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On the uncoupled problem of stress-assisted diffusion through a linear elastic solid

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Summary. In this paper, the uncoupled version of Aifantis bilinear stress-assisted theory of diffusion through a linear elastic solid is considered. In analogy to thermoelasticity the basic equations and certain special representations of the general problem are presented. The general three-dimensional problem is reduced to a problem of body and surface forces and the reciprocal diffuso-elastic theorem is established. Analytical solutions of particular diffusion problems are derived and a complex formulation of the two-dimensional elastodiffusion problem is given. A crack elastodiffusion problem is considered as an application of the complex representation.

1 Introduction

Theoretical and experimental studies on the diffusion problem in solids have shown the significant effect of the stress state of bodies on the diffusion process [1]–[10]. The proposed theories are based on the extension and/or the modification of the first Fick law for pure diffusion. In these theories, the effect of the stress field is introduced by considering that the mass flux of the diffusing substance is a linear function of the trace gradient of the stress tensor, exactly as occurs with the gradient of concentration. The additional assumption of the linear dependence of the coefficient of concentration gradient from the trace of stress tensor from the concentration leads to a generalized consideration of the problem. Such a consideration is proposed by Aifantis' theory [6]–[9], which is based on the principle of rational mechanics and generalizes and unifies all previous theories.

According to previous theories, the mathematical model of an elastodiffusion theory is represented by the equation of diffusive flux, the equation of conservation of mass and the equations of the stress and strain states of the body. A general form of diffusion equations is given by [9]

$$\boldsymbol{J} = -(D + N\sigma) \operatorname{grad} \rho - (L + M\rho) \operatorname{grad} \sigma \tag{1.1.1}$$

$$\dot{\rho} + \operatorname{div} \boldsymbol{J} = 0 \tag{1.1.2}$$

where $\boldsymbol{J}$ is the diffusive flux vector, σ is the trace of the stress tensor σ_{ij} due to mechanical loading and diffusion, ρ is the concentration of the substance, D, N, L and M are scalar constants and grad, div are the gradient and divergence operators.

In the framework of linear elasticity of an isotropic and homogeneous medium an alternative process will be adopted for the derivation of equations of previous theories related to the strain and stress state of the body. We consider that the strains e_{ij}^d due to diffusion are introduced into the body as initial strains and we assume that they have the form

$$e_{ij}^d = \gamma_d \rho \delta_{ij} \tag{1.2}$$

where γ_d is the coefficient of linear diffusion expansion and δ_{ij} is the Kronecker delta ($\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$). The introduction of the initial strain e_{ij}^d into the body produces an elastic strain state e'_{ij} and a stress state σ_{ij}. Thus, the elastodiffusion problem has now been reduced to a distortion problem of elasticity whose basic equations are [12]

$$\sigma_{ij} = \lambda e \delta_{ij} + 2\mu \varepsilon_{ij} - \gamma \rho \delta_{ij} \tag{1.3.1}$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{1.3.2}$$

$$\sigma_{ji,j} = 0 \tag{1.3.3}$$

where u_i represent the displacement components, e is the volume strain ($e = e_{kk}$), λ and μ are the Lamé constants and $\gamma = (3\lambda + 2\mu)\gamma_d$. Equation (1.3.1) expresses the stress-strain relations, Eq. (1.3.2) the usual relations of strains and displacements and Eq. (1.3.3) the equations of internal equilibrium. In this work, all the tensor quantities are related to a system of rectangular axes x_i and depend on the vector position $\boldsymbol{x}$ and the time t. The Latin indices will take the values 1, 2 or 3, the indices after the comma will denote differentiation with respect to the corresponding coordinates and the dot differentiation with respect to time t. The usual summation convention is used ($e_{kk} = e_{11} + e_{22} + e_{33}$).

The unknown quantities σ_{ij}, e_{ij}, u_i and ρ are determined by solving the system of Eqs. (1.1) and (1.3) using the appropriate boundary and initial conditions. It is obvious that Eqs. (1.1) and (1.3) are coupled and that the coupling of concentration and stresses is due to the last term of the right side of (1.1.1). As a first approximation we will subsequently consider the uncoupled problem, in which we assume that the concentration field is only affected by the stresses of mechanical loading.

