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# Fluctuation problem in electromagnetic cascades

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The fluctuation problem in electromagnetic cascades is examined in the light of the new approach to cascade theory and it is shown that the method originally proposed by JANOSSY is best suited to deal with it. A method of obtaining explicit expressions for the second moment of the distribution is given and the differential equations obtained by this method turn out to be simpler and amenable to numerical computation.

## 1. Introduction

The fluctuation problem of electron–photon cascades has been studied extensively in the past few years. Methods of stochastic theory have been applied in various ramifications to obtain the moments of the number distribution at any particular thickness[1]. Recently it has been found[2, 3] that it is more convenient to deal with the distribution of the number of particles produced between 0 and $t$ with reference to the energy at the point of production (herein after referred to as primitive energy). Thus it is possible to define $\pi_i(n_1, n_2, E, E_0; t)$ as the probability $n_1$ electrons and $n_2$ photons are produced between 0 and $t$ in a shower excited by a primary of $i$-th type of energy $E_0$ ($i=1,2$ refer respectively to an electron and a photon), the primitive energy of each of the electrons and photons being above $E$. In an earlier contribution[3], RAMAKRISHNAN and the author have discussed the method of obtaining expressions for the moments of $\pi_i$ using product density techniques. The mean numbers for large and small thicknesses have been obtained on the basis of these solutions[4–6]. It was observed in reference 5 that the theoretical mean numbers are less than those calculated on the basis of recent experimental data on high energy cascades and as a plausible

[1] For a comprehensive review on the subject see RAMAKRISHNAN, A., and P.M. MATHEWS: Progr. Theor. Phys. **11**, 95 (1954).

[2] FAY, H.: Private communication.

[3] RAMAKRISHNAN, A., and S.K. SRINIVASAN: Proc. Ind. Acad. Sci. A **44**, 263 (1956).

[4] SRINIVASAN, S.K., and N.R. RANGANATHAN: Proc. Ind. Acad. Sci. A **45**, 69, 268 (1957).

[5] SRINIVASAN, S.K., J.C. BUTCHER, B.A. CHARTRES and H. MESSEL: Nuovo Cim. **9**, 77 (1958).

[6] SRINIVASAN, S.K.: Unpublished.

argument it was pointed out that at such high energies[7] tridents may play a role. However the fluctuations about the mean must also be studied before we could draw any inference regarding the role of tridents. In this paper we shall discuss a convenient method of calculation of fluctuations

In reference 3, it was shown that the mean square number of electrons produced between 0 and $t$ can be expressed as a double Mellin integral where the integrand contains terms involving Melling transforms of the product density of degree two of photons and a mixed second order product density of electrons and photons at $t$. Mellin transform solution of these product densities have been explicitly obtained by BHABHA and RAMAKRISHNAN[8]. However the number of terms occuring in these expressions is enormously large and the numerical evaluation of the Mellin integral is quite a difficult task. In fact the mean square number of electrons and photons based on the results of reference 8 have been calculated only for fairly large thicknesses[1,9] where a number of terms which differ by an order magnitude can be neglected so that evaluation for small thicknesses is still a problem in spite of fast modern computing devices. We wish to show in this paper that the difficulty could be overcome by dealing with $\pi_i$ directly rather than connect its moments with the so called density functions of different orders.

## 2. Differential equation for $\pi_i$

Let $\pi_i(n, E, E_0; t)$ be the probability that $n$ electrons are produced between 0 and $t$ by a primary of energy $E_0$. As usual, we shall assume that the probability per unit thickness of matter that (i) an electron of energy $E$ radiates a quantum and drops to an energy between $E'$ and $E'+dE'$ is $R_1(E'|E)\,dE'$ and (ii) a photon of energy $E$ annihilates into an electron positron pair one of which has an energy between $E'$ and $E'+dE'$ is $R_2(E'|E)\,dE'$.

When screening is complete, $R_1$ and $R_2$ are given by [10]

$$R_1(E'|E) = \left\{\frac{E-E'}{E} - \left(\frac{4}{3}+\alpha\right)\left(1-\frac{E}{E-E'}\right)\right\}\frac{1}{E}, \tag{1}$$

$$R_2(E'|E) = \left\{1-\left(\frac{4}{3}+\alpha\right)\left(\frac{E'}{E}-\frac{E'^2}{E^2}\right)\right\}\frac{1}{E}. \tag{2}$$

We shall take only these two processes into account and neglect collision loss. In view of the homogeneous nature of the cross sections (1) and (2),

[7] See for example FAY, H.: Nuovo Cim. **5**, 293 (1957).

