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# On the Connection between the One-Dimensional $S = 1/2$ Heisenberg Chain and Haldane–Shastry Model

**V. I. Inozemtsev**[1]

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Extra integrals of motion and the Lax representation are found for interacting spin systems with the Hamiltonian $H = (J/2) \sum_{j,k=1, j \neq k}^{N} \mathcal{P}(j-k)\, \boldsymbol{\sigma}_j \boldsymbol{\sigma}_k$, where one of the periods of the Weierstrass $\mathcal{P}$ function is equal to $N$. The Heisenberg and Haldane–Shastry chains appear as limiting cases of these systems at some values of the second period. The simplest eigenvectors and eigenvalues of $H$ corresponding to the scattering of two spin waves are presented explicitly for these finite-dimensional systems and for their infinite-dimensional version.

**KEY WORDS:** Integrability; spin chains; magnons; elliptic functions.

## 1. INTRODUCTION

This paper is devoted to the study of the problem of the integrability of one-dimensional, $S = 1/2$ spin chains with the Hamiltonian

$$H = \frac{J}{2} \sum_{\substack{j,k=1 \\ j \neq k}}^{N} h(j-k)\, \boldsymbol{\sigma}_j \boldsymbol{\sigma}_k, \qquad h(x) = h(-x), \quad x \in \mathbb{Z} \tag{1}$$

which have long been used as a model of ferromagnetism and antiferromagnetism. The simplest possible model of the type (1) is the famous periodic Heisenberg chain[(1)] with interaction only between nearest neighbors,

$$h(x) = \delta_{1x} + \delta_{N-1,x}, \qquad 0 < x < N \tag{2}$$

It is well known that this model can be included in the Yang–Baxter scheme and has a transfer matrix with a dependence on a complex

[1] Laboratory of Theoretical Physics, JINR, Moscow, USSR.



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parameter. All the local integrals of motion can be generated as derivatives of the logarithm of the transfer matrix with respect to this parameter evaluated at a fixed point.[2] The spectrum is relatively complicated and can be obtained by solving the set of transcendental equations of the Bethe ansatz.

Recently Haldane[3] and Shastry[4] have constructed a number of eigenvectors of the Hamiltonian (1) with the "potential"

$$h(x) = \frac{\pi^2}{N^2 \sin^2(\pi x/N)} \tag{3}$$

The spectrum of this model seems to be completely equidistant, and most of the energy levels are highly degenerate. There is no doubt about the integrability of such a system, but the extra integrals of motion have not been found.

Nothing is known about the analog of the transfer matrix and the connection of the model with the Yang–Baxter equations.

It is natural to suppose that the integrability of spin-1/2 chains like (1) and of one-dimensional systems of interacting particles in classical mechanics is based on essentially the same Lie-algebraic grounds. One can expect a deep analogy between them, as mentioned also in ref. 4. One of the purposes of this paper is to exploit methods known in classical dynamics for the investigation of quantum systems (1). I show that the "potentials" (2) and (3) are connected in a simple but unusual way. The systems (1) share some common properties with them. Some extra integrals of motion for the Haldane–Shastry model are also presented.

## 2. THE LAX REPRESENTATION

The integrability in classical dynamics in most cases is associated with the existence of the Lax representation of the equations of motion, i.e., the equivalence of these equations to the bilinear matrix relation

$$\frac{dL}{dt} = \{H_{\text{cl}}, L\} = [L, M] \tag{4}$$

where $L$ and $M$ are quadratic (possibly infinite) matrices depending on dynamical variables, $[\cdots]$ is the matrix commutator, $H_{\text{cl}}$ is the classical Hamiltonian, and $\{\cdots\}$ is the Poisson bracket. As a consequence of (4), all the invariants of $L$, for example, $I_k = \text{tr}(L^k)$, $k \in \mathbb{Z}$, belong to the variety of classical integrals of motion.

For systems of particles interacting with each other through pair potentials, the structure of the matrices $L$ and $M$ was established in ref. 5.

In the case of interacting spins it is natural to construct operator-valued matrices $L$ and $M$ obeying the quantum analog of Eq. (4),

$$[H, L]_{jk} = [L, M]_{jk} = \sum_l (L_{jl} M_{lk} - M_{jl} L_{lk}) \tag{5}$$

where the elements of the matrix $[H, L]$ on the left-hand side are commutators of the Hamiltonian and matrix elements of $L$.[2] The proper modification of the classical ansatz[5] in this case is the following: the dimensions of $L$ and $M$ equal the number of spins $N$, and

