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Continued Fractions for Some Alternating Series

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Abstract. We discuss certain simple continued fractions that exhibit a type of "self-similar" structure: their partial quotients are formed by perturbing and shifting the denominators of their convergents. We prove that all such continued fractions represent transcendental numbers. As an application, we prove that Cahen's constant

$$C = \sum_{i \geqslant 0} \frac{(-1)^i}{S_i - 1}$$

is transcendental. Here (S_n) is *Sylvester's sequence* defined by $S_0 = 2$ and $S_{n+1} = S_n^2 - S_n + 1$ for $n \geqslant 0$. We also explicitly compute the continued fraction for the number C; its partial quotients grow doubly exponentially and they are all squares.

I. Introduction

In this paper we discuss certain continued fractions that exhibit a type of "self-similar" structure: their partial quotients are formed by perturbing and shifting the denominators of their convergents. The real numbers represented by these continued fractions can also be described as the sum of an alternating series. We prove the transcendence of the numbers by an appeal to Roth's theorem. As an example, we prove that Cahen's constant is transcendental.

II. The Main Result

If a_0 is an integer and $a_1, a_2, \ldots$ is an infinite sequence of positive integers, then, as usual, we let

$$x = [a_0, a_1, a_2, \ldots]$$

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be the real number whose partial quotients are $a_0, a_1, a_2, \ldots$, and

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$$

be the n-th convergent to the continued fraction.

A well-known theorem (see, e.g. ROBERTS [1977, p. 101]) says that

$$\frac{p_n}{q_n} = a_0 + \frac{1}{q_0 q_1} - \frac{1}{q_1 q_2} + \ldots + \frac{(-1)^{n-1}}{q_{n-1} q_n}. \tag{1}$$

Let $w_0, w_1, w_2, \ldots$ be an infinite sequence of positive integers. Put $a_0 = 0$, $a_1 = w_0$, and $a_{n+2} = w_{n+1} q_n$ for $n \geqslant 0$. Hence, $q_0 = 1$, $q_1 = w_0$, and $q_{n+2} = q_n(w_{n+1} q_{n+1} + 1)$ for $n \geqslant 0$.

Using equation (1), we see

$$\frac{p_{n+1}}{q_{n+1}} = \sum_{0 \leqslant i \leqslant n} \frac{(-1)^i}{q_i q_{i+1}} = [0, w_0, w_1 q_0, w_2 q_1, \ldots, w_n q_{n-1}].$$

Hence, letting $n \to \infty$, we find the following way to express the continued fraction as an alternating series:

Proposition 1.

$$x = \sum_{i \geqslant 0} \frac{(-1)^i}{q_i q_{i+1}} = [0, w_0, w_1 q_0, w_2 q_1, w_3 q_2, \ldots]. \tag{2}$$

Notice that the sequence of partial quotients in the continued fraction for x is formed by shifting the sequence of the denominators of the convergents, and then perturbing by the sequence (w_n).

We now prove that all such x are transcendental. To do so, we require the following simple lemma:

Lemma 2. *For all sequences of positive integers (w_n), and all $n \geqslant 3$, at least one of the following inequalities holds:*

$$w_n q_{n-1} \geqslant \sqrt{q_n};$$

$$w_{n+1} q_n \geqslant \sqrt{q_{n+1}};$$

or

$$w_{n+2} q_{n+1} \geqslant \sqrt{q_{n+2}}.$$

Proof. Assume that none of the inequalities hold so $w_n q_{n-1} <$

$< \sqrt{q_n}$, $w_{n+1} q_n < \sqrt{q_{n+1}}$, and $w_{n+2} q_{n+1} < \sqrt{q_{n+2}}$. Square each inequality and multiply together to get

$$w_n^2 w_{n+1}^2 w_{n+2}^2 q_{n-1}^2 q_n^2 q_{n+1}^2 < q_n q_{n+1} q_{n+2}.$$

Hence

$$w_n^2 w_{n+1}^2 w_{n+2}^2 q_{n-1}^2 q_n q_{n+1} < q_{n+2}.$$

Since $n \geqslant 3$, $q_{n-1} > 1$, so

$$q_n(w_{n+1} q_{n+1} + 1) < q_n(4w_{n+1} q_{n+1}) \leqslant q_n(q_{n-1}^2 w_{n+1} q_{n+1}) \leqslant \\ \leqslant w_n^2 w_{n+1}^2 w_{n+2}^2 q_{n-1}^2 q_n q_{n+1} < q_{n+2},$$

a contradiction. ∎

We can now complete the transcendence proof:

Theorem 3. *For all sequences of positive integers* (w_n), *the number* x *in Proposition 1 is transcendental.*

Proof. Let x be the number in equation (2), with continued fraction expansion $x = [a_0, a_1, a_2, \ldots]$. By a classical theorem (see, e.g., HARDY and WRIGHT [1985]), we have

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1} q_n^2}.$$

By definition of x, however, $a_{n+1} = w_n q_{n-1}$. Hence

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{w_n q_{n-1} q_n^2}.$$

But by Lemma 1, $w_n q_{n-1} \geqslant \sqrt{q_n}$ for infinitely many n, so we see that

$$\left| x - \frac{p_n}{q_n} \right| < q_n^{-5/2}$$

infinitely often. Hence by a theorem of ROTH [1955], x is transcendental. ∎

III. Two Examples

A. Suppose we put $w_i = 1$ for all $i \geqslant 0$. Then $q_0 = 1$, $q_1 = 1$, and $q_{n+2} = q_n(q_{n+1} + 1)$ for $n \geqslant 0$. Here is a short table of this sequence:

$n =$	0	1	2	3	4	5	6	7	8	...
$q_n =$	1	1	2	3	8	27	224	6075	1361024	...

Hence we find

$$\sum_{i \geqslant 0} \frac{(-1)^i}{q_i q_{i+1}} = [0, 1, q_0, q_1, q_2, \ldots],$$

and this number is transcendental.

It is perhaps worthwhile to comment on the asymptotic behavior of the sequence (q_n) in this example. Using the techniques from AHO and SLOANE [1973] or GREENE and KNUTH [1982], it can be shown that there exist two constants c_1 and c_2 such that

$$q_n = \lfloor c_1^{\alpha^n} c_2^{\beta^n} \rfloor$$

for $n \geqslant 0$, where $\alpha = \frac{1}{2}(1 + \sqrt{5})$ and $\beta = \frac{1}{2}(1 - \sqrt{5})$. The approximate values of c_1 and c_2 are 1.3505061 and 1.4298155, respectively.

B. Suppose we put $w_0 = 1$ and $w_{i+1} = q_i$ for $i \geqslant 0$. Then we find $q_0 = 1$, $q_1 = 1$, and $q_{n+2} = q_n^2 q_{n+1} + q_n$ for $n \geqslant 0$.

Define $b_n = q_n q_{n+1}$. Then we prove by induction that $b_n = S_n - 1$, where (S_n) is *Sylvester's sequence*: $S_0 = 2$, and $S_{n+1} = S_n^2 - S_n + 1$ for $n \geqslant 0$. This is clearly true for $n = 0$. Now

$$\begin{aligned} S_{n+1} - 1 &= S_n^2 - S_n = (b_n + 1)^2 - (b_n + 1) = \\ &= (q_n q_{n+1})^2 + q_n q_{n+1} = q_{n+1}(q_n^2 q_{n+1} + q_n) = \\ &= q_{n+1} q_{n+2}, \end{aligned}$$

and the proof is complete by induction.

It is also easy to prove by induction that $q_n > 2^{2^{n-3}}$ for $n \geqslant 3$. Here are the first few values of these sequences:

$n =$	0	1	2	3	4	5	6	...
$q_n =$	1	1	2	3	14	129	25298	...
$b_n =$	1	2	6	42	1806	3263442	10650056960806	...
$S_n =$	2	3	7	43	1807	3263443	10650056960807	...

Hence we find

$$\begin{aligned} C = \sum_{i \geqslant 0} \frac{(-1)^i}{S_i - 1} &= [0, 1, q_0^2, q_1^2, q_2^2, \ldots] = \\ &= [0, 1, 1, 1, 4, 9, 196, 16641, 639988804, \ldots]. \end{aligned}$$

The number C is *Cahen's constant*. Theorem 3 shows that C is transcendental.