[8] BHABHA, H. J., and A. RAMAKRISHNAN: Proc. Ind. Acad. Sci. A **32**, 141 (1950).

[9] RAMAKRISHNAN, A., and S.K. SRINIVASAN: Progr. Theor. Phys. **13**, 95 (1955).

[10] BETHE, H.A., and W. HEITLER: Proc. Roy. Soc. Lond. A **146**, 83 (1934).

$\pi_i(n, E, E_0; t)$ is a function only of $E/E_0$ and denoting it by $\pi_i(n, \varepsilon; t)$ $(\varepsilon = E/E_0)$ we obtain (see for example, reference [11])

$$\left.\begin{aligned}\frac{\partial \pi_1(n,\varepsilon;t)}{\partial t} = &-\pi_1(n,\varepsilon;t)\int_0^1 R_1(\varepsilon')\,d\varepsilon' + \\ &+\sum_{m=0}^{\infty}\int_0^1 R_1(\varepsilon')\,\pi_1\left(m,\frac{\varepsilon}{\varepsilon'};t\right)\pi_2\left(n-m,\frac{\varepsilon}{1-\varepsilon'};t\right)d\varepsilon',\end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned}\frac{\partial \pi_2(n,\varepsilon,t)}{\partial t} = &-\pi_2(n,\varepsilon,t)\int_0^1 R_1(\varepsilon')\,d\varepsilon' + \\ &+\int_0^{\varepsilon} R_2(\varepsilon')\,\pi_2\left(n-1,\frac{\varepsilon}{1-\varepsilon'},t\right)d\varepsilon' + \\ &+\sum_{m=0}^{\infty}\int_{\varepsilon}^{1-\varepsilon} R_2(\varepsilon')\,\pi_1\left(m,\frac{\varepsilon}{\varepsilon'},t\right)\pi_2\left(n-m-2,\frac{\varepsilon}{1-\varepsilon'},t\right)d\varepsilon' + \\ &+\int_{1-\varepsilon}^{1} R_2(\varepsilon')\,\pi_1\left(n-1,\frac{\varepsilon}{\varepsilon'},t\right)d\varepsilon'\end{aligned}\right\} \tag{4}$$

with the initial conditions

$$\pi_1(n,\varepsilon,0) = \pi_2(n,\varepsilon,0) = \delta_0^n. \tag{5}$$

We observe that (3) holds good for the entire range $0 \leqq \varepsilon \leqq 1$, while (4) is valid only for the range $0 \leqq \varepsilon \leqq \frac{1}{2}$. For $\varepsilon > \frac{1}{2}$, $\pi_2$ satisfies the equation*

$$\left.\begin{aligned}\frac{\partial \pi_2(n,\varepsilon,t)}{\partial t} = &-\pi_2(n,\varepsilon,t)\int_0^1 R_2(\varepsilon')\,d\varepsilon' + \delta_0^n\int_{1-\varepsilon}^{\varepsilon} R_2(\varepsilon')\,d\varepsilon' + \\ &+\int_0^{1-\varepsilon} R_2(\varepsilon')\,\pi_1\left(n-1,\frac{\varepsilon}{1-\varepsilon'},t\right)d\varepsilon' + \\ &+\int_{\varepsilon}^{1} R_2(\varepsilon')\,\pi_1\left(n-1,\frac{\varepsilon}{\varepsilon'},t\right)d\varepsilon'.\end{aligned}\right\} \tag{4'}$$

Comparing equations (3) and (4) with the corresponding ones in reference [11], we find (3) is identical with the equation of JANOSSY. The difference is brought out by (4) where linear terms are integrated over

* This has been observed by Professor G. MOLIÈRE in a private communication and the author is grateful to him for supplying the correct equation for the special case $\varepsilon \geqq \frac{1}{2}$.

[11] JANOSSY, L.: Proc. Phys. Soc. Lond. A **63**, 241 (1950).

partial ranges. This is due to the situation that at the regeneration point one or two electrons with energy above $\varepsilon$ are produced. If $\varepsilon' = E'/E_0$ falls in the interval $(\varepsilon, 1-\varepsilon)$ then both the electrons produced have an energy above $\varepsilon$ and the second term on the right and side of (4) corresponds to this situation. If however $\varepsilon'$ falls outside the interval $(\varepsilon, 1-\varepsilon)$, only one of the electrons have an energy above $\varepsilon$ and this is taken care of by the last two terms in (4).