The failure analysis of a body immersed in a corrosive or hydrogen environment taking into account the mechanical stress-assisted diffusion theory constitutes an important research locus with a great practical interest. The foundation of a failure criterion according to the modern considerations of fracture mechanics requests extensive theoretical and experimental studies on the suitable mechanical crack models. Such a criterion based on the maximum concentration of the corrosive species in the vicinity of the crack tips has been proposed by Aifantis [16]. Concentration solutions of specific crack problems under steady state conditions have been presented in [16] and [18].

In this work the uncoupled problem of diffusion through an isotropic, homogeneous and linear elastic matrix is theoretically studied. In Section 2 the uncoupled problem is described and its basic equations are presented. The elasto-diffusion problem is reduced to a problem of body and surface forces and the reciprocal diffusoelastic theorem is established. In Section 3, the concentration field is derived for particular boundary value problems using Aifantis' equations. In Section 4, the two-dimensional problem is formulated in terms of two holomorphic complex functions and an application to a crack problem is given.

2 Basic equations of the uncoupled problem

Following Aifantis [9], we consider an isotropic, homogeneous and linear elastic body, whose elastic state due to a mechanical loading is described by the strains e_{ij}^0 and stresses σ_{ij}^0. The presence of a diffusion process in the solid induces an extra state of stresses σ_{ij}^+ and strains e_{ij}^+, which are not negligible with respect to the initial elastic strains e_{ij}^0. In the uncoupled problem we assume that the extra state is effected only by the initial elastic state depending on the mechanical loading of the body. The superscripts (0) and (+) will denote quantities characterized by the mechanical state and the extra state due to diffusion, respectively. Thus, from Eqs. (1.1) it follows that

$$\boldsymbol{J} = -(D + N\sigma^0)\,\mathrm{grad}\,\rho + (L + M\rho)\,\mathrm{grad}\,\sigma^0 \tag{2.1.1}$$

$$\dot{\rho} = (D + N\sigma^0)\,\nabla^2\rho - (M - N)\,\mathrm{grad}\,\rho\,\mathrm{grad}\,\sigma^0 \tag{2.1.2}$$

where ∇^2 is the Laplace operator. The concentration field is completely determined by (2.1.2), when appropriate initial and boundary conditions are given. Of course, the initial elastic state of the body e^0_{ij}, σ^0_{ij} has previously been obtained by solving the relative mechanical problem. The boundary condition of the form

$$\boldsymbol{J}\cdot\boldsymbol{n} = -(D + N\sigma^0)\frac{\partial\rho}{\partial n} + (L + M\rho)\frac{\partial\sigma^0}{\partial n} = s(\boldsymbol{x}, t), \qquad \boldsymbol{x}\in A, \qquad t > 0 \tag{2.2}$$

represents the normal diffusive flux on the surface A of the body V, where $\boldsymbol{n}$ is the outward unit normal to surface A of the body and $S(x, t)$ is a given function. When the body is insulated on A, then $S(\boldsymbol{x}, t) = 0$. If the concentration is given on A, the boundary condition is expressed by the formula

$$\rho = h(\boldsymbol{x}, t), \qquad \boldsymbol{x}\in A, \qquad t > 0 \tag{2.3}$$

where $h(\boldsymbol{x}, t)$ is a given function. The initial condition determines the concentration field at $t = 0$ and has the form

$$\rho = g(\boldsymbol{x}), \qquad \boldsymbol{x}\in V, \qquad t = 0 \tag{2.4}$$

where $g(\boldsymbol{x})$ is a given function.