$$\begin{aligned} L_{jk} &= (1-\delta_{jk})\, f(j-k)(1+\boldsymbol{\sigma}_j \boldsymbol{\sigma}_k) \\ M_{jk} &= (1+\boldsymbol{\sigma}_j \boldsymbol{\sigma}_k)(1-\delta_{jk})\, g(j-k) + \delta_{jk} \sum_{s \neq j}^{N} z(j-s)(1+\boldsymbol{\sigma}_j \boldsymbol{\sigma}_s) \end{aligned} \tag{6}$$

where $\delta$ is the usual Kronecker symbol (all the diagonal elements of $L$ are equal to zero), and $f$, $g$, and $z$ are unknown functions of the argument $x \in \mathbb{Z}$. It is easy to show by direct substitution of (1) and (6) into (5) that the "quantum Lax representation" exists if the following conditions are satisfied for all nonzero $x, y \in \mathbb{Z}$:

$$z(x) = -h(x) \tag{7a}$$

$$f(x)\, g(y) - f(y)\, g(x) = f(x+y)[h(y) - h(x)] \tag{7b}$$

$$f(x)\, g(-x) - f(-x)\, g(x) = f(x+N)\, g(-x-N) - f(-x-N)\, g(x+N) \tag{7c}$$

The first two conditions appear also in the classical theory, where the arguments $x$ and $y$ are arbitrary real or complex numbers. The last condition of periodicity, (7c), appears only for the spin chains and is completely absent for continuum systems.

The general solution to (7a) and (7b) is well known.[6] Up to a trivial exponential factor, it is given by the formulas

$$\begin{aligned} h(x) = -z(x) = -f(x)\, f(-x) + \text{const} &= \mathscr{P}(x) + \text{const} \\ g(x) &= -\frac{df(x)}{dx} \end{aligned} \tag{8a}$$

$$f(x) = \frac{\sigma(x-\alpha)}{\sigma(x)\, \sigma(\alpha)} \exp[x\zeta(\alpha)] \tag{8b}$$

[2] Note that (5) differs slightly from the equation of Calogero *et al.*,[8] who used the definition $[L, M]_{jk} = \frac{1}{2} \sum_l \{L_{jl}, M_{lk}\} - \{M_{jl}, L_{lk}\}$.

where $\mathscr{P}(x)$, $\zeta(x)$, and $\sigma(x)$ are the usual Weierstrass elliptic functions

$$\mathscr{P}(x)=\frac{1}{x^2}+\sum_{\gamma\in\Gamma}{}'\left[\frac{1}{(x-\gamma)^2}-\frac{1}{\gamma^2}\right]$$

$$\zeta'(x)=-\mathscr{P}(x),\qquad \zeta(x)-\frac{1}{x}\to 0\quad\text{at}\quad x\to 0 \tag{9}$$

$$\sigma'(x)=\zeta(x)\,\sigma(x),\qquad \frac{\sigma(x)}{x}\to 1\quad\text{at}\quad x\to 0$$

The sum in (9) runs over all points of the lattice $\Gamma$ on the complex plane, $\Gamma=\{m_1\omega_1+m_2\omega_2\}$ $[m_1,\ m_2\in\mathbb{Z};\ \omega_1,\ \omega_2\in\mathbb{C};\ \mathrm{Im}(\omega_2/\omega_1)\neq 0]$ except for the origin of the coordinate system $m_1=m_2=0$. The "spectral parameter" $\alpha$ is defined on a complex torus $\mathbb{C}/\Gamma$.

The function (8b) also obeys the elliptic Lamé equation.[6] The last condition (7c) is satisfied if and only if one of the periods of the Weierstrass functions (for example, $\omega_1$) is equal to $N$. The choice of the second period $\omega_2$ in the imaginary axis

$$\omega_2\equiv\omega=i\kappa,\qquad \mathrm{Im}\,\kappa=0 \tag{10}$$

guarantees that $\mathscr{P}(x)$ is real at real $x$. For definiteness we shall choose $\kappa$ to be positive. Finally, we have shown that the systems (1) with a real "potential" $h(x)=\mathscr{P}(x)$ depending on an arbitrary real parameter $\kappa$ have a "quantum Lax representation" of the type (5)–(8).

Let us consider some limiting situations. As the second period $\omega\to\infty$, the asymptotic behavior of the $\mathscr{P}$ function is

$$\mathscr{P}(x)\,|_{\omega\to\infty}=\frac{\pi^2}{N^2}\left(\frac{1}{\sin^2(\pi x/N)}-\frac{1}{3}\right) \tag{11}$$

and we obtain, up to a trivial term proportional to the square of the total spin $\mathbf{S}$ commuting with all the Hamiltonians (1), the Haldane–Shastry chain. Another situation when the Weierstrass functions degenerate into trigonometric ones is the limit of a small second period. One finds

$$\mathscr{P}(x)\,|_{\omega\to 0}=\frac{\pi^2}{\kappa^2}\left[\frac{1}{3}+4(e^{-(2\pi/\kappa)\,|x|}+e^{-(2\pi/\kappa)\,|N-x|}+e^{-(2\pi/\kappa)\,|N+x|})\right]$$
$$+\ O(e^{-(4\pi/\kappa)\,|x|}+e^{-(4\pi/\kappa)\,|N-x|}+e^{-(4\pi/\kappa)\,|N+x|}),\quad |x|<N,\ x\in\mathbb{Z}$$

