Peristaltic transport of a power-law fluid: application to the ductus efferentes of the reproductive tract

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Abstract: The problem of peristaltic transport of a non-Newtonian (power-law) fluid in uniform and non-uniform two-dimensional channels has been investigated under zero Reynolds number with long wavelength approximation. A comparison of the results with those for a Newtonian fluid model shows that the magnitude of pressure rise, under a given set of conditions, is smaller in the case of the non-Newtonian fluid (power-law index $n < 1$) at zero flow rate. Further, the pressure rise is smaller as n decreases from 1 at zero flow rate, is independent of n at a certain value of flow rate and becomes greater if flow rate increases further. Also, at a given flow rate, an increase in wavelength leads to a decrease in pressure rise and increase in the influence of non-Newtonian behaviour. Pressure rise in the case of non-uniform geometry, is found to be much smaller than the corresponding value in the case of uniform geometry. Finally, the analysis is applied and compared with observed flow rates in the ductus efferentes of the male reproductive tract.

Key words: Peristaltic transport, power-law fluid, long wavelength approximation, ductus efferentes

1. Introduction

The study of the mechanism of peristalsis, in both mechanical and physiological situations, has recently become the object of scientific research. Since the first investigation of Latham [1], several theoretical and experimental attempts have been made to understand peristaltic action in different situations. All such investigations seem to differ in various details. A review of much of the early literature is presented in an article by Jaffrin and Shapiro [2] and in a monograph by Rath [3]. A summary of most of the experimental and theoretical investigations reported so far, with details of the geometry, fluid, Reynolds number, wavelength parameter, wave amplitude parameter, and wave shape, has been presented in a recent paper by Srivastava and Srivastava [4].

Most of these theoretical investigations have been carried out by assuming that the blood and other physiological fluids behave like Newtonian fluids. Although this approach provides a satisfactory understanding of the peristaltic mechanism in the ureter, it fails to give a good understanding of the peristaltic mechanism in small blood vessels, lymphatic vessels, intestine, ductus efferentes of the male reproductive tract, or in transport of spermatozoa in the cervical canal. It is known that most of physiological fluids behave like non-Newtonian fluids, but it appears that no rigorous attempt has been made to understand the problem of non-Newtonian fluids since the inital investigation by Raju and Devanathan [5, 6]. Recently, some research on non-Newtonian fluids are reported in the literature [3, 7–10]. It has been pointed out that the flow behaviour of blood in vessels of small diameter (0.02 cm) and at low shear rates ($< 20\ s^{-1}$) can be represented by a power-law fluid [11, 12]. Lew et al. [13] suggested chyme as a non-Newtonian material having plastic-like properties. In addition, physiological organs are generally observed to be a non-uniform duct [14–16]. In particular, vas deferens in the rhesus monkeys is in the form of a diverging tube with a ratio of exit to inlet dimensions of approximately four [17]. Hence, peristaltic analysis of a Newtonian fluid in a uniform geometry cannot be applied when explaining the mechanism of transport of fluid in most bio-systems. Recently, Gupta and Seshadri [18], Srivastava and Srivastava [19] and Srivastava et al. [20] studied peristaltic transport of a Newtonian fluid in non-uniform geometries.

With the above discussion in mind, we propose to study the peristaltic transport of a power law fluid in uniform and non-uniform two-dimensional channels. The applicability of the results to the flow rates observed in the ductus efferentes of the male reproductive tract is discussed.

2. Formulation and Analysis

Consider the flow of a power-law fluid through a two-dimensional channel of non-uniform thickness with a sinusoidal wave travelling down its wall. The geometry of the wall surface (cf. figure 1) is described as

$$H(x',t') = a(x') + b\sin\frac{2\pi}{\lambda}(x'-ct'), \tag{1}$$

with

$$a(x') = a_0 + kx', \tag{2}$$

where $a(x')$ is the half-width of the channel at any axial distance x' from inlet, a_0 is the half-width at inlet, $k(\ll 1)$ is a constant whose magnitude depends on the length of the channel and exit and inlet dimensions, b is the amplitude of the wave, λ is the wavelength, c is the propagation velocity, and t' is the time.