### 3. Moments of $\pi_i$

Defining $G_i(u, \varepsilon, t)$ as

$$G_i(u, \varepsilon, t) = \sum_{n=0}^{\infty} u^n \pi_i(n, \varepsilon, t) \tag{6}$$

we obtain

$$\left.\begin{aligned} \frac{\partial G_i(u, \varepsilon, t)}{\partial t} = & - G_i(u, \varepsilon, t) \int_0^1 R_i(\varepsilon')\, d\varepsilon' + \\ & + u^{i-1} \int_0^{\varepsilon} G_{3-i}\left(u, \frac{\varepsilon}{1-\varepsilon'}, t\right) R_i(\varepsilon')\, d\varepsilon' + \\ & + u^{2i-2} \int_{\varepsilon}^{1-\varepsilon} G_1\left(u, \frac{\varepsilon}{\varepsilon'}, t\right) G_{3-i}\left(u, \frac{1-\varepsilon'}{\varepsilon}, t\right) \cdot R_i(\varepsilon')\, d\varepsilon' + \\ & + u^{i-1} \int_{1-\varepsilon}^{1} G_1\left(u, \frac{\varepsilon}{\varepsilon'}, t\right) R_i(\varepsilon')\, d\varepsilon' . \end{aligned}\right\} \tag{7}$$

(7) is again valid for the entire range of $\varepsilon$ only for $i=1$. When $i=2$, (7) covers only the region $0 < \varepsilon \leqq \frac{1}{2}$. For $\varepsilon > \frac{1}{2}$, $G_2(u, \varepsilon, t)$ satisfies the equation

$$\left.\begin{aligned} \frac{\partial G_2(u, \varepsilon, t)}{\partial t} = & - G_2(u, \varepsilon, t) \int_0^1 R_2(\varepsilon')\, d\varepsilon' + \\ & + \int_{1-\varepsilon}^{\varepsilon} R_2(\varepsilon') d\varepsilon' + u \int_0^{1-\varepsilon} R_2(\varepsilon')\, G_1\left(u, \frac{\varepsilon}{1-\varepsilon'}, t\right) d\varepsilon' + \\ & + u \int_{\varepsilon}^{1} R_2(\varepsilon')\, G_1\left(u, \frac{\varepsilon}{\varepsilon'}, t\right) d\varepsilon' . \end{aligned}\right\} \tag{7'}$$

The $m$-th moment of the number of electrons produced between 0 and $t$ is given by

$$\mathsf{E}\{[n_i(\varepsilon, t)]^m\} = \left(u \frac{\partial}{\partial u}\right)^m G_i(u, \varepsilon, t)\Big|_{u=1} . \tag{8}$$

Differentiating both sides of (6) with respect to $u$ at $u=1$, we obtain

$$\left.\begin{aligned}\frac{\partial n_i(\varepsilon,t)}{\partial t} = &- n_i(\varepsilon,t)\int_0^1 R_i(\varepsilon')\,d\varepsilon' + \\ &+ (i-1)\left[\int_0^{1-\varepsilon} R_i(\varepsilon')\,d\varepsilon' + \int_0^{\varepsilon} R_i(\varepsilon')\,d\varepsilon'\right] + \\ &+ \int_{\varepsilon}^{1} R_i(\varepsilon')\,n_1\left(\frac{\varepsilon}{\varepsilon'},t\right)d\varepsilon' + \\ &+ \int_0^{1-\varepsilon} R_i(\varepsilon')\,n_{3-i}\left(\frac{\varepsilon}{1-\varepsilon'},t\right)d\varepsilon'.\end{aligned}\right\} \quad (9)^{\star}$$

Writing

$$\left.\frac{\partial^2 G_i(u,\varepsilon,t)}{\partial u^2}\right|_{u=1} = N_i(\varepsilon,t) \quad (10)$$

we obtain by double differentiation of (6) at $u=1$

$$\left.\begin{aligned}\frac{\partial N_1(\varepsilon,t)}{\partial t} = &- N_1(\varepsilon,t)\int_0^1 R_1(\varepsilon')\,d\varepsilon' + \int_{\varepsilon}^{1} R_1(\varepsilon')\,N_1\left(\frac{\varepsilon}{\varepsilon'},t\right)d\varepsilon' + \\ &+ \int_0^{1-\varepsilon} R_1(\varepsilon')\,N_2\left(\frac{\varepsilon}{1-\varepsilon'},t\right)d\varepsilon' + L_1(\varepsilon,t),\end{aligned}\right\} \quad (11)$$

