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A note on the number of functional digraphs

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In a recent paper [2], HARARY obtained the formula

$$v(x) = \exp \sum_{m=1}^{\infty} \frac{1}{m} \sum_{k=2}^{\infty} Z(C_k, T(x^m))$$

for the counting series $v(x)$ of all functional digraphs[1]), and conjectured that it might be possible to simplify it. In this note we show how this can be done.

Since

$$Z(C_k) = \frac{1}{k} \sum_{d|k} \varphi(d) f_d^{k/d},$$

we have

$$\begin{aligned}
v(x) &= \exp \sum_{m=1}^{\infty} \left\{\frac{1}{m} \sum_{k=2}^{\infty} Z(C_k, T(x^m))\right\} \\
&= \exp \sum_{m=1}^{\infty} \left\{\frac{1}{m} \sum_{k=1}^{\infty} \frac{1}{k} \sum_{d|k} \varphi(d)\, [T(x^{md})]^{k/d} - \frac{1}{m} T(x^m)\right\} \\
&= \exp \sum_{m=1}^{\infty} \left\{\frac{1}{m} \sum_{d=1}^{\infty} \sum_{r=1}^{\infty} \frac{1}{dr} \varphi(d)\, [T(x^{md})]^r - \frac{1}{m} T(x^m)\right\} \\
&= \exp \sum_{m=1}^{\infty} \left\{\frac{1}{m} \sum_{d=1}^{\infty} \left(\frac{1}{d} \varphi(d) \sum_{r=1}^{\infty} \frac{1}{r} [T(x^{md})]^r\right) - \frac{1}{m} T(x^m)\right\} \\
&= \exp \sum_{m=1}^{\infty} \left\{\frac{1}{m} \sum_{d=1}^{\infty} \left(\frac{-\varphi(d)}{d} \sum_{r=1}^{\infty} \log(1 - T(x^{md}))\right) - \frac{1}{m} T(x^m)\right\} \\
&= \prod_{m=1}^{\infty} \exp \left\{\frac{1}{m} \sum_{d=1}^{\infty} \left(\frac{-\varphi(d)}{d} \sum_{r=1}^{\infty} \log(1 - T(x^{md}))\right) \right. . \\
&\qquad \exp\left\{- \sum_{m=1}^{\infty} \frac{1}{m} T(x^m)\right\} \\
&= \prod_{m=1}^{\infty} \prod_{d=1}^{\infty} \exp\left\{\frac{-\varphi(d)}{md} \log(1 - T(x^{md}))\right\} \cdot \exp\left\{- \sum_{m=1}^{\infty} \frac{1}{m} T(x^m)\right\} \\
&= \prod_{m=1}^{\infty} \prod_{d=1}^{\infty} \{1 - T(x^{md})\}^{-\varphi(d)/md} \cdot \exp\left\{- \sum_{m=1}^{\infty} \frac{1}{m} T(x^m)\right\} \\
&= \prod_{n=1}^{\infty} \prod_{d|n} \{1 - T(x^n)\}^{-\varphi(d)/n} \cdot \exp\left\{- \sum_{m=1} \frac{1}{m} T(x^m)\right\} \\
&= \prod_{n=1}^{\infty} \{1 - T(x^n)\}^{-\frac{1}{n} \sum_{d|n} \varphi(d)} \cdot \exp\left\{- \sum_{m=1}^{\infty} \frac{1}{m} T(x^m)\right\}.
\end{aligned}$$

[1]) For the nomenclature and notation see Harary's paper.

But $\sum_{d|n} \varphi(d) = n$ (see [3], p. 52), and

$$\exp \sum_{m=1}^{\infty} \frac{1}{m} T(x^m) = \frac{1}{x} T(x)$$

(see [2]). Hence

$$v(x) = \frac{x}{T(x)} \prod_{n=1}^{\infty} \{1 - T(x^n)\}^{-1}.$$

This counting series enumerates functional digraphs which contain no slings. If slings are allowed then the second summation in Harary's formula goes from $k = 1$ upwards, and we obtain the counting series for the function $fcn(n)$ of DAVIS [1]. The simplification proceeds as before, and we have

$$\sum fcn(n) x^n = \prod_{n=1}^{\infty} \{1 - T(x^n)\}^{-1}.$$

Using the result

$$T(x) = x + x^2 + 2x^3 + 4x^4 + 9x^5 + 20x^6 + 48x^7 + \\ + 115x^8 + 286x^9 + 719x^{10} + 1842x^{11} + 4766x^{12} + \cdots .$$

I find that

$$v(x) = 1 + x^2 + 2x^3 + 6x^4 + 13x^5 + 40x^6 + 100x^7 + 291x^8 + \\ + 797x^9 + 2273x^{10} + 6389x^{11} + \cdots$$

and

$$\sum fcn(n) x^n = 1 + x + 3x^2 + 7x^3 + 19x^4 + 47x^5 + 130x^6 + \\ + 343x^7 + 951x^8 + 2615x^9 + 7318x^{10} + 20491x^{11} + 57902x^{12} + \cdots .$$

References

[1] DAVIS, R. L.: The number of structures of finite relations. Proc. Am. Math. Soc. 4, 486—495 (1953).
[2] HARARY, F.: The number of functional digraphs. Math. Ann. 138, 203—210 (1959).
[3] HARDY, G. H., and E. M. WRIGHT: An introduction to the theory of numbers. Oxford 1945.

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