When the concentration field is determined by the solution of the relative boundary value problem, the secondary state expressed by σ^+_{ij}, e^+_{ij} and $u_i{}^+$ is obained by solving the following differential system [17]:

$$\sigma^+_{ij} = \lambda e^+\delta_{ij} + 2\mu e^+_{ij} - \gamma\rho\delta_{ij} \tag{2.5.1}$$

$$e^+_{ij} = \frac{1}{2}(u^+_{i,j} + u^+_{j,i}) \tag{2.5.2}$$

$$\sigma^+_{ji,j} = 0. \tag{2.5.3}$$

The system of Eqs. (2.5) should be completed by the boundary conditions. Thus, if surface A consists of two parts A_s and A_d, the boundary conditions are

$$\sigma^+_{ij} n_j = 0, \qquad \boldsymbol{x}\in A_s \tag{2.6.1}$$

$$u_i{}^+ = U_i(\boldsymbol{x}, t), \qquad \boldsymbol{x}\in A_d, \qquad t > 0 \tag{2.6.2}$$

where n_j shows the direction cosines of $\boldsymbol{n}$. The system (2.5) can be reduced in terms of the displacements in the form

$$(\lambda + \mu)\,u^+_{k,ki} + \mu\nabla^2 u_i{}^+ = \gamma\rho_{,i} \tag{2.7}$$

or, in terms of the stresses in the form of the Beltrami-Michell equations

$$\nabla^2\sigma_{ij}^+ + \frac{1}{1+\nu}\sigma_{kk,ij}^+ + \frac{\gamma_d E}{1+E}\left(\frac{1+\nu}{1-\nu}\nabla^2\rho\delta_{ij} + \rho_{,ij}\right) = 0 \tag{2.8}$$

where

$$E = \frac{3\lambda + 2\mu}{\lambda + \mu}, \qquad \nu = \frac{\lambda}{2(\lambda + \mu)}.$$

It is assumed that the diffusion-induced state of strain is "coherent" and that the derived strains e_{ij}^+ should satisfy the compatibility conditions

$$e_{ij,kl}^+ + e_{kl,ij}^+ = e_{jl,ik}^+ + e_{ik,jl}^+. \tag{2.9}$$

2.1 *The ordinary problem*

The uncoupled problem can be reduced to an equivalent problem of body and surface forces *(ordinary problem)* which are specified in terms of the concentration ρ of the original problem. Writing Eq. (2.5.1) in the form

$$\sigma_{ij}' = \lambda e^+\delta_{ij} + 2\mu e_{ij}^+ \tag{2.10}$$

where

$$\sigma_{ij}' = \sigma_{ij}^+ + \gamma\rho\delta_{ij} \tag{2.11}$$

and considering that the stress and strain state of the ordinary problem is expressed by σ_{ij} and e_{ij}^+, respectively, we are seeking body forces X_i and surface forces $\bar{X}_i$ which satisfy the equations

$$\sigma_{ij,i}' + X_j = 0, \qquad \boldsymbol{x} \in V \tag{2.12.1}$$

$$\sigma_{ij}' n_i = \bar{X}_j, \qquad \boldsymbol{x} \in A. \tag{2.12.2}$$

These equations express the equilibrium and boundary condition of the body for the ordinary state. Introducing Eq. (2.11) into (2.12) we obtain

$$X_i = -\gamma\rho_i, \qquad \boldsymbol{x} \in V \tag{2.13.1}$$

$$\bar{X}_i = \gamma\rho n_i, \qquad \boldsymbol{x} \in A. \tag{2.13.2}$$

Therefore, the theorems and the methods which have been established for the ordinary problems, can be used directly for the diffuso-elastic problems. Thus, a direct result is the uniqueness of the solution of the diffuso-elastic problem. Using the fact of existing solutions for the ordinary problems as well as the diffuso-elastic reciprocal theorem which will be established below, immediate solutions for the diffuso-elastic problem can be obtained.

2.2 *The reciprocal theorem*

The reciprocal theorem of classical elasticity will now apply [11]. As the first state of the theorem is taken to be the previous ordinary problem, the second state is a set of surface and body forces $\bar{X}_i''$ and X_i'', respectively, which produce elastic strains e_{ij}'', stresses σ_{ij}'' and displacements u_i''.