Using the long wavelength approximation and neglecting the inertia terms, the appropriate equations describing the flow in the laboratory frame of reference are

$$\frac{\partial p'}{\partial x'} = -\frac{\partial \tau_{y'x'}}{\partial y'}, \tag{3}$$

$$\frac{\partial p'}{\partial y'} = 0, \tag{4}$$

where p' is the pressure, y' and x' are the radial and axial coordinates respectively, and $\tau_{y'x'}$ the shear stress normal to y' in the x' direction,

In the case of a power-law fluid model, $\tau_{y'x'}$ is given as

$$\tau_{y'x'} = m\left(-\frac{\partial u'}{\partial y'}\right)^n, \tag{5}$$

where m is the consistency and n the power-law index. (When $n=1$, then $m=\mu$ is the Newtonian viscosity of the fluid). In view of eq. (5), eq. (3) assumes non-dimensional form as

$$\frac{dp}{dx} = -\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)^n, \tag{6}$$

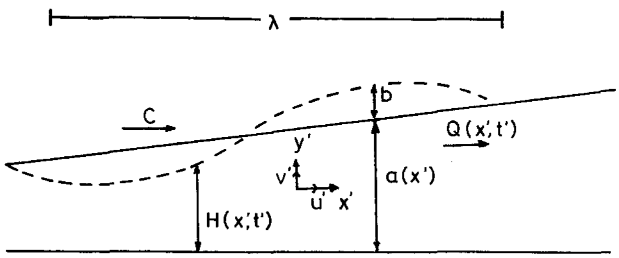

Fig. 1. Peristaltic transport in a non-uniform channel

where

$$y = \frac{y'}{a_0}, \quad x = \frac{x'}{\lambda}, \quad u = \frac{u'}{c}, \quad t = \frac{ct'}{\lambda}, \quad p = p'\frac{a_0^{n+1}}{m\lambda c^n}. \tag{7}$$

The dimensionless boundary conditions are:

$$\frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = 0, \tag{8}$$

$$u = 0 \quad \text{at} \quad y = h = 1 + \frac{\lambda k x}{a_0} + \phi\sin 2\pi(x-t), \tag{9}$$

with $h = H/a_0$ and ϕ (amplitude ratio) $= b/a_0 < 1$.

Integrating eq. (6) and using the boundary conditions (8) and (9), one finds the expression for the velocity profile as

$$u(x,y,t) = \frac{n}{n+1}\left(-\frac{dp}{dx}\right)^{\frac{1}{n}}\left[h^{\frac{n+1}{n}} - y^{\frac{n+1}{n}}\right]. \tag{10}$$

The instantaneous volume flow rate $Q(x,t)$ is given by

$$Q(x,t) = \int_0^h u\,dy = \frac{n}{2n+1}\left(-\frac{dp}{dx}\right)^{\frac{1}{n}} h^{\frac{2n+1}{n}} \tag{11}$$

or

$$\frac{dp}{dx} = -\left(\frac{2n+1}{n}\right)^n \frac{Q^n(x,t)}{h^{2n+1}}. \tag{12}$$

The pressure rise $\Delta p_L(t)$ and the friction force $F_L(t)$ (at the wall) in the channel of length L, in their non-dimensional forms, are given by

$$\Delta p_L(t) = \int_0^{L/\lambda}\left(\frac{dp}{dx}\right)dx, \tag{13}$$

$$F_L(t) = \int_0^{L/\lambda} h\left(-\frac{dp}{dx}\right)dx. \tag{14}$$

Use of eqs. (9) and (12) in eqs. (13) and (14) yields

$$\Delta p_L(t) = -\left(\frac{2n+1}{n}\right)^n \cdot \int_0^{L/\lambda} \frac{Q^n(x,t)}{\left[1 + \frac{\lambda k x}{a_0} + \phi\sin 2\pi(x-t)\right]^{2n+1}}dx, \tag{15}$$

$$F_L(t) = \left(\frac{2n+1}{n}\right)^n \cdot \int_0^{L/\lambda} \frac{Q^n(x,t)}{\left[1 + \frac{\lambda k x}{a_0} + \phi\sin 2\pi(x-t)\right]^{2n}}dx. \tag{16}$$