By adding to the Hamiltonian (1) with $h(x)=\mathscr{P}(x)$ the term

$-(J\pi^2/6\kappa^2)(4\mathbf{S}^2-3N)$, performing the "renormalization" of the constant $J$ in (1), $J \to (J\kappa^2/4\pi^2)\exp(2\pi/\kappa)$, and taking the limit $\omega \to 0$, we obtain

$$h_0(x) = \lim_{\omega \to 0}\left(\frac{\kappa^2}{4\pi^2}\exp\frac{2\pi}{\kappa}\right)\left[\mathscr{P}(x) - \frac{\pi^2}{3\kappa^2}\right]$$
$$= \delta_{1x} + \delta_{N-1,x}, \qquad x \in \mathbb{Z}, \quad 0 < x < N$$

which is the "potential" of the periodic Heisenberg chain. We see that both models (2) and (3) can be obtained from (8a) as some limits and also have a Lax representation of the type (5)–(8). The situation is completely analogous to the case of continuum classical models where the periodic Toda and Sutherland particle systems can be treated as the limits of systems with interaction through an elliptic potential.[7]

Finally, when the number of spins and, consequently, the real period of $\mathscr{P}$ tend to infinity,

$$\mathscr{P}(x)\,|_{N\to\infty} = \frac{\pi^2}{\kappa^2}\left(\frac{1}{\sinh^2(\pi x/\kappa)} + \frac{1}{3}\right)$$

and we get a model for an infinite one-dimensional magnetic chain with short-range interaction depending on the parameter $\kappa$,

$$h_\infty(x) = \frac{\pi^2}{\kappa^2\sinh^2(\pi x/\kappa)} \tag{12}$$

Taking the limit $\kappa \to 0$ after a trivial renormalization of $J$ in (1), $J \to (J\kappa^2/4\pi^2)\,e^{2\pi/\kappa}$, we obtain an infinite Heisenberg chain treated by Bethe.[1]

## 3. THE EXTRA INTEGRALS OF MOTION

In contrast to the classical models, the existence of the Lax matrices does not guarantee that the invariants of $L$ would be integrals of motion in the quantum case. For a matrix $L$ of the form (6) the situation is even more pessimistic: it is easy to show that the first two invariants $\mathrm{tr}(L^k)$ are trivial $c$-numbers, i.e., they do not depend on spin operators $\{\boldsymbol{\sigma}_j\}$. One needs a new way to construct nontrivial integrals.

Let us consider the $2N \times 2N$ operator-valued matrix

$$(A)_{jk,\alpha\beta} = (1-\delta_{jk})\,f(j-k)(t_j + t_k)_{\alpha\beta} \tag{13}$$

where $t_j = \frac{1}{2}(\mathbb{1} + \boldsymbol{\sigma}_0\boldsymbol{\sigma}_j)$ and the Greek indices of $A$ stand for elements of the extra Pauli matrices $\{\boldsymbol{\sigma}_0\}$. The matrices $t_j$ have the usual properties, $t_j^2 = \mathbb{1}$,

$\mathrm{tr}_{(0)}\, t_j t_k = \frac{1}{2}(1 + \boldsymbol{\sigma}_j \boldsymbol{\sigma}_k)$ ($\mathrm{tr}_{(0)}$ denotes the trace over the indices of $\boldsymbol{\sigma}_0$, and the multiplication of $t$'s is performed so that $\{\boldsymbol{\sigma}_j\}$ are treated as operator coefficients of $\boldsymbol{\sigma}_0$). For the operator (13) there is no analog of the Lax equation (5). However, it is easy to show that, up to the square of the total spin $\mathbf{S}^2$,

$$\mathrm{Tr}\, \Lambda^2 \simeq H = \frac{1}{2} \sum_{j,k=1, j\neq k}^{N} h(j-k)\, \boldsymbol{\sigma}_j \boldsymbol{\sigma}_k, \qquad h(j-k) = f(j-k)\, f(k-j)$$

where Tr denotes the trace over both the Latin and Greek indices of $\Lambda$. The calculation of the next invariant of $\Lambda$, $\mathrm{Tr}\, \Lambda^3$, gives, up to a constant additive term,

$$\begin{aligned}\mathrm{Tr}\, \Lambda^3 \simeq & -\frac{i}{2} \sum_{j\neq k\neq l}^{N} f(j-k)\, f(k-l)\, f(l-j)(\sigma_j \sigma_k \sigma_l) \\ & + \frac{7}{4} \sum_{j\neq k}^{N} \boldsymbol{\sigma}_j \boldsymbol{\sigma}_k \\ & \times \sum_{l\neq j,k}^{N} [f(j-k)\, f(k-l)\, f(l-j) + f(k-j)\, f(l-k)\, f(j-l)]\end{aligned} \tag{14}$$

where the operator $(\sigma_j \sigma_k \sigma_l) \equiv \boldsymbol{\sigma}_j \cdot (\boldsymbol{\sigma}_k \times \boldsymbol{\sigma}_l)$ is completely antisymmetric in the indices $(jkl)$.