From the application of the theorem we obtain

$$\int_A \bar{X}_i''u_i^+ \, dA + \int_V X_i''u_i^+ \, dV = \frac{K}{3} \int_V \rho\sigma'' \, dV \tag{2.14}$$

where σ'' is the trace of the stress tensor of the second state and K is the bulk modulus $\left(K = \lambda + \dfrac{2}{3}\mu\right)$.

We will now consider an interesting application of the preceding theorem. Let the second state be that which is induced by a uniform normal (tensile) loading p'' over the whole surface of the body. Then, at any point in the body, we have

$$\sigma''_{11} = \sigma''_{22} = \sigma''_{33} = p'', \qquad \sigma'' = 3p'' \tag{2.15}$$

and the theorem yields

$$\Delta V^+ = K\gamma\Delta m^+ \tag{2.16}$$

where Δm^+ and ΔV^+ are the changes of mass and volume of the body due to diffusion. Thus, the coefficient of linear expansion γ can be calculated from (2.16), when changes Δm^+ and ΔV^+ are measured experimentally.

3 Concentration distributions

In this Section particular cases of the diffusion problems are studied and analytical solutions of these problem for various initial and boundary condition are presented. Specifically, the one-dimensional diffusion in a sheet plane, the diffusion in a cylinder and the diffusion in an infinite plate are examined.

3.1 The one dimensional problem

Consider the case of diffusion through a plane sheet of thickness l, whose surfaces $x_1 = 0$ and $x_1 = l$ are maintained at constant concentrations ρ_1 and ρ_2, respectively. The initial concentration is taken to be $g(x_1) = 0$, while the initial stress state is given by means of the trace of the stress tensor $\sigma_0(x_1)$. Since the trace of the stress tensor in the linear elasticity with zero body forces is a harmonic function, it follows that

$$\sigma_0(x_1) = p_1 x_1 + p_0 \tag{3.1}$$

where p_1 and p_0 are given real constants. In the one-dimensional case the diffusion equation (2.1.2) can be written

$$(x_1 + \varepsilon)\frac{\partial^2 \rho}{\partial x_1} - (\alpha - 1)\frac{\partial \rho}{\partial x_1} = \frac{\partial^2 \rho}{\partial \tau^2} \tag{3.2}$$

with

$$\varepsilon = \frac{D}{Np_1} + \frac{p_0}{p_1}, \qquad \tau = Np_1 t, \qquad \alpha = \frac{M}{N}$$

when p_1 is a non-zero constant. In the steady state ($\partial\rho/\partial\tau = 0$) from (3.2) it is easy to deduce that the solution ρ_{ss} is given by

$$\frac{\rho_{ss}(x_1) - \rho_1}{\rho_2 - \rho_1} = \frac{\varepsilon^\alpha - (x_1 + \varepsilon)^\alpha}{\varepsilon^\alpha - (l + \varepsilon)^\alpha}. \tag{3.3}$$

By using the method of separation of variables the derived solution of (3.2) has the form

$$\rho(x_1, \tau) = \rho_{ss}(x_1) + (x_1 + \varepsilon)^{\alpha/2} \sum_{m=1}^{\infty} \exp(-\lambda_m{}^2\tau)\, A_m C_{\alpha\alpha}(\lambda_m r_0, \lambda_m r) \tag{3.4}$$

with

$$A_m = \int_{r_0}^{r_1} r f(r)\, C_{\alpha\alpha}(\lambda_m r_0, \lambda_m r)\, dr \Big/ \int_{r_0}^{r_1} r[C_{\alpha\alpha}(\lambda_m r_0, \lambda_m r)]^2\, dr$$

$$f(r) = -\left(\frac{2}{r}\right)^\alpha \rho_{ss}(r)$$

$$r = 2\sqrt{x + \varepsilon}, \quad r_0 = 2\sqrt{\varepsilon}, \quad r_1 = 2\sqrt{1 + \varepsilon},$$