Setting $k = o$ in eqs. (15) and (16), we obtain expressions for the pressure rise and friction force in a uniform channel. In addition, with $n=1$, the expressions reduce to the same results as those of Shapiro et al. [21] and Lardner and Shack [22], when the eccentricity of the elliptical motion of cilia tips is zero in their analysis. Further, with $n = 1$, eq. (15) reduces to that obtained by Gupta and Seshadri [18] for a Newtonian fluid of constant viscosity. The analytical interpretations and comparisons of our analyses with other theories are difficult to make at this stage, as the integrals in eqs. (15) and (16) are not integrable in closed form, neither for non-uniform nor for uniform geometry ($k = o$). Thus, further studies of our analysis are only possible after numerical evaluation of these integrals.

3. Numerical results and discussion

In order to be able to discuss the results obtained above quantitatively, we assume the form of instantaneous volume flow rate $Q(x, t)$, periodic in $(x-t)$ as [18, 19].

$$Q^n(x, t) = (\bar{Q})^n + \phi \sin 2\pi (x-t), \tag{17}$$

where $\bar{Q}$ is the time-average of the flow over one period of the wave. this form of $Q(x, t)$ has been assumed in view of the fact that the constant value of $Q(x, t)$ gives $\Delta p_L(t)$, always negative, and hence there will be no pumping action. Using this form of $Q(x, t)$, we now compute the dimensionless pressure rise $\Delta p_L(t)$ and friction force $F_L(t)$ over the channel length for various values of the dimensionless flow average $\bar{Q}$, amplitude ratio ϕ, and the power-law index n. The average rise in pressure $\Delta \bar{p}_L$, and friction force $\bar{F}_L$ are then evaluated by averaging $\Delta p_L(t)$ and $F_L(t)$ over one period. Using the following values of the parameters [18, 19].

$$a_0 = 0.012 \text{ cm}, \quad L = \lambda = 20 \text{ cm}, \quad k = 3a_0/L = 0.018,$$

the integrals in eqs. (15) and (16) are numerically evaluated. Figures 2 and 3 represent the variation of dimensionless pressure rise with dimensionless time for $\phi = 0.8$ and power-law indexes $n = 1$, 2/3, 1/3 for non-uniform and uniform channels, respectively. A comparison of the results with those of a Newtonian fluid model shows that the pressure rise decreases as the power law index n decreases from 1. The difference between two corresponding values of the Newtonian and non-Netwonian pressure rise becomes greater with decreasing flow rate and becomes maximum at zero flow rate. It is also observed that as n tends towards 1, the values for the non-Newtonian fluid approach the corresponding values for the Newtonian fluid in both uniform and non-uniform geometries. Figures 2 and 3 further reveal that the magnitude of the pressure rise in the case of the non-uniform channel is much smaller than the corresponding value in the case of the uniform channel. It can also be seen that the effect