$$\left.\begin{aligned}\frac{\partial N_2(\varepsilon,t)}{\partial t} = &- N_2(\varepsilon,t)\int_0^1 R_2(\varepsilon')\,d\varepsilon' + \int_{\varepsilon}^{1} R_2(\varepsilon')\,N_1\left(\frac{\varepsilon}{\varepsilon'},t\right)d\varepsilon' + \\ &+ \int_0^{1-\varepsilon} R_2(\varepsilon')\,N_1\left(\frac{\varepsilon}{1-\varepsilon'},t\right)d\varepsilon' + L_2(\varepsilon,t)\end{aligned}\right\} \quad (12)$$

where $L_1(\varepsilon,t)$ and $L_2(\varepsilon,t)$ are given by

$$L_1(\varepsilon,t) = 2\int_0^1 n_1\left(\frac{\varepsilon}{\varepsilon'},t\right)n_2\left(\frac{\varepsilon}{1-\varepsilon'},t\right)R_1(\varepsilon')\,d\varepsilon', \quad (13)$$

$$\left.\begin{aligned}L_2(\varepsilon,t) = &\,2\int_0^1 n_1\left(\frac{\varepsilon}{\varepsilon'},t\right)n_1\left(\frac{\varepsilon}{1-\varepsilon'},t\right)R_2(\varepsilon')\,d\varepsilon' + \\ &+ 2\int_{\varepsilon}^{1}\left[n_1\left(\frac{\varepsilon}{\varepsilon'},t\right)+n_1\left(\frac{\varepsilon}{1-\varepsilon'},t\right)\right]R_2(\varepsilon')\,d\varepsilon' + \\ &+ 2\int_0^{1-\varepsilon}\left[n_1\left(\frac{\varepsilon}{\varepsilon'},t\right)+n_1\left(\frac{\varepsilon}{1-\varepsilon'},t\right)\right]R_2(\varepsilon')\,d\varepsilon' + \\ &+ 2\int_{\varepsilon}^{1-\varepsilon} R_2(\varepsilon')\,d\varepsilon'\cdot\left[1-H\left(\varepsilon-\frac{1}{2}\right)\right]\end{aligned}\right\} \quad (14)$$

* For convenience of notation we shall drop the expectation symbol **E**.

where $H(\lambda)$ is the Heaviside unit function. (9) can be easily solved by Mellin transform technique. Defining

$$p_i(s,t) = \int_0^1 \varepsilon^{s-1} n_i(\varepsilon,t)\, d\varepsilon \tag{15}$$

we obtain

$$\frac{\partial p_1(s,t)}{\partial t} = -A(s)\, p_1(s,t) + C(s)\, p_2(s,t). \tag{16}$$

$$\frac{\partial p_2(s,t)}{\partial t} = -D p_2(s,t) + B(s)\, p_1(s,t) + \frac{B(s)}{s} \tag{17}$$

where $A$, $B$, $C$, $D$ are given by (see for example reference [11])

$$\left.\begin{aligned} A(s) &= \int_0^1 R_1(\varepsilon)\,(1-\varepsilon^s)\, d\varepsilon, & B(s) &= 2\int_0^1 R_2(\varepsilon)\,\varepsilon^s\, d\varepsilon \\ C(s) &= \int_0^1 R_1(\varepsilon)\,(1-\varepsilon)^s\, d\varepsilon, & D &= \int_0^1 R_2(\varepsilon)\, d\varepsilon. \end{aligned}\right\} \tag{18}$$

$A$, $B$, $C$, $D$ have been explicitly obtained by BHABHA and CHAKRABARTI[12] and tabulated numerically by JANOSSY and MESSEL[13].

(16) and (17) can be solved by the use of matrix calculus (see for example LEFCHETZ[14]). The explicit solution is given by

$$p_1(s,t) = \frac{B(s)\,C(s)}{s\{\mu(s)-\lambda(s)\}}\left[\frac{1-e^{-\lambda(s)t}}{\lambda(s)} - \frac{1-e^{-\mu(s)t}}{\mu(s)}\right] \tag{19}$$

$$p_2(s,t) = \frac{B(s)}{s\{\mu(s)-\lambda(s)\}}\left[\frac{\mu(s)-D}{\lambda(s)}\left(1-e^{-\lambda(s)t}\right) + \frac{D-\lambda(s)}{\mu(s)}\left(1-e^{-\mu(s)t}\right)\right] \tag{20}$$