where the function $C_{\mu\nu}(\xi\alpha, \xi b)$ is defined by

$$C_{\mu\nu}(\xi\alpha, \xi b) = J_\mu(\xi\alpha)\, Y_\nu(\xi b) - J_\nu(\xi b)\, Y_\mu(\xi\alpha) \tag{3.5}$$

and $\lambda_\varkappa$ is a positive root of

$$C_{\mu\nu}(\lambda_m r_0, \lambda_m r) = 0. \tag{3.6}$$

In (3.5) $J_\mu(\xi\alpha)$ and $Y_\nu(\xi b)$ are Bessel functions of the first and second kind of order μ and ν, respectively. When the initial concentration is a non-zero function $g(x_1)$, then the function $f(r)$ has the form

$$f(r) = \left(\frac{2}{r}\right)^\alpha [g(r) - \rho_{ss}(r)]. \tag{3.7}$$

3.2 Diffusion in a cylinder

We consider a circular cylinder with inner and outer radii a and b, respectively, which is subjected to hydrostatic pressure P_1 on the internal and P_2 on the external surface. We suppose that the initial concentration is $g(r)$ ($a \leqq r \leqq b$) and the internal and external surfaces are kept at the constant concentration ρ_1 and ρ_2, respectively. In this case the trace of initial stress tensor [11] is

$$\sigma^0 = 2\frac{P_1 a^2 - P_2 b^2}{b^2 - a^2}. \tag{3.8}$$

The solution of Eq. (2.1) in cylindrical polar coordinates r, θ ($x_1 = r\cos\theta$ and $x_2 = r\sin\theta$) is different from that of Carslaw and Jaeger [14] only in the diffusion coefficient D, which is substituted by

$$D^* = D + 2N\frac{P_1 a^2 - P_2 b^2}{b^2 - a^2}. \tag{3.9}$$

3.3 *Diffusion in an infinite and a semi-infinite plate*

From the theory of two-dimensional elasticity it is known that the trace of the stress tensor can be expressed in the form [14]

$$\sigma^0 = \sigma_{11}^0 + \sigma_{22}^0 = W_0(z) + \bar{W}_0(\bar{z}) \tag{3.10}$$

where $W_0(z)$ is a holomorphic function defined in the region of the z-plane ($z = x_1 + ix_2$) corresponding to material.

Setting

$$W(z) = \frac{1}{2} + \beta W_0(z), \tag{3.11}$$

Eq. (2.1.1) in terms of variables z and $\bar{z}$ in steady state is written

$$2[W(z) + \bar{W}(\bar{z})]\frac{\partial^2 \rho}{\partial z\, \partial \bar{z}} + (1 - \alpha)\left(\frac{\partial \rho}{\partial z}\frac{d\bar{W}}{d\bar{z}} + \frac{\partial \rho}{\partial \bar{z}}\frac{dW}{dz}\right) = 0. \tag{3.12}$$

The holomorphic function $W(z)$ can be used to map conformally the region of the z-plane corresponding to material onto a region of the W-plane. Thus, in the W-plane, Eq. (3.12) is written

$$2(W + \bar{W})\frac{\partial^2 \rho}{\partial W\, \partial \bar{W}} + (1 - \alpha)\left(\frac{\partial \rho}{\partial W} + \frac{\partial \rho}{\partial \bar{W}}\right) = 0 \tag{3.13}$$

whose one obvious solution is

$$\rho(W, \bar{W}) = C_1(W + \bar{W})^{\alpha} + C_2 \tag{3.14}$$

where C_1, C_2 are constant coefficients. The previous complex formulation will now be used to determine the concentration field in two interesting problems of mechanics.