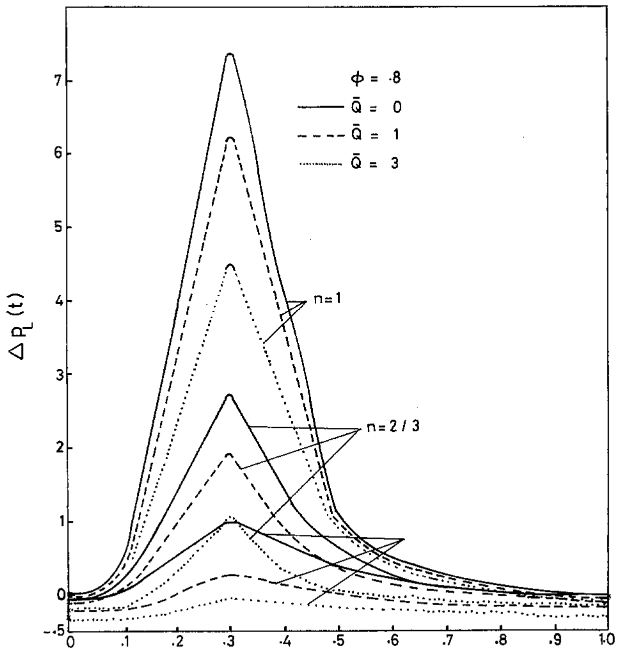

Fig. 2. Variation of pressure rise over the length of a non-uniform channel

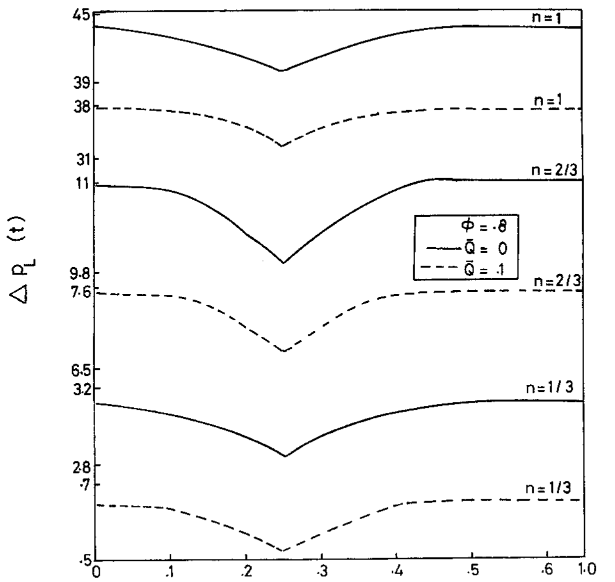

Fig. 3. Variation of pressure rise over the length of a uniform channel

of increasing the flow rate is to reduce the pressure rise for various values of ϕ and n. The pressure flow rate relationship in the non-uniform channel, for various values of ϕ and n, is shown in figure 4. As expected, the curves show a linear relation between pressure rise and flow rate for