Direct calculation of the commutator of these invariants gives the following result: if $f(x)$ and $h(x)$ obey the conditions (7b) and (7c) guaranteeing the existence of the Lax representation (5), then

$$[\mathrm{Tr}\, \Lambda^2, \mathrm{Tr}\, \Lambda^3] \equiv 0 \tag{15}$$

The terms quartic in the spin operators disappear in the commutator if the functional equation (7b) is satisfied. The terms of third and second order in spin operators are absent if the periodicity (7c) also holds.

The use of an explicit form of $f(x)$ in (8b) simplifies (14). With the help of addition theorems for $\mathscr{P}$, $\zeta$ functions and the sigma-function formulas, we have

$$\frac{\sigma(x-\alpha)\, \sigma(x+\alpha)}{\sigma^2(x)\, \sigma^2(\alpha)} = \begin{vmatrix} 1 & \mathscr{P}(x) \\ 1 & \mathscr{P}(\alpha) \end{vmatrix}$$

$$\frac{\sigma(x-\alpha)\, \sigma(y-\alpha)\, \sigma(x-y)}{\sigma^3(x)\, \sigma^3(y)\, \sigma^3(\alpha)} = \frac{1}{2} \begin{vmatrix} 1 & \mathscr{P}(x) & \mathscr{P}'(x) \\ 1 & \mathscr{P}(y) & \mathscr{P}'(y) \\ 1 & \mathscr{P}(\alpha) & \mathscr{P}'(\alpha) \end{vmatrix}$$

One can show that the second term in (14) is proportional to the square of S, and

$$\operatorname{Tr} \Lambda^3 \simeq -\frac{i}{2}\,[\hat{I}_1 \mathscr{P}(\alpha) + \hat{I}_2] + \text{const} \cdot \mathbf{S}^2 \tag{16}$$

where

$$\hat{I}_1 = \sum_{j \neq k \neq l}^{N} [\zeta(j-k) + \zeta(k-l) + \zeta(l-j)](\sigma_j \sigma_k \sigma_l)$$

$$\hat{I}_2 = \sum_{j \neq k \neq l}^{N} \{2[\zeta(j-k) + \zeta(k-l) + \zeta(l-j)]^3 + \mathscr{P}'(j-k) + \mathscr{P}'(k-l) + \mathscr{P}'(l-j)\}(\sigma_j \sigma_k \sigma_l)$$

Both the operators $\hat{I}_1$ and $\hat{I}_2$ commute with the Hamiltonian because of Eqs. (15) and (16) and the arbitrariness of the "spectral parameter" $\alpha$. They are functionally independent. So the spin models are principally different at this point from classical particle systems, where the trace of the $(k+1)$th power of the $L$ matrix contains only one integral independent of the integrals in traces of the first $k$ powers of $L$.

For the trigonometric degeneration corresponding to the Haldane–Shastry model there are simpler combinations of the limits of $\hat{I}_1$, $\hat{I}_2$:

$$\hat{I}_s = \sum_{j \neq k \neq l}^{N} \varphi_s(j-k)\, \varphi_s(k-l)\, \varphi_s(l-j)(\sigma_j \sigma_k \sigma_l), \qquad s = 1, 2$$

$$\varphi_1(x) = \coth \frac{\pi x}{N}, \qquad \varphi_2(x) = \left(\sinh \frac{\pi x}{N}\right)^{-1}$$

So, the first four terms of the decomposition of the operator

$$\tau(\lambda, \alpha) = \operatorname{Tr}\{\exp[\lambda \Lambda(\alpha)]\} \tag{17}$$

in the parameter $\lambda$ give the integrals of motion for the model with the "potential" (8a). It is likely that (17) can be treated as the generating function of these integrals depending on two parameters, $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{C}/\Gamma$. One may suppose that this operator is a formal analog of the transfer matrix for this model (and the Haldane–Shastry model as a limiting case). There is not yet a full proof of this hypothesis finally confirming the integrability of the model.