Infinite plate with crack. Consider an infinite plate on z-plane containing a rectilinear crack of length $2l$. The crack is symmetrically located on the x_1-axis and its edges are maintaining at a constant concentration ρ_0. When the plate is subjected to infinity with stresses σ_{11}^{∞}, σ_{22}^{∞} the function $W_0(z)$ has the form

$$W_0(z) = \frac{\sigma_{22}^{\infty} z}{\sqrt{z^2 - l^2}} + \frac{1}{2}(\sigma_{11}^{\infty} - \sigma_{22}^{\infty}). \tag{3.15}$$

The concentration is determined from Eq. (3.14) whose unknown coefficients are calculated from the boundary condition of the concentration on the crack edges and from the reasonable assumption that the diffusive flux must be bounded along the crack boundary including the end points. The derived solution is

$$\rho(z, \bar{z}) = \rho_0\big(1 + AX(z, \bar{z})\big)^{\alpha} \tag{3.16}$$

where

$$X(z, \bar{z}) = \frac{z}{\sqrt{z^2 - l^2}} + \frac{\bar{z}}{\sqrt{\bar{z}^2 - l^2}} \tag{3.17.1}$$

$$A = \frac{\beta \sigma_{22}^{\infty}}{1 + \beta(\sigma_{11}^{\infty} - \sigma_{22}^{\infty})}. \tag{3.17.2}$$

Considering the asymptotic behaviour of (3.16) in the neighbourhood of the crack tip we obtain the solution of Unger and Aifantis [16]. In the case where the crack opened by uniform internal pressure p the solution is given by (3.16) with

$$A = \frac{\beta p}{1 - \beta p}. \tag{3.18}$$

Half-plane with concentrated force. When the elastic material occupies the half-plane $x_2 \geqq 0$ of z-plane and a tensile concentrated force P is applied along x_2-axis at the origin of the axis, then the complex potential has the form [14]

$$W_0(z) = \frac{iP}{\pi z}. \tag{3.19}$$

If the boundary $x_2 = 0$ is kept at a constant concentration ρ_0 and there is no initial concentration then in the same manner as previously, we find

$$\rho(z, \bar{z}) = \rho_0 \left[1 + \frac{i\beta P}{\pi} \left(\frac{1}{z} - \frac{1}{\bar{z}} \right) \right]^\alpha \tag{3.20}$$

or, in polar coordinates r, θ $(z = r \exp(i\theta))$

$$\rho(r, \theta) = \rho_0 \left(1 + \frac{2\beta P}{\pi r} \sin\theta \right)^\alpha. \tag{3.21}$$

4 Complex representation of the two-dimensional problem

By using the ordinary problem of Section 3 and according to Muskhelishvili [15] the two-dimensional problem uncouples in terms of two holomorphic functions $W^+(z)$ and $w^+(z)$ as well as of the concentration $\rho(z, \bar{z})$. The components of stresses and displacements in terms of these functions are given by

$$\sigma_{11}^+ + \sigma_{22}^+ = W^+(z) + \bar{W}^+(\bar{z}) - n\gamma\rho(z, \bar{z}) \tag{4.1.1}$$

$$\sigma_{22}^+ - \sigma_{11}^+ + 2i\sigma_{12}^+ = \bar{z}W^{+\prime\prime}(z) + w^{+\prime}(z) - n\gamma \int \frac{\partial \rho}{\partial \bar{z}}\, dz \tag{4.1.2}$$

$$4\mu(u^+ + iv^+) = \varkappa \int W^+(z)\, dz - \bar{z}\bar{W}^+(\bar{z}) - \int \bar{w}^+(\bar{z})\, d\bar{z} + n\gamma \int \rho\, dz \tag{4.1.3}$$

where $n = 2(\varkappa - 1)/(1 + \varkappa)$, $\varkappa = 3 - 4\nu$ for plane strain and $\varkappa = (3 - \nu)/(1 + \nu)$ for generalized plane stress, ν is Poisson's ratio. Thus, the problem has been reduced to a well known expression and for its solution the equally well known methods of complex elasticity can be used. In the following the previous formulation will be applied to a crack problem.