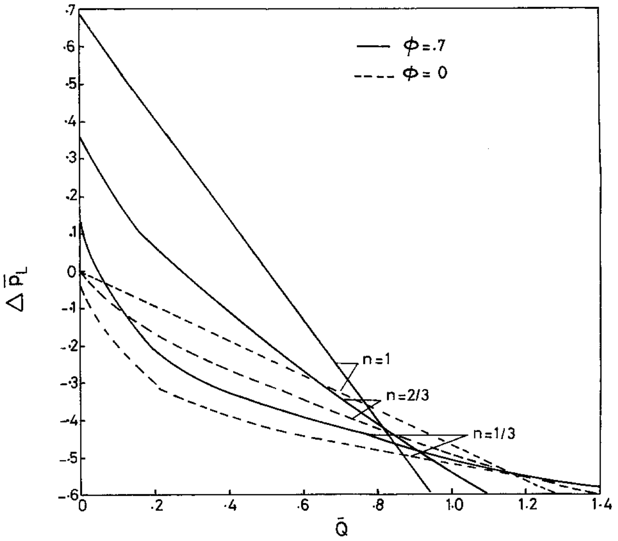

Fig. 4. Pressure vs flow rate for a non-uniform channel

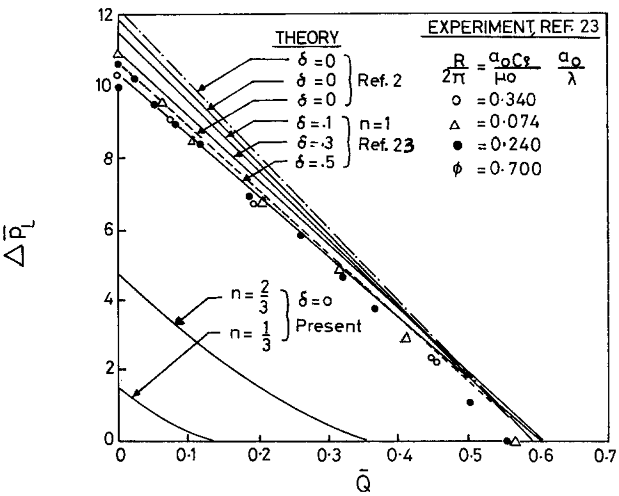

Fig. 5. Pressure vs flow rate for a two-dimensional uniform channel. ——, two-dimensional theory for $R=0$, $\alpha=0$; —·—·—, two-dimensional theory corrected for wall ends according to [2]; -----, two-dimensional theory corrected for wall ends and inactive pumping regions [2]. Data points – experimental results of Weinberg et al. [24]. δ represents a viscosity variation parameter [23], R is the Reynolds number and α the wave number

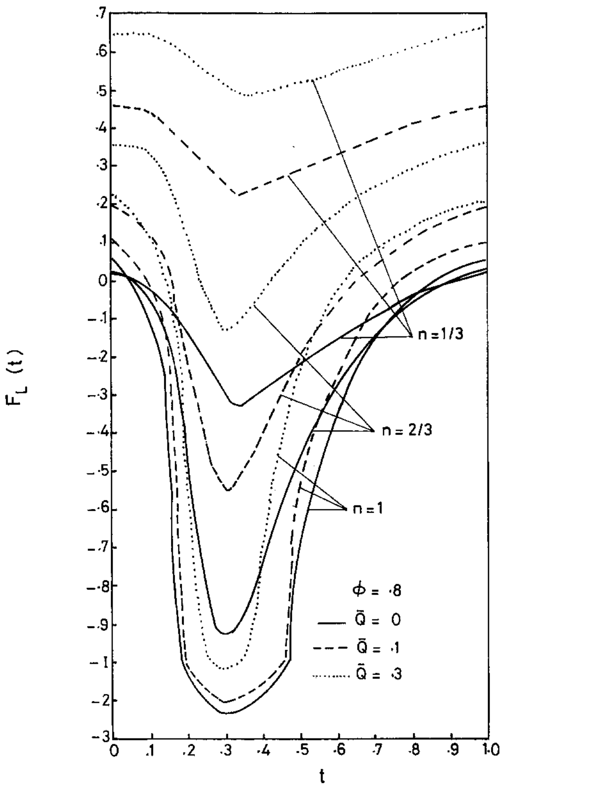

Fig. 6. Variation of friction force over the length of a non-uniform channel

the Newtonian fluid ($n=1$), and a non-linear relation for the non-Newtonian fluid ($n=2/3$, $1/3$). It is clear that an increase in flow rate reduces the pressure rise; thus maximum flow rate is achieved at zero pressure rise, and maximum pressure rise occurs at zero flow rate. Also, the effect of increasing the amplitude ratio is an increase in the pressure rise for both Newtonian and non-Newtonian fluids in uniform as well as in non-uniform geometries. Furthermore, the pressure rise becomes smaller as n decreases from 1, at zero flow rate, is independent of n at certain value of flow rate, and becomes greater if the flow rate increases further. A comparison of the present analysis with the theoretical results of Jaffrin and Shapiro [2] and Srivastava et al. [23] and with the experimental results of Weinberg et al. [24] is given in figure 5. Finally, the friction forces $F_L(t)$ and $\bar{F}$ are plotted in figures 6 and 7 respectively, these shows that the friction forces have the opposite behavior compared to pressure rise.