## 4. THE SIMPLEST EIGENVECTORS

Here we shall consider only the ferromagnetic case and investigate the state vectors corresponding to one or two spin waves. Let us denote by $|0\rangle$

the state in which all spins have the same projection on the $Z$ axis. Let the operator $a_j^+$ transform $|0\rangle$ to the state in which the sign of the projection of the $j$th spin on the $Z$ axis is opposite. It is convenient to begin the consideration with a slightly simpler case of the infinite chain (12).

We shall use a Hamiltonian differing from (1) and (12) by a constant term,

$$\tilde{H}_\infty = -\frac{1}{2} \sum_{\substack{j,k=-\infty \\ j \neq k}}^{\infty} \frac{\pi^2}{\kappa^2 \sinh^2[(\pi/\kappa)(j-k)]} \left( \frac{\boldsymbol{\sigma}_j \boldsymbol{\sigma}_k - 1}{2} \right), \qquad \tilde{H}_\infty |0\rangle = 0 \tag{18}$$

The calculation of the energy of a spin wave with momentum $p$,

$$\psi_p = \sum_{k=-\infty}^{\infty} \exp(ipk)\, a_k^+ |0\rangle \tag{19}$$

is based on the formula

$$\begin{aligned} F(z) &= \sum_{k=-\infty}^{\infty} \frac{\pi^2 e^{ikp}}{\kappa^2 \{\sinh[(\pi/\kappa)(k+z)]\}^2} \\ &= -\frac{\tilde{\sigma}(z+r_p)}{\tilde{\sigma}(z-r_p)} \exp\left[\frac{pz}{\pi} \tilde{\zeta}\left(\frac{\omega}{2}\right)\right] \\ &\quad \times \left\{ \tilde{\mathscr{P}}(z) - \tilde{\mathscr{P}}(r_p) + \left[\tilde{\zeta}(r_p) - \frac{2r_p}{\omega} \tilde{\zeta}\left(\frac{\omega}{2}\right)\right] \right. \\ &\quad \times \left. \left( \frac{\tilde{\mathscr{P}}'(z) - \tilde{\mathscr{P}}'(r_p)}{\tilde{\mathscr{P}}(z) - \tilde{\mathscr{P}}(r_p)} - \frac{\tilde{\mathscr{P}}''(r_p)}{\tilde{\mathscr{P}}'(r_p)} \right) \right\} \end{aligned} \tag{20}$$

Henceforth $\omega = i\kappa$ and $r_p = -\omega p/4\pi$; $\tilde{\mathscr{P}}$, $\tilde{\zeta}$, and $\tilde{\sigma}$ are Weierstrass functions with periods $(1, \omega)$. The derivation of (20) is based on the quasiperiodicity of the sum on its left-hand side,

$$F(z+\omega) = F(z), \qquad F(z+1) = \exp(-ip)\, F(z)$$

on the structure of its only singularity at the point $z=0$ on a torus obtained by a factorization of a complex $z$ plane on the lattice of periods $(1, \omega)$, and on the Liouville theorem for elliptic functions. The substitution of (19) into the equation $\tilde{H}_\infty \psi_p = \varepsilon_p^{(1)} \psi_p$ gives

$$\begin{aligned} \varepsilon_p^{(1)} &= \tilde{\mathscr{P}}(r_p) + \frac{\tilde{\mathscr{P}}''(r_p)}{\tilde{\mathscr{P}}'(r_p)} \left[\tilde{\zeta}(r_p) - \frac{2r_p}{\omega} \tilde{\zeta}\left(\frac{\omega}{2}\right)\right] \\ &\quad + 2\left[\tilde{\zeta}(r_p) - \frac{2r_p}{\omega} \tilde{\zeta}\left(\frac{\omega}{2}\right)\right]^2 + \frac{2}{\omega} \tilde{\zeta}\left(\frac{\omega}{2}\right) \end{aligned} \tag{21}$$

Taking the limit $\kappa \to 0$, after the multiplication of (21) by $(\kappa^2/4\pi^2)\exp(2\pi/\kappa)$ we get the standard dispersion relation for the spin wave in an infinite Heisenberg chain.

Before constructing two-magnon states, note that formula (20) admits the following evident generalization ($l \in \mathbb{Z}$):

$$\sum_{k=-\infty}^{\infty} \frac{\pi^2 e^{ikp}}{\kappa^2\{\sinh[(\pi/\kappa)(k+z)]\}^2} \coth\frac{\pi}{\kappa}(k+l+z)$$
$$= -\frac{\tilde{\sigma}(z+r_p)}{\tilde{\sigma}(z-r_p)} \coth\left(\frac{\pi l}{\kappa}\right) \exp\left[\frac{pz}{\pi}\tilde{\zeta}\left(\frac{\omega}{2}\right)\right]$$
$$\times\left\{\tilde{\mathscr{P}}(z)-\tilde{\mathscr{P}}(r_p)+\left[\tilde{\zeta}(r_p)-\frac{2r_p}{\omega}\tilde{\zeta}\left(\frac{\omega}{2}\right)+\frac{\pi}{\kappa\sinh(2\pi l/\kappa)}(1-e^{-ipl})\right]\right.$$
$$\left.\times\left(\frac{\tilde{\mathscr{P}}'(z)-\tilde{\mathscr{P}}'(r_p)}{\tilde{\mathscr{P}}(z)-\tilde{\mathscr{P}}(r_p)}-\frac{\tilde{\mathscr{P}}''(r_p)}{\tilde{\mathscr{P}}'(r_p)}\right)\right\} \tag{22}$$