We now make some historical remarks. To the best of our knowledge, the sequence (S_n) was first mentioned by SYLVESTER [1880a, 1880b]. Detailed work on this sequence was done by GOLOMB [1963b]; AHO and SLOANE [1973]; and FRANKLIN and GOLOMB [1975]. (Aho and Sloane attribute the sequence to E. Lucas, but this appears to be in error.) Other papers that mention the sequence, in a variety of contexts, include CARMICHAEL [1915, pp. 115—116]; KELLOGG [1921]; TAKENOUCHI [1921]; CURTISS [1922]; CURTISS [1929]; SALZER [1947]; ERDÖS [1950]; GOLOMB [1963a]; ERDÖS and STRAUS [1964]; ZAKS, PERLES, and WILLS [1982]; HENSLEY [1983]; ODONI [1985]; and LAGARIAS and ZIEGLER [1990]. It is sequence #331 in SLOANE [1973]. Also see SALZER [1948].

The number

$$C = \sum_{i \geqslant 0} \frac{(-1)^i}{S_i - 1} \doteq .64341\ 05462\ 88338$$

was apparently first discussed by CAHEN [1891], who proved it was irrational. The number C was also mentioned by REMEZ [1951].

It is perhaps worthwhile to point out that the related number

$$C' = \sum_{i \geqslant 0} \frac{(-1)^i}{S_i}$$

is also transcendental. This follows because $2C = C' + 1$, as is easily proved by induction. Furthermore, the continued fraction for C' is

$$\begin{aligned} C' &= [0, 3, 2h_1^2, 2h_2^2, 2h_3^2, \ldots] = \\ &= [0, 3, 2, 18, 98, 33282, 319994402, \ldots], \end{aligned}$$

where $h_0 = 1$ and $S_n - 1 = 2h_{n-1}h_n$ for $n \geqslant 1$. A proof of this can be found in an earlier version of this paper; see SHALLIT [1990].

IV. A Converse to Proposition 1

Let (x_i) be a sequence of positive integers, and suppose $0 < x < 1$ is a real irrational number such that $x = \sum_{i \geqslant 0}(-1)^i/x_i$. Let

$x = [0, a_1, a_2, \ldots]$ be the continued fraction expansion of x, and, as usual, put

$$\frac{p_n}{q_n} = [0, a_1, a_2, \ldots, a_n].$$

We prove the following "converse" to Proposition 1:

Theorem 4. *Assume that*

$$\sum_{0 \leqslant i \leqslant n} \frac{(-1)^i}{x_i} = \frac{p_{n+1}}{q_{n+1}}$$

for all $n \geqslant 0$, *and that* $x_n \mid x_{n+1}$ *for all* $n \geqslant 0$. *Then (i)* $x_n = q_n q_{n+1}$ *for* $n \geqslant 0$ *and (ii)* $q_n \mid a_{n+2}$ *for* $n \geqslant 0$.

Proof. Part (i) is easily verified for $n = 0$. We now prove it for $n \geqslant 1$. We find

$$\frac{p_{n+1}}{q_{n+1}} = \sum_{0 \leqslant i \leqslant n} \frac{(-1)^i}{x_i} = \frac{p_n}{q_n} + \frac{(-1)^n}{x_n}.$$

Hence

$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{x_n}.$$

By a classical theorem on continued fractions (see, e.g., HARDY and WRIGHT [1985, Theorem 150]), we have

$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_{n+1} q_n},$$

and hence $x_n = q_{n+1} q_n$.

To prove (ii), we note that $x_n \mid x_{n+1}$ implies that $q_n q_{n+1} \mid q_{n+1} q_{n+2}$; hence $q_n \mid q_{n+2}$. But by a standard identity in continued fractions, $q_{n+2} = a_{n+2} q_{n+1} + q_n$, so $q_n \mid a_{n+2} q_{n+1}$. By another standard identity, $\gcd(q_n, q_{n+1}) = 1$, so $q_n \mid a_{n+2}$. This completes the proof. ■

V. Concluding Remarks

There are only a few "naturally-occurring" transcendentals for which the continued fraction is explicitly known. Expansions for $e^{2/k}$ and $\tan(1/k)$ (integer $k \geqslant 1$) are well known. For some others, see BÖHMER [1926]; DANILOV [1972]; DAVISON [1977]; ADAMS and DAVISON

[1977]; SHALLIT [1979]; KMOŠEK [1979]; KÖHLER [1980]; SHALLIT [1982]; PETHÖ [1982]; BLANCHARD and MENDÈS FRANCE [1982]; VAN DER POORTEN and SHALLIT [1990]; and TAMURA [1990].

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