4. Application to transport of semen in the ductus efferentes of the male reproductive tract

In this section, we discuss whether our theoretical analysis of peristaltic flow in a uniform two-dimensional channel is applicable to an explanation of semen transport in the ductus efferentes of the male reproductive tract. On the basis of experimental observations [25–27] on the flow rates in the rete testis of rat, ram, and bull, Lardner and Shack [22] estimated the approximate flow

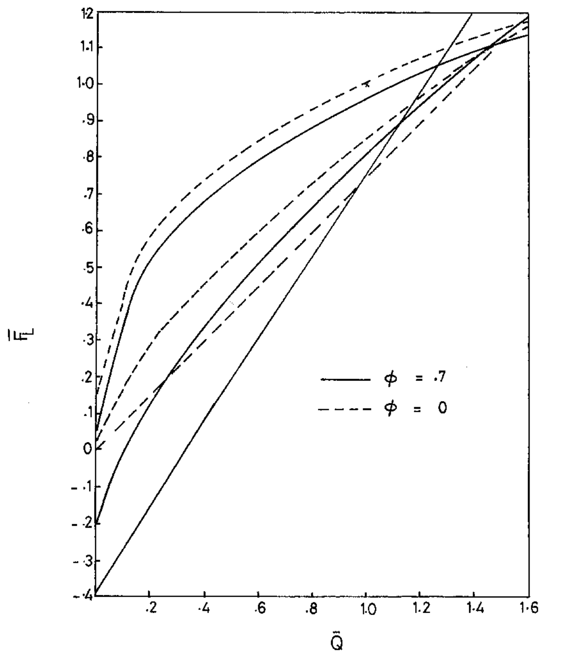

Fig. 7. Friction force vs flow rate for a non-uniform channel

Table 1. Flow rate versus pressure drop in ductus efferentes of the male reproductive tract

	Newtonian ($n=1$) [29]			Non-Newtonian ($n=1/3$), this work		
Pressure drop ($-\Delta\bar{p}$)	0	0.05	0.1	0	0.05	0.1
Flow rate ($\mathrm{ml/h} \cdot 10^{-3}$)	0.084	0.18	0.27	0.49	0.68	0.97
Error (%)	98.59	97.07	95.55	91.83	88.66	83.83

rate in human rete testis per ductus efferentes as $6 \cdot 10^{-3}$ ml/h. The approximate value of various parameters for flow in ductus efferentes, reported in Lardner and Shack [22], are

$$a_0 = 0.005 \text{ cm}, \quad c = 0.02 \text{ cm/s}, \quad \lambda = 0.05 \text{ cm}, \quad \phi = 0.1 .$$

These values justify the use of long wavelength and zero Reynolds number theory of the present analysis. Lardner and Shack [22] calculated the flow rate, assuming that no pressure gradient exists in the ductus efferentes. But in the ductus efferents, water is known to be pushed into the tubes across the membranes, creating a positive flux [28]. Thus, the flux may exceed $\bar{Q}_{max} (= \bar{Q}$, at $\Delta\bar{p} = 0)$ and hence $\Delta\bar{p} < 0$. Using these data, $\bar{Q}$ is calculated. Finally, the various values of flow rate ($= \bar{Q} \times \pi a_0^2 c$) in ml/h, and their differences with observed values are presented in table 1.

The considerable difference between the theoretical and the observed values shows that the peristalsis cannot account for the total flow rate in ductus efferents. There must be some other important factors, responsible for the transport of semen, one may probably be the cillia, which keeps semen moving towards the epididymis [30]. Theoretical investigations concerning the effect of cillia on fluid transport in ductus efferentes and other biological organs, include the work of Lardner and Shack [22], Liron [28], Winet and Blake [31], and Blake and Winet [32]. However, there is no doubt that the peristalsis aids in moving semen in ductus efferentes. Considerably more theoretical and experimental investigations are however necessary to understand adequately the mechanism involved in transport of semen in ductus efferentes.

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