The scheme of the proof for (22) is the same as for (20). The structure of this formula shows that the two-magnon state is described by the vector

$$\psi^{(\infty)}_{p_1p_2} = \sum_{\substack{k_1,k_2=-\infty \\ k_1\neq k_2}}^{\infty} \left[e^{i(p_1k_1+p_2k_2)}\sinh\frac{\pi}{\kappa}\left(k_1-k_2+\frac{\kappa}{\pi}\gamma\right)\right.$$
$$\left.+\, e^{i(p_2k_1+p_1k_2)}\sinh\frac{\pi}{\kappa}\left(k_1-k_2-\frac{\kappa}{\pi}\gamma\right)\right]$$
$$\times\left[\sinh\frac{\pi}{\kappa}(k_1-k_2)\right]^{-1} a^+_{k_1}a^+_{k_2}\,|0\rangle \tag{23}$$

substitution of which in $\tilde{H}_\infty\psi^{(\infty)}_{p_1p_2} = \varepsilon^{(2)}_{p_1p_2}\psi^{(\infty)}_{p_1p_2}$ gives

$$\varepsilon^{(2)}_{p_1p_2} = \varepsilon^{(1)}_{p_1} + \varepsilon^{(1)}_{p_2} \tag{24}$$

where the $\varepsilon^{(1)}_{p_s}$ are calculated according to (21), and the phase $\gamma$ is connected with momenta $p_1$ and $p_2$ by the relation

$$\coth\gamma = \frac{\kappa}{2\pi}\left[\tilde{\zeta}\left(\frac{p_2\omega}{2\pi}\right)-\tilde{\zeta}\left(\frac{p_1\omega}{2\pi}\right)+\frac{p_1-p_2}{\pi}\tilde{\zeta}\left(\frac{\omega}{2}\right)\right]$$

In the limit $\omega \to 0$ this is just the expression for the Bethe phase in the Orbach parametrization. As for the infinite Heisenberg ferromagnet, according to (24), the additivity of magnon energies takes place.

In the case of finite spin systems, consider the Hamiltonian

$$\tilde{H} = -\frac{1}{2} \sum_{\substack{j \neq k \\ j,k=1}}^{N} \mathscr{P}(j-k) \frac{\boldsymbol{\sigma}_j \boldsymbol{\sigma}_k - 1}{2}, \qquad \tilde{H}\,|0\rangle = 0$$

In the same way as for (20) and (22), one can obtain the formulas for the sum of Weierstrass functions,

$$\begin{aligned}\sum_{k=0}^{N-1} \exp\left(\frac{2\pi i}{N} mk\right) \mathscr{P}(k+z) &= -\frac{\tilde{\sigma}(z+r_m)}{\tilde{\sigma}(z-r_m)} \exp\left(\frac{2\tilde{\zeta}(\omega/2)\, mz}{N}\right)\Bigg[\tilde{\mathscr{P}}(z) - \tilde{\mathscr{P}}(r_m) \\ &\quad + \left[\tilde{\zeta}(r_m) - \frac{2r_m}{\omega}\tilde{\zeta}\left(\frac{\omega}{2}\right)\right]\left(\frac{\tilde{\mathscr{P}}'(z) - \tilde{\mathscr{P}}'(r_m)}{\tilde{\mathscr{P}}(z) - \tilde{\mathscr{P}}(r_m)} - \frac{\tilde{\mathscr{P}}''(r_m)}{\tilde{\mathscr{P}}'(r_m)}\right)\Bigg] \end{aligned} \qquad (25a)$$

