G \to F$

$$\begin{array}{ccccccc} G & \xrightarrow{p} & G/B & \underset{\sim}{\xleftarrow{\text{mult.}}} & E \times K & \xrightarrow{\mathrm{prox}} & E & \underset{\sim}{\xleftarrow{\sigma}} & T \\ & \searrow^{g} & & & & & {\scriptstyle p}\downarrow\wr & & \wr\downarrow{\scriptstyle L(p)} \\ & & & & & & F & \underset{\sim}{\xleftarrow{\tau}} & R \end{array}$$

The mapping $G \to F \times L$, $x \to (g(x), (g(x))^{-1} \cdot x)$ is an inverse mapping of the multiplication $F \times L \to G$.

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Herbert Abels
Fakultät für Mathematik der Universität
D-4800 Bielefeld
Kurt-Schumacher-Str.
Federal Republic of Germany

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