4.1 Infinite plate with straight collinear cracks

Consider an elastic plate, which occupies the z-plane and has a finite number of straight cracks S_m along the x_1-axis with end points a_m, b_m $(m = 1, 2, \ldots, n)$. We will now introduce a new function $\Omega(z)$ instead of $W(z)$ defined by

$$\Omega^+(z) = -\bar{W}^+(z) - z\bar{W}^{+\prime}(z) - \bar{w}^+(z). \tag{4.2}$$

From Eqs. (4.1) and (4.2) we obtain

$$2(\sigma_{22}^{+} - i\sigma_{12}^{+}) = W^{+}(z) - \Omega^{+}(\bar{z}) + (z - \bar{z})\,\bar{W}^{+\prime}(\bar{z}) - n\gamma \int \frac{\partial \rho}{\partial \bar{z}}\, dz \tag{4.3.1}$$

$$4\mu(u^{+} + iv^{+}) = \varkappa \int W^{+}(z)\, dz + \int \Omega^{+}(\bar{z})\, d\bar{z} - (z - \bar{z})\,\bar{W}^{+}(\bar{z}) + n\gamma \int \rho(z, \bar{z})\, dz. \tag{4.3.2}$$

The following boundary condition must hold on the edges of the crack:

$$(\sigma_{22}^{+} - i\sigma_{12}^{+})^{L} = (\sigma_{22}^{+} - i\sigma_{12}^{+})^{R} = 0, \qquad \sigma \text{ on } S \tag{4.4}$$

where S is the union of S_m and L, R denote the boundary values of the function for $y \to +0$, $y \to -0$, respectively. Using the relationships (4.3.1) and (4.4), we arrive at the following boundary value problems to determine the unknown sectionally holomorphic functions:

$$[W^{+}(x)]^{L} - [\Omega^{+}(x)]^{L} = n\gamma R^{L}(x, x) \tag{4.5.1}$$

$$[W^{+}(x)]^{L} - [\Omega^{+}(x)]^{R} = n\gamma R^{R}(x, x) \tag{4.5.2}$$

where

$$R(z, \bar{z}) = \rho(z, \bar{z}) + \int \frac{\partial \rho}{\partial \bar{z}}\, dz. \tag{4.6}$$

Adding and substracting Eqs. (4.5) we have

$$[W^{+}(x) - \Omega^{+}(x)]^{L} + [W^{+}(x) - \Omega^{+}(x)]^{R} = 2n\gamma q_1(x) \tag{4.7.1}$$

$$[W^{+}(x) + \Omega^{+}(x)]^{R} - [W^{+}(x) + \Omega^{+}(x)]^{R} = 2n\gamma q_2(x) \tag{4.7.2}$$

where

$$q_1(x) = R^{L}(x, x) + R^{R}(x, x) \tag{4.8.1}$$

$$q_2(x) = R^{L}(x, x) - R^{R}(x, x). \tag{4.8.2}$$

If the functions $q_1(x)$, $q_2(x)$ satisfy the Hölder condition [15] on S and are bounded at infinity, then the general solution of the boundary problems is given by Muskhelishvili [15]:

$$W^{+}(z) + \Omega^{+}(z) = \frac{2n\gamma}{\pi i} \int_{S} \frac{q_2(t)}{(t - z)}\, dt + R_{\infty} \tag{4.9.1}$$

$$W^{+}(z) - \Omega^{+}(z) = \frac{2n\gamma x(z)}{\pi i} \int_{S} \frac{q_1(t)\, dt}{x^{L}(t)\,(t - 2)} + 2P_n(z)\, x(z) \tag{4.9.2}$$

where

$$x(z) = \{(z - a_1)(z - a_2)\cdots(z - a_n)(z - b_1)(z - b_2)\cdots(z - b_n)\}^{-1/2}$$

$$P(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^{n}.$$

$x(z)$ is the Plemeli function defined on S and that branch for which $\lim_{z \to \infty} z^{n} x(z) = 1$ is considered. The constant R_{∞} and the polynomial coefficients $\alpha_0, \alpha_1, \ldots, \alpha_n$ are calculated by using the one-valued condition of displacements and the behaviour of complex potential at infinity. The behaviour of $W^+(z)$, $\Omega^+(z)$ at infinity can be determined from Eqs. (4.1.1), (4.3.1) taking into account the behaviour of concentration ρ at infinity.

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