where $m \in \mathbb{Z}$, $m < N$; $r_m = -\omega m/2N$, and

$$\begin{aligned}\sum_{k=0}^{N-1} \mathscr{P}(k+z) \frac{\sigma(k-l+\gamma+z)}{\sigma(k-l+z)} \exp(i\alpha k) &= -\frac{\sigma(l-\gamma)}{\sigma(l)} \frac{\tilde{\sigma}(z+r_{\alpha\gamma})}{\tilde{\sigma}(z-r_{\alpha\gamma})} \\ &\quad \times \exp\left\{\frac{z}{2\pi i}\left[\zeta\left(\frac{N}{2}\right)\tilde{\zeta}\left(\frac{\omega}{2}\right)\gamma + i\zeta\left(\frac{\omega}{2}\right)\alpha\right]\right\} \\ &\quad \times \left\{\tilde{\mathscr{P}}(z) - \tilde{\mathscr{P}}(r_{\alpha\gamma}) + \left(\frac{\tilde{\mathscr{P}}'(z) - \tilde{\mathscr{P}}'(r_{\alpha\gamma})}{\tilde{\mathscr{P}}(z) - \tilde{\mathscr{P}}(r_{\alpha\gamma})} - \frac{\tilde{\mathscr{P}}''(r_{\alpha\gamma})}{\tilde{\mathscr{P}}'(r_{\alpha\gamma})}\right)\right. \\ &\quad \times \left[\tilde{\zeta}(r_{\alpha\gamma}) + \frac{\zeta(l-\gamma) - \zeta(l)}{2} - \frac{e^{i\alpha l}\sigma(\gamma)\,\sigma(l)}{2\sigma(l-\gamma)}\mathscr{P}(l)\right. \\ &\quad \left.\left. + \frac{\zeta(N/2)\,\tilde{\zeta}(\omega/2)\gamma + i\zeta(\omega/2)\alpha}{4\pi i}\right]\right\} \end{aligned} \qquad (25b)$$

where $\alpha$ and $\gamma$ are connected by

$$\exp[i\alpha N + \gamma\zeta(N/2)] = 1, \qquad l \in \mathbb{Z}$$

$$r_{\alpha\gamma} = -(4\pi)^{-1}\,[\alpha\omega + i^{-1}\gamma\zeta(\omega/2)]$$

The expression for the energy of the spin wave (19) for which the quasimomenta $\{p\}$ are quantized according to the periodicity condition $p_m = (2\pi/N)m$, $0 \leqslant m \leqslant N-1$, $m \in \mathbb{Z}$, can be easily found from (25a),

$$\varepsilon^{(1)}(p_m) = \varphi(r_m), \qquad r_m = -\frac{\omega m}{2N}$$

$$\varphi(r) = \tilde{\mathscr{P}}(r) + \frac{\tilde{\mathscr{P}}''(r)}{\tilde{\mathscr{P}}'(r)}\left[\tilde{\zeta}(r) - \frac{2r}{\omega}\zeta\left(\frac{\omega}{2}\right)\right]$$
$$+ 2\left[\tilde{\zeta}(r) - \zeta\left(\frac{\omega}{2}\right)\frac{2r}{\omega}\right]^2 + \frac{2}{\omega}\left[\zeta\left(\frac{\omega}{2}\right) - N\zeta\left(\frac{\omega}{2}\right)\right]$$

Let us search for the vectors of two-magnon states in a form analogous to (23),

$$\psi_{p_1 p_2}^{(N)} = \sum_{\substack{k_1, k_2 = 1 \\ k_1 \neq k_2}}^{N} \left[ e^{i(p_1 k_1 + p_2 k_2)} \frac{\sigma(k_1 - k_2 + \gamma)}{\sigma(k_1 - k_2)} \right.$$
$$\left. + e^{i(p_2 k_1 + p_1 k_2)} \frac{\sigma(k_1 - k_2 - \gamma)}{\sigma(k_1 - k_2)} \right] a_{k_1}^{+} a_{k_2}^{+} |0\rangle \tag{26}$$

The quasimomenta $p_1$ and $p_2$ and the phase $\gamma$ must be determined from the periodicity conditions and $\tilde{H}\psi_{p_1 p_2}^{(N)} = \varepsilon_{p_1 p_2}^{(2)} \psi_{p_1 p_2}^{(N)}$. By using Eq. (25b), one makes sure that (26) is just the eigenvector of $\tilde{H}$ with the eigenvalue

$$\varepsilon_{p_1 p_2}^{(2)} = \varphi(r_{p_1 \gamma}) + \varphi(r_{p_2 \gamma}) + \mathscr{P}(\gamma) - \zeta^2(\gamma)$$

where

$$r_{p_1 \gamma} = -(4\pi)^{-1}\,[p_1 \omega + i^{-1}\gamma\zeta(\omega/2)]$$
$$r_{p_2 \gamma} = -(4\pi)^{-1}\,[p_2 \omega + i^{-1}\gamma\zeta(\omega/2)]$$

and $(p_1, p_2, \gamma)$ is an arbitrary solution of the system of transcendental equations

$$\exp\left[ip_1 N + 2\gamma\zeta\left(\frac{N}{2}\right)\right] = 1$$

$$\exp\left[ip_2 N - 2\gamma\zeta\left(\frac{N}{2}\right)\right] = 1$$

$$\tilde{\zeta}(2r_{p_1\gamma}) - \tilde{\zeta}(2r_{p_2\gamma}) + \frac{4\tilde{\zeta}(\omega/2)}{\omega}(r_{p_2\gamma} - r_{p_1\gamma}) + \frac{4\zeta(\omega/2)\gamma}{\omega} - 2\zeta(\gamma) = 0$$

In the limit $\omega \to 0$ these equations coincide with the equations of the Bethe ansatz for the quasimomenta of two-magnon states in a periodic Heisenberg chain.