where $\lambda(s)$ and $\mu(s)$ are eigen values of the matrix

$$\begin{pmatrix} -A(s) & C(s) \\ B(s) & -D \end{pmatrix}.$$

The mean number of electrons produced between 0 and $t$ is given by

$$n_1(\varepsilon,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\varepsilon^{-s} B(s)\, C(s)}{S[\mu(s)-\lambda(s)]}\left\{\frac{1-e^{-\lambda(s)t}}{\lambda(s)} - \frac{1-e^{-\mu(s)t}}{\mu(s)}\right\} ds. \tag{21}$$

$$\left.\begin{aligned} n_2(\varepsilon,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} &\frac{\varepsilon^{-s} B(s)}{S[\mu(s)-\lambda(s)]} \times \\ &\times \left\{\frac{\mu(s)-D}{\lambda(s)}\left(1-e^{-\lambda(s)t}\right) + \frac{D-\lambda(s)}{\mu(s)}\left(1-e^{-\mu(s)t}\right\}\right. ds \end{aligned}\right\} \tag{22}$$

in agreement with the results obtained earlier by the author using product density techniques. The numerical evaluation of $n_1(\varepsilon,t)$ for small values of $t$ has been discussed in reference[5].

[12] BHABHA, H. J., and S. K. CHAKRABARTI: Proc. Roy. Soc. **181**, 267 (1942).

[13] JANOSSY LEONIE, and H. MESSEL: Proc. Roy. Irish Acad. Sci. **54** A, 217 (1951).

[14] LEFCHETZ S., Lectures on Differential Equations: Princeton University Press (1947).

### 4. Evaluation of mean square

To obtain the mean square number we must deal with (11) and (12). Defining $P_i(s,t)$ as the Mellin transform of $N_i(\varepsilon,t)$ with respect to $\varepsilon$ we have

$$\frac{\partial P_1(s,t)}{\partial t} = -A(s)\,P_1(s,t) + C(s)\,P_2(s,t) + L_1(s,t), \tag{23}$$

$$\frac{\partial P_2(s,t)}{\partial t} = -D\,P_2(s,t) + B(s)\,P_1(s,t) + L_2(s,t). \tag{24}$$*

We observe that (23) and (24) are similar to (16) and (17), the difference being only in the inhomogeneous terms. The complete solution for $P_1$ and $P_2$ can be written in the matrix notation as

$$\begin{pmatrix} P_1(s,t) \\ P_2(s,t) \end{pmatrix} = M(s)\,\Lambda(s,t) \int_0^t \Lambda^{-1}(s,t')\,M^{-1}(s) \begin{pmatrix} L_1(s,t') \\ L_2(s,t') \end{pmatrix} dt' \tag{25}$$

where $M$ and $\Lambda$ are defined by

$$M(s) = \begin{pmatrix} -C(s) & -C(s) \\ -A(s)+\lambda(s) & -A(s)+\mu(s) \end{pmatrix}, \tag{26}$$

$$\Lambda(s,t) = \begin{pmatrix} e^{-\lambda(s)t} & 0 \\ 0 & e^{-\mu(s)t} \end{pmatrix}. \tag{27}$$

Finally the mean square number of electrons with energy above $\varepsilon$ can be expressed as

$$\mathsf{E}\{[n_i(\varepsilon,t)]^2\} = \mathsf{E}\{n_i(\varepsilon,t)\} + \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} P_i(s,t)\,\varepsilon^{-s}\,ds. \tag{28}$$

As has been pointed out in section 1, the first term offers no difficulty in numerical computation and it has been evaluated for small and fairly large thicknesses. To evaluate the second term, we can adopt the following procedure. First a table of mean numbers for various $\varepsilon$ and $t$ can be formed. This can be done even with the help of desk machines. Once we are in possession of these tables $L_1(s,t)$ and $L_2(s,t)$ can be tabulated. Using these it is not very difficult to tabulate the right hand side of (25) against $s$ and $t$. The contour integral on the right hand side of (28) can be calculated either directly [see for example reference (4) or by the method of steepest descents. For fairly small thickness it is also wothwhile dealing with equations (11) and (12) directly especially when one has access to a digital computer. We propose to evaluate the Mellin integral occuring in equation (28) and the numerical mean square numbers for various values of $\varepsilon$ and $t$ will be presented in a subsequent contribution wherein the contributions from multiple processes like tridents and double bremmstrahlung will also be estimated.

* We have used the same symbol $L_i(s,t)$ for the Mellin Transform of $L_i(\varepsilon,t)$.