The investigation of states with a larger number of magnons can be performed in the case of the infinite chain on the basis of the summation formula for trigonometric series generalizing (22),

$$\sum_{k=-\infty}^{\infty} \frac{\pi^2 e^{ikp}}{\kappa^2 \{\sinh[(\pi/\kappa)(k+z)]\}^2} \prod_{\lambda=1}^{n} \coth\frac{\pi}{\kappa}(k+z+l_\lambda)$$
$$= -\frac{\tilde{\sigma}(z+r_p)}{\tilde{\sigma}(z-r_p)} \exp\left[\frac{pz}{\pi}\tilde{\zeta}\left(\frac{\omega}{2}\right)\right]$$
$$\times\left(\prod_{\lambda=1}^{n} \coth\frac{\pi l_\lambda}{\kappa}\right)\left[\tilde{\wp}(z)-\tilde{\wp}(r_p)+\left(\frac{\tilde{\wp}'(z)-\tilde{\wp}'(r_p)}{\tilde{\wp}(z)-\tilde{\wp}(r_p)}-\frac{\tilde{\wp}''(r_p)}{\tilde{\wp}'(r_p)}\right)\right.$$
$$\times\left(\tilde{\zeta}(r_p)-\frac{2r_p}{\omega}\tilde{\zeta}\left(\frac{\omega}{2}\right)\right)$$
$$+\frac{\pi}{\kappa}\left\{\sum_{\nu=1}^{n}\left(\sinh\frac{2\pi l_\nu}{\kappa}\right)^{-1}-\sum_{\nu=1}^{n}\frac{\exp(-ipl_\nu)}{2[\sinh(\pi l_\nu/\kappa)]^2}\right.$$
$$\left.\left.\times\left[\prod_{\varepsilon\neq\nu}^{n}\coth\frac{\pi}{\kappa}(l_\varepsilon-l_\nu)\right]\left(\prod_{\lambda=1}^{n}\coth\frac{\pi l_\lambda}{\kappa}\right)^{-1}\right\}\right)\right]$$

where $\{l_\lambda\}$ are nonzero integers, and $\prod_{\lambda>\mu}^{n}(l_\lambda - l_\mu) \neq 0$. There is an analogous formula for the summation of a finite series like (25) containing elliptic functions. However, in contrast to the infinite chain, this formula is not useful for the construction of eigenvectors of $\tilde{H}$. The situation bears a strong resemblance to quantum systems of particles on a line. In this case the wave functions can be easily found for the trigonometric Sutherland systems, but for elliptic potentials of pair interactions the single known result is a solution of the Lamé equation for two-particle systems. At the classical level, the trajectories of particle systems in the elliptic case were found by Krichever[6] by the methods of algebraic geometry and the solution contained the multidimensional Riemann theta functions. To my knowledge, nobody at this time has indicated a way of solving the corresponding quantum problem.

## 5. SUMMARY

In this paper the simplest properties of the spin model generalizing the Heisenberg and Haldane–Shastry chains were found. The most important problem for further investigation is the proof of the hypothesis on the existence of the generating function of integrals of motion (17) and finding its connection with the Yang–Baxter equations. As for purely calculational schemes, it would be interesting to indicate a simple way of constructing states with an arbitrary number of magnons, especially for the periodic chain.

It would be interesting also to investigate the possibility of the destruction of $SU(2)$ symmetry of the Hamiltonian. In particular, one can expect in the $XXZ$ case, as in the Haldane–Shastry model, the conservation of integrability for values of the anisotropy parameter $\Delta = m(m+1)/2$, $m \in \mathbb{Z}$, $m > 1$. These numbers appear in the equations determining the Legendre polynomials and Lamé functions as parameters at which these equations have solutions without any branch points. For problems of finding the eigenvectors of a Hamiltonian like (1), this corresponds to the possibility of analytical summation of series of the type

$$\sum_{k=-\infty}^{\infty} \frac{\pi^2}{\kappa^2} \frac{\exp(ikp)}{\{\sinh[(\pi/\kappa)(k+z)]\}^2} P\left[\coth \frac{\pi}{\kappa}(k+z)\right]$$

and their generalizations like (25) and (27) ($P$ denotes an arbitrary polynomial). It is likely that the corresponding formulas would be lengthy and